ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA

LUBLIN-POLONIA

VOL. XLVII, 9 SECTIO A 1993

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Quasisymmetric Functions and Quasihomographies

Abstract. J. Zając introduced in [8] quasihomographies $\mathcal{A}_\Gamma(K)$ as automorphisms of a circle Γ changing the cross-ratio of points on Γ in a uniformly bounded manner according to the formulas (2.1) and (2.4). In this paper a comparison between $\mathcal{A}_{\mathbb{T}}(K)$ and the class $Q(M)$ of M-quasisymmetric automorphisms of the unit circle $\mathbb{T}$ is presented.

Quasisymmetric functions. Quasisymmetric functions appear in the problem of the boundary correspondence under quasiconformal (abbr.: qc.) mappings of Jordan domains in the extended plane $\widehat{\mathbb{C}}$. Let $G \subset \widehat{\mathbb{C}}$ be a Jordan domain and let f be a qc. self-mapping of G . As shown by Ahlfors [1], f has a homeomorphic extension on the closure $\overline{G}$ of G . In other words, a qc. automorphism of a Jordan domain $G \subset \widehat{\mathbb{C}}$ generates an automorphism of the boundary curve ∂G . Here and in what follows an automorphism of an orientable manifold S means a homeomorphic sense-preserving self-mapping of S . We denote the class of automorphisms of S by $\mathrm{Aut}(S)$. The problem of characterizing the elements of $\mathrm{Aut}(\partial G)$ generated by a mapping $w \in \mathrm{Aut}(G)$ was solved three years later by Beurling and Ahlfors [2].

Because of Brouwer's fixed point theorem every $f \in \mathrm{Aut}(\overline{G})$, $G \subset \widehat{\mathbb{C}}$ being a Jordan domain, has a fixed point z_0 . For $z_0 \in \partial G$ and a conformal mapping Ψ of G onto the upper half-plane U such that $\Psi(z_0) = \infty$ the composition $\Psi \circ f \circ \Psi^{-1} \in \mathrm{Aut}(\overline{U})$ has ∞ as a fixed point. Then the generated automorphism of $\mathbb{R} = \partial U$ is a continuous strictly increasing function φ which satisfies $\varphi(-\infty) = -\infty$, $\varphi(+\infty) = +\infty$. We have

Theorem A [2]. *A continuous strictly increasing function φ on the real axis $\mathbb{R}$ coincides with the boundary values of a qc. automorphism w of the upper half-plane U with a fixed point ∞ if and only if there exists $\rho \ge 1$ such that*

$$\rho^{-1} \le \frac{\varphi(x+t) - \varphi(x)}{\varphi(x) - \varphi(x-t)} \le \rho \tag{1.1}$$

holds for any $x,\ t \in \mathbb{R},\ t > 0$.

More precisely, if ρ and φ in (1.1) are given then the construction presented in [2] yields a K-qc. mapping $w \in \mathrm{Aut}(U)$, $w(x) = \varphi(x)$ on $\mathbb{R}$, with $K = K(\rho) \le 8\rho(1+\rho)^2$ (which is not the best possible estimate); see e. g. [7].

Conversely, if $w(z)$ is a K-qc. automorphism of U such that $w(\infty)=\infty$ and $w(x)=\varphi(x)$ on $\mathbb{R}$ then (1.1) holds with $\rho=\lambda(K)$, where

$$\lambda(K)=[\mu^{-1}(\pi K/2)]^{-2}-1 \tag{1.2}$$

and $\mu(r)$ denotes the module of the unit disk $\mathbb{D}$ slit along $[0,r]$. The estimate $\rho=\lambda(K)$ in (1.1) is sharp.

Following Kelingos [3] any $\varphi\in\mathrm{Aut}(\mathbb{R})$ satisfying (1.1) is called a ρ-quasisymmetric (abbr.: qs.) function on $\mathbb{R}$ and the relevant class of functions is denoted by $\mathcal{H}(\rho)$. The classes $\mathcal{H}(\rho)$ are not compact (in the sense of Arzelà theorem) but their subclasses $\mathcal{H}_0(\rho)=\{\varphi\in\mathcal{H}(\rho):\varphi(0)=\varphi(1)-1=0\}$ are compact. Any $\varphi\in\mathcal{H}(\rho)$ has the form $\varphi(x)=[\varphi(1)-\varphi(0)]\varphi_0(x)+\varphi(0)$, $\varphi_0\in\mathcal{H}_0(\rho)$.

Evidently, the assumption on a conformal mapping $\Psi:G\mapsto U$ to have a fixed point $z_0\in\partial G$, is inessential. For any K-qc. $f\in\mathrm{Aut}(\overline{G})$ there exist conformal mappings Ψ_1, Ψ_2 of G onto U such that $\Psi_1\circ f\circ\Psi_2\in\mathrm{Aut}(\overline{U})$ is K-qc. in U and has ∞ as a fixed point.

Suppose now $f\in\mathrm{Aut}(\overline{G})$ is K-qc. in G and has a fixed point $z_0\in G$. If Ψ is a conformal mapping of G onto the unit disk $\mathbb{D}$ such that $\Psi(z_0)=0$ then $h=\Psi\circ f\circ\Psi^{-1}\in\mathrm{Aut}(\overline{\mathbb{D}})$ is K-qk. in $\mathbb{D}$ and $h(0)=0$. In this situation a counterpart of Theorem A can be stated as

Theorem B [4]. *An automorphism g of the unit circle $\mathbb{T}$ coincides with the boundary values of a quasiconformal automorphism of the unit disk $\mathbb{D}$ if and only if there exists a constant $M\geq 1$ such that the condition*

$$\frac{|g(\alpha_1)|}{|g(\alpha_2)|}\leq M \tag{1.3}$$

holds for any pair of disjoint adjacent open subarcs α_1, α_2 of $\mathbb{T}$ with equal length $|\alpha_1|=|\alpha_2|$.

Let $Q(M)$ denote the class of all $g\in\mathrm{Aut}(\mathbb{T})$ which satisfy (1.3). If $g\in Q(M)$ and $g(e^{i\theta})=\exp[i\varphi(\theta)]$ then (1.3) implies $\varphi(\theta)\in\mathcal{H}(M)$ after φ has been extended on $\mathbb{R}$ by the condition $\varphi(\theta+2\pi)=2\pi+\varphi(\theta)$. The Beurling-Ahlfors construction and a subsequent exponentiation result in a K-qc. automorphism h of $\mathbb{D}$ such that $h(0)=0$, $h(t)=g(t)$ on $\mathbb{T}$ and $K=K(M)\leq 8M(1+M)^2$. For details cf. [4].

Conversely, let $S(K)$ denote the class of all K-qc. automorphisms of $\mathbb{D}$ and define $S_r(K)=\{h\in S(K):|h(0)|\leq r\}$, $0\leq r<1$. Suppose $h\in S_0(K)$. Then h may be considered as an automorphism of a doubly connected domain $\mathbb{D}\setminus\{0\}$ which may be lifted under a locally conformal mapping $z\mapsto -i\log z$ on the universal covering surface U of $\mathbb{D}\setminus\{0\}$ as a K-qc. automorphism $\tilde{h}$ of U. We have $\tilde{h}|\mathbb{R}\in\mathcal{H}(M)$, where $\tilde{h}(x)-x$ is 2π-periodic and $M=\lambda(K)$ by Theorem A. The exponentiation implies (1.3) with $g=h|\mathbb{T}$; cf. [4].

We now prove that the assumption $h\in S_0(K)$ can be weakened.

Lemma . *Suppose $h\in S_r(K)$, $0\leq r<1$. If*

$$K_r=(1+r)(1-r)^{-1} \tag{1.4}$$

then (1.3) *holds with* $g = h|\mathbb{T}$ *and* $M = \lambda(KK_r)$.

Proof. If $h \in S_r(K)$ then $h(0) = z_0$ and $|z_0| \le r$. For $w = w(z) = i(1+z)(1-z)^{-1}$ define

$$W(z) = (1-|z_0|^2)^{-1}[(1-z_0)w + z_0(1-\overline{z}_0)\overline{w}].$$

It is easily verified that the function

$$z \mapsto L(z, z_0) = [W(z) - i][W(z) + i]^{-1} \tag{1.5}$$

is a qc. automorphism of $\mathbb{D}$ which satisfies $L(z_0, z_0) = 0$ and $L(t, z_0) = t$ for any $t \in \mathbb{T}$. The complex dilatation of L, i.e.

$$\frac{\overline{\partial} L}{\partial L} = \frac{z_0(1-\overline{z}_0)}{1-z_0}\,\frac{\overline{w'(z)}}{w'(z)} = z_0\frac{1-\overline{z}_0}{1-z_0}\left(\frac{1-\overline{z}_0 z}{1-z_0\overline{z}}\right)^2$$

satisfies $|\overline{\partial} L/\partial L| = |z_0|$ and hence L is K_r–qc., where K_r is given by (1.4). The mapping $F = L(\cdot, z_0) \circ h$ has the same boundary values on $\mathbb{T}$ as h. Moreover, $F \in S_0(KK_r)$ and hence, by Theorem B, $F|\mathbb{T} = h|\mathbb{T} \in Q(M)$ with $M = \lambda(KK_r)$ which ends the proof.

The mappings $g \in Q(M)$ will be called M–qs. automorphisms of $\mathbb{T}$. It seems that the ρ-qs. functions on $\mathbb{R}$ represent in a natural way the boundary correspondence for qc. automorphisms of G with a fixed point on ∂G, while the M-qs. automorphisms of $\mathbb{T}$ are quite natural in case a fixed point is an interior point. Note that no boundary point is distinguished in the latter case.

2. Quasihomographies. Let $Q = U(x_1, x_2, x_3, x_4)$ be a quadrilateral consisting of the upper half-plane U with the vertices x_k on $\mathbb{R}$ indexed in the increasing order. Its module $M(Q)$ is a characteristic conformal invariant. The vertices x_k can be mapped under a suitable conformal automorphism of U onto $-r$, -1, 1, r. Since their "modified cross-ratio"

$$[x_1x_2x_3x_4] := \left\{\frac{x_3 - x_2}{x_3 - x_1} : \frac{x_4 - x_2}{x_4 - x_1}\right\}^{1/2} \in (0,1) \tag{2.1}$$

remains unchanged, it must be equal $2\sqrt{r}(1+r)^{-1}$. If $\mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2 t)^{-1/2}\,dt$ then

$$M(Q) = \frac{\mathcal{K}(\sqrt{1-r^2})}{2\mathcal{K}(r)} = \frac{\mu(r)}{\pi},$$

where $\mu(r)$ denotes the module of the ring domain $\mathbb{D} \setminus [0, r]$.

On the other hand

$$\mu(r) \equiv 2\mu(2\sqrt{r}(1+r)^{-1}) = 2\mu([x_1x_2x_3x_4])$$

cf. [7; p.60]. Hence the relation between these two characteristic conformal invariants follows:

$$M(Q) = \frac{2}{\pi}\mu([x_1x_2x_3x_4]), \tag{2.2}$$

cf.e.g. [7; p.81].

As observed by J.Zając [8], this equality provides a control on the behaviour of the modified cross-ratio under qc. automorphisms of U. Due to the invariance of both sides in (2.2) under homographies an analogous equality holds if U is replaced by an arbitrary disk. If f is a K-qc. automorphism of U then (2.2) obviously implies

$$K^{-1}\mu([x_1x_2x_3x_4]) \le \mu([f(x_1)f(x_2)f(x_3)f(x_4)]) \le K\mu([x_1x_2x_3x_4]) . \tag{2.3}$$

By means of the distortion function $\varphi_K(t) = \mu^{-1}(\mu(t)/K)$, $K > 0$, we obtain from (2.3)

$$\varphi_{1/K}([x_1x_2x_3x_4]) \le [f(x_1)f(x_2)f(x_3)f(x_4)] \le \varphi_K([x_1x_2x_3x_4]) . \tag{2.4}$$

This suggests the following

Definition. ([8], [10]). Given an oriented circle Γ in the extended plane $\widehat{\mathbb{C}}$, an automorphism f of Γ is called a quasihomography of order K (notation: $f \in \mathcal{A}_\Gamma(K)$) if (2.4) holds for any quadruple of points $x_k \in \Gamma$ whose order is compatible with the orientation of Γ.

The identity $\varphi_{K_1} \circ \varphi_{K_2} = \varphi_{K_1K_2}$ implies the following nice properties of quasihomographies which have no counterparts for $\mathcal{H}(\rho)$ and $Q(M)$:

(i) If $f_j \in \mathcal{A}_\Gamma(K_j)$, $j = 1,2$, then $f_1 \circ f_2 \in \mathcal{A}_\Gamma(K_1K_2)$;

(ii) if $f \in \mathcal{A}_\Gamma(K)$ then also $f^{-1} \in \mathcal{A}_\Gamma(K)$.

If $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and the usual order on $\mathbb{R}$ is replaced by the cyclic order on $\overline{\mathbb{R}}$ invariant under Moebius automorphisms of $\overline{U}$ then the class $\mathcal{A}_{\overline{\mathbb{R}}}(K)$ shows to be an obvious generalization of $\mathcal{H}(\rho)$.

Let Ψ_1, Ψ_2 be Moebius automorphisms of $\overline{U}$ and let $f \in \mathcal{A}_{\overline{\mathbb{R}}}(K)$. Then (2.3) remains true if we replace f by $\Psi_1 \circ f$ and then x_k by $\Psi_2(t_k)$. Thus $f \in \mathcal{A}_{\overline{\mathbb{R}}}(K)$ implies $\Psi_1 \circ f \circ \Psi_2 \in \mathcal{A}_{\overline{\mathbb{R}}}(K)$, i.e. the class $\mathcal{A}_{\overline{\mathbb{R}}}(K)$ is closed w.r.t. the outer and inner composition with Moebius automorphisms of $\overline{U}$.

There is an obvious connection between the classes $\mathcal{A}_{\overline{\mathbb{R}}}(K)$ and $\mathcal{H}(\rho)$. If $f \in \mathcal{A}_{\overline{\mathbb{R}}}(K)$ then, by taking suitable Ψ_1, Ψ_2 we obtain $\varphi = \Psi_1 \circ f \circ \Psi_2 \in \mathcal{A}_{\overline{\mathbb{R}}}(K)$ which satisfies $\varphi(\infty) = \infty$. Substituting in (2.3) $f = \varphi$ and $x_4 = \infty$ we obtain (1.1) with $\rho = \lambda(K)$, cf. [8]. Conversely, any $\varphi \in \mathcal{H}(\rho)$ has a $K(\rho)$-qc. extension $w(z) \in \mathrm{Aut}(\overline{U})$, where we can take $K(\rho) = 8\rho(1+\rho)^2$. Then by (2.2) and (2.3) we easily verify that $\varphi \in \mathcal{A}_{\overline{\mathbb{R}}}(K(\rho))$. Consequently, $\mathcal{A}_{\overline{\mathbb{R}}}(K)$ and $\mathcal{H}(\rho)$ are, in some sense, equivalent.

If Γ is an arbitrary oriented circle in the extended plane then there exists a homography Ψ such that $\Gamma = \Psi(\overline{\mathbb{R}})$ and the orientation is preserved under Ψ. If $f \in \mathcal{A}_{\overline{\mathbb{R}}}(K)$ then obviously $F = \Psi \circ f \circ \Psi^{-1} \in \mathcal{A}_\Gamma(K)$ and this defines an isomorphism of $\mathcal{A}_{\overline{\mathbb{R}}}(K)$ and $\mathcal{A}_\Gamma(K)$.

3. The classes $S(K)$, $\mathcal{A}_{\mathbb{T}}(K)$, and $Q(M)$. In this section the subclasses $\mathcal{A}_{\mathbb{T}}(K)$ and $Q(M)$ of $\mathrm{Aut}(\mathbb{T})$ are considered. Since any $h \in S(K)$ has a homeomorphic extension on $\overline{\mathbb{D}}$, we may also consider the class $S(K)|\mathbb{T} \subset \mathrm{Aut}(\mathbb{T})$ consisting of all $h|\mathbb{T}$ where $h \in S(K)$. A subclass $\mathcal{M}$ of $\mathrm{Aut}(\mathbb{T})$ is called closed if for any sequence $h_n \in \mathcal{M}$ convergent at any $t \in \mathbb{T}$ the limit function $h \in \mathcal{M}$. A subclass $\mathcal{M}$ of $\mathrm{Aut}(\mathbb{T})$ is compact if it is closed and equicontinuous on $\mathbb{T}$.

In [9] and [10] the author was dealing with the relation between the subclasses $\mathcal{A}_{\mathbb{T}}(K)$ and $Q(M)$ of Aut($\mathbb{T}$) .

He claims [10; p.404] that the condition (1.3) does not characterize the boundary values of an arbitrary K–qc.automorphism of $\mathbb{D}$. However, this is not true. By our Lemma, for an arbitrary $G \in S(K)$ such that $G(0) = z_0$ we have $G|\mathbb{T} \in Q(M)$, where $M = \lambda(KK_r)$, $r = |z_0|$ and K_r is defined by (1.4). The author's claim was based on the following Example [9; p.422]. Let (h_n) be the sequence of Moebius automorphisms defined by the equalities: $h_n(1) = 1$, $h_n(i) = \exp[\pi in/(n+1)]$, $h_n(-1) = -1$. If α_1, α_2 are subarcs of $\mathbb{T}$ with end-points 1, i and $i, -1$, resp., then $h_n \in \mathcal{A}_{\mathbb{T}}(1)$ and $|h_n(\alpha_1)|/|h_n(\alpha_2)| = n$. The sequence (h_n) is pointwise convergent on $\mathbb{T}$ to the function $h(t)$, where $h(t) = 0$ for $t \in \mathbb{T} \setminus \{-1, 1\}$, $h(1) = 1$, $h(-1) = -1$, so that $h \notin \mathcal{A}_{\mathbb{T}}(1)$. What does this example actually prove is that the classes $\mathcal{A}_{\mathbb{T}}(K)$ and $S(K)$ are not closed for any $K \geq 1$ which is their common serious drawback. Since any class $Q(M)$ is compact due to Arzelà theorem (cf.[5]), the problems of maximizing the l.h.s. of (1.3) in $\mathcal{A}_{\mathbb{T}}(K)$, or $S(K)$, are ill-posed.

In order to find a relation betwenn the classes $S(K)$ and $Q(M)$ we have to confine ourselves to a suitable subclass $\tilde{S}(K)$ of $S(K)$. Note that the condition: "there exists a sequence $h_n \in \tilde{S}(K)$ such that $\lim |h_n(0)| = 1$ ", implies $\tilde{S}(K)$ to be non-closed. In fact, if there exists a convergent subsequence (h_{n_k}) , its limit function $h \notin S(K)$, so it maps $\mathbb{D}$ on a set consisting of one, or two points, cf. [7; p.74]. Hence a natural assumption on $\tilde{S}(K)$ to be closed is that there exists $r \in [0, 1)$ such that $|h(0)| \leq r$ for any $h \in \tilde{S}(K)$. Then our Lemma yields the desired relation: *If* $h \in S_r(K)$ *then* $h|\mathbb{T} \in Q(M)$ *with* $M = \lambda(KK_r)$.

Given $f \in Q(M)$ we obtain as in [4] a K–qc. extension F of f onto $\mathbb{D}$ such that $F(0) = 0$ and $k = K(M) \leq 8M(1 + M)^2$. Using the equality (2.2) and the well-known behaviour of $M(Q)$ under K–qc. mappings we obtain at once $f \in \mathcal{A}_{\mathbb{T}}(K(M))$, i.e. $Q(M) \subset \mathcal{A}_{\mathbb{T}}(K(M))$.

Given $f \in \mathcal{A}_{\mathbb{T}}(\rho)$ *there exists* $M = M(f, \rho)$ *such that* $f \in Q(M)$, as proved in [9] without reference to qc. extension of f. We now sketch a simple alternative proof.

As we have already seen, there is the following relation between the classes $S(K)$ and $Q(M)$:

(i) If $F \in S(K)$, $F(0) = z_0$, $|z_0| \leq r < 1$ and $K_r = (1 + r)/(1 - r)$, then $F|\mathbb{T} \in Q(M)$, where $M = \lambda(KK_r)$.

(ii) Denote by $S^*(K)$ the family of all $G \in S(K)$ such that $G(t_k) = t_k = \exp(2\pi ik/3)$, $k = 0, 1, 2$. We have evaluated in [6] a number $r(K) < 1$ such that $|G(0)| \leq r(K)$ for any $G \in S^*(K)$.

Suppose $f \in \mathcal{A}_{\mathbb{T}}(\rho)$ is given and denote by Ψ the Moebius transformation such that $\Psi(t_k) = f(t_k)$, $k = 0, 1, 2$. Let F be a $K(\rho)$–qc. extension of F on the unit disk. Then obviously $G = \Psi^{-1} \circ F \in S^*(K(\rho))$. Due to (ii) we have $|G(0)| \leq r(K(\rho))$. The disk $|z| \leq r(K(\rho))$ is mapped under Ψ on a disk contained in $|w| \leq r(f, \rho) < 1$. We have by (ii) $|F(0)| = |(\Psi \circ G)(0)| \leq r(f, \rho)$ and hence by (i) $F|\mathbb{T} = f \in Q(M)$, where

$$M = \lambda[K(\rho)(1 + r(f, \rho))(1 - r(f, \rho))^{-1}] .$$

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(received December 15, 1993)