ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XXXIII, 16 SECTIO A 1979

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On a Minimization of Functionals in Banach Spaces

O minimalizacji funkcjonałów w przestrzeniach Banacha

О минимизации функционалов в банаховых пространствах

1. INTRODUCTION

Throughout this paper X will denote a real Banach space with the norm $\|\cdot\|$.

Let a function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ be continuous, $\gamma(t) \nearrow +\infty$, $t \to +\infty$; $\gamma(t) \searrow 0$, $t \to 0$. Set

$$\gamma_1(s) := \int_0^s \gamma(t)dt, \quad s \geq 0,$$

and, for $t \in \langle 0,1 \rangle$, $s \geq 0$,

$$\Gamma(t,s) := t\,\gamma_1((1-t)s) + (1-t)\,\gamma_1(ts) .$$

Let Ψ be a fixed real functional defined on X, such that

(i) There exists a function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, nondecreasing continuous and satisfying:

$$|x_i| \leq r \to |\Psi(x_1) - \Psi(x_2)| \leq \beta(r) \|x_1 - x_2\|, \quad x_i \in X; \quad i = 1,2 . \tag{1.1}$$

(ii) For any $t \in (0,1)$ and $x,h \in X$

$$\Psi(x + th) - \Psi(x) \leq t[\Psi(x + h) - \Psi(x)] - \Gamma(t,\|h\|). \tag{1.2}$$

Let Φ denote a fixed real functional defined on X, such that

(i) There exists a function $\beta' : \mathbb{R}_+ \to \mathbb{R}_+$, nondecreasing and continuous, such that

$$\|x_i\| \leq r \Rightarrow |\Phi(x_1) - \Phi(x_2)| \leq \beta'(r)\|x_1 - x_2\|, \quad x_i \in X, \quad i = 1,2 . \tag{1.3}$$

(ii) For any $t \in (0,1)$ and $x,h \in X$

$$\Phi(x + th) - \Phi(x) \leq t[\Phi(x + h) - \Phi(x)]. \tag{1.4}$$

In this paper we shall consider the problem of the existence of a minimum (unconditional and conditional - with the condition depending on Ψ) of the functional Φ. The results obtained in sections 2 and 3 will be applied to the theory of partial differential equations.
We shall make use of the results obtained by T. Leżański [1]. Therefore, we restate at the moment several results from [1].

LEMMA 1.1. Ψ <u>is bounded from below and</u> $\Psi(x) \to +\infty$ <u>if</u> $\|x\| \to +\infty$.

LEMMA 1.2. <u>If</u> $x_1,x_2 \in X$, <u>then</u>

$$\|x_1 - x_2\| \leq \sum_{i=1}^{2} \gamma_1^{-1}(\Psi(x_i) - d), \tag{1.5}$$

where $d := \inf_{x \in X} \Psi(x)$.

LEMMA 1.3. *If*

(1.6) $$\Psi(x + t(y - x)) - \Psi(x) \geq 0 \quad \text{for} \quad t \in \langle 0,1 \rangle \quad \text{and} \quad x,y \in X$$

then

(1.7) $$\| x - y \| \leq \gamma_1^{-1}(\Psi(y) - \Psi(x)).$$

THEOREM 1.4. *There exists a unique element* $\tilde{x} \in X$ *such that* $\Psi(\tilde{x}) = d = \inf_{x \in X} \Psi(x)$. *Moreover*

(1.8) $$\bigwedge_{x \in X} \| x - \tilde{x} \| \leq \gamma_1^{-1}(\Psi(x) - d)$$

2. CONDITIONAL MINIMUM OF THE FUNCTIONAL Φ

Set

$$Z := \{ x \in X : \Psi(x) \leq 0 \}.$$

In this section we consider the problem of minimization of the functional Φ on Z.

For $\lambda > 0$ we define

(2.1) $$\Phi_\lambda(x) := \frac{1}{\lambda} \Phi(x) + \Psi(x), \qquad x \in X.$$

First note that, for any fixed $\lambda > 0$, the functional Φ_λ satisfies the assumptions (i) and (ii) (with Ψ replaced by Φ_λ). Therefore, in view of Theorem 1.4, for any fixed $\lambda > 0$ there exists a unique element $x_\lambda \in X$ such that $\Phi_\lambda(x_\lambda) = \inf_{x \in X} \Phi_\lambda(x)$, i.e.

$$\frac{1}{\lambda}\,\Phi(x_\lambda) + \Psi(x_\lambda) \leq \frac{1}{\lambda}\,\Phi(x) + \Psi(x), \qquad x \in X \tag{2.2}$$

LEMMA 2.1. [1]. The following implications hold:

(I) $\Psi(x) \leq \Psi(x_\lambda) \Rightarrow \Phi(x) \geq \Phi(x_\lambda)$;

(II) $\Phi(x_\lambda) \geq \Phi(x) \Rightarrow \Psi(x_\lambda) \leq \Psi(x)$, $\quad x \in X$

P r o o f. From (2.2) we obtain $\frac{1}{\lambda}(\Phi(x_\lambda) - \Phi(x)) + (\Psi(x_\lambda) - \Psi(x)) \leq 0$, which implies (I) and (II). The proof is complete.

For $\lambda > 0$ we set

$$\varphi(\lambda) := \Phi(x_\lambda)$$
$$\psi(\lambda) := \Psi(x_\lambda).$$

LEMMA 2.2. The function φ is nondecreasing and bounded from above; the function ψ is nonincreasing and bounded from below.

P r o o f. Assume that $\lambda > 0$, $\mu > 0$ and put $x = x_\mu$ in (2.2):

$$\Psi(x_\lambda) \leq \frac{1}{\lambda}\Phi(x_\mu) + \Psi(x_\mu) - \frac{1}{\lambda}\Phi(x_\lambda)$$

In view of the analogous inequality:

$$\Psi(x_\mu) \leq \frac{1}{\mu}\Phi(x_\lambda) + \Psi(x_\lambda) - \frac{1}{\mu}\Phi(x_\mu)$$

we obtain

$$\Psi(x_\lambda) \leq \frac{1}{\lambda}\Phi(x_\mu) + \frac{1}{\mu}\Phi(x_\lambda) + \Psi(x_\lambda) - \frac{1}{\mu}\Phi(x_\mu) - \frac{1}{\lambda}\Phi(x_\lambda)$$

Hence

$$0 \leq \frac{1}{\mu}(\Phi(x_\lambda) - \Phi(x_\mu)) - \frac{1}{\lambda}(\Phi(x_\lambda) - \Phi(x_\mu)),$$

i.e.

$$(\Phi(x_\lambda) - \Phi(x_\mu))(\frac{1}{\mu} - \frac{1}{\lambda}) \geq 0.$$

If $0 < \mu \leq \lambda$ we then have $\Phi(x_\mu) \leq \Phi(x_\lambda)$, i.e. the function φ is nondecreasing. Hence it follows from Lemma 2.1 (II) (with $x = x_\mu$) that ψ is nonincreasing.

The function ψ is bounded from below in view of Lemma 1.1. Moreover, it follows from Theorem 1.4 that

$$\psi(\lambda) = \Psi(x_\lambda) \geq \Psi(\tilde{x}) = \inf_{x \in X} \Psi(x), \qquad \lambda > 0.$$

Hence it follows from Lemma 2.1 (I) (with $x = \check{x}$) that

$$\varphi(\lambda) = \Phi(x_\lambda) \leq \Phi(\tilde{x}) \qquad \text{for all} \quad \lambda > 0,$$

i.e. the function φ is bounded from above, so the proof is complete.

COROLLARY 2.3. For any $\lambda > 1$ we have $\Phi(x_\lambda) \geq \Phi(x_1)$.

LEMMA 2.4. The functions φ and ψ are continuous.

P r o o f (due to T. Leżański). Assume that $\lambda > 0$, $\mu > 0$. For $x \in X$ we set

$$\tilde{\Phi}_\lambda(x) := \Phi(x) + \lambda(\Psi(x) - d),$$

where $d = \inf_{x \in X} \Psi(x)$. In view of the inequality (2.2), the functional $\tilde{\Phi}_\lambda$ attains at the point $x_\lambda \in X$ its minimum. Furthermore, we obtain

$$\tilde{\Phi}_\mu(x_\lambda) = \Phi(x_\lambda) + \mu(\Psi(x_\lambda) - d) =$$

$$= (\Phi(x_\lambda) + \lambda\Psi(x_\lambda) - \lambda d) - \lambda\Psi(x_\lambda) + \lambda d + \mu\Psi(x_\lambda) - \mu d =$$

$$= \tilde{\Phi}_\lambda(x_\lambda) + (\mu-\lambda)(\Psi(x_\lambda) - d) \leq \tilde{\Phi}_\lambda(x_\mu) + (\mu-\lambda)(\Psi(x_\lambda) - d) =$$
$$= \Phi(x_\mu) + \lambda(\Psi(x_\mu) - d) + (\mu-\lambda)(\Psi(x_\lambda) - d) =$$
$$= (\Phi(x_\mu) + \mu\Psi(x_\mu) - \mu d) - \mu\Psi(x_\mu) + \lambda\Psi(x_\mu) + \mu\Psi(x_\lambda) - \lambda\Psi(x_\lambda) =$$
$$= \tilde{\Phi}_\mu(x_\mu) + (\mu-\lambda)(\Psi(x_\lambda) - \Psi(x_\mu)).$$

Hence

$$0 \leq \tilde{\Phi}_\mu(x_\lambda) - \tilde{\Phi}_\mu(x_\mu) \leq (\lambda-\mu)(\Psi(x_\mu) - \Psi(x_\lambda)) <$$
$$\leq |\lambda-\mu|\,(\Psi(x_\mu) - d) .$$

Let us fix μ and put $\mu = \mu_0$. It follows that

$$0 \leq \Phi_{\mu_0}(x_\lambda) - \Phi_{\mu_0}(x_{\mu_0}) < \frac{|\lambda-\mu_0|}{\mu_0}(\Psi(x_{\mu_0}) - d)$$

Hence $\Phi_{\mu_0}(x_\lambda) - \Phi_{\mu_0}(x_{\mu_0}) \longrightarrow 0$ when $|\lambda-\mu_0| \longrightarrow 0.$ Then by (1.5) (with $x_1 = x_\lambda$, $x_2 = x_{\mu_0}$, $\Psi = \Phi_{\mu_0}$) it follows that $\|x_\lambda - x_{\mu_0}\| \longrightarrow 0$ when $|\lambda-\mu_0| \longrightarrow 0.$

Now choose $r_1 > 0$ such that $\|x_{\mu_0}\| \leq r_1$. If λ is close enough to μ_0, it follows that $\|x_\lambda - x_{\mu_0}\| \leq r_1$ and then $\|x_\lambda\| \leq \|x_\lambda - x_{\mu_0}\| + \|x_{\mu_0}\| \leq 2r_1$. Hence we can apply the assumption (i′) (with x_1 replaced by x_λ and x_2 replaced by x_{μ_0}):

$$|\Phi(x_\lambda) - \Phi(x_{\mu_0})| \leq \beta'(2r_1)\,\|x_\lambda - x_{\mu_0}\|$$

This means that φ is continuous at any point $\mu_0 > 0$.

The continuity of ψ is obtained by using (i). This concludes the proof of the lemma.

LEMMA 2.5. <u>There exists the limit</u> $\lim_{\lambda\to+\infty} \Psi(x_\lambda)$ <u>and</u>

$$\lim_{\lambda\to+\infty} \Psi(x_\lambda) = \Psi(\tilde{x}) = \inf_{x\in X} \Psi(x)$$

P r o o f. It follows from Lemma 2.2 that there exists $p := \lim_{\lambda \to +\infty} \Psi(x_\lambda)$. We shall prove that $p = \Psi(\tilde{x})$.

Assume that $0 < \mu \leq \lambda$. Since the function φ is non-decreasing (Lemma 2.2), $\Phi(x_\mu) - \Phi(x_\lambda) \leq 0$. Hence, by using (ii′), it follows that $\Phi(x_\lambda + t(x_\mu - x_\lambda)) - \Phi(x_\lambda) \leq t[\Phi(x_\mu) - \Phi(x_\lambda)] \leq 0$, i.e.

$$\Phi(x_\lambda + t(x_\mu - x_\lambda)) \leq \Phi(x_\lambda),$$

where $t \in (0,1)$. Hence it follows from Lemma 2.1 (II) that

$$\Psi(x_\lambda + t(x_\mu - x_\lambda)) - \Psi(x_\lambda) \geq 0.$$

Therefore, by Lemma 1.3,

$$\|x_\lambda - x_\mu\| \leq \gamma_1^{-1}(\Psi(x_\mu) - \Psi(x_\lambda)). \tag{2.3}$$

Consequently $\|x_\lambda - x_\mu\| \longrightarrow 0$ when $\lambda, \mu \longrightarrow +\infty$, $\mu < \lambda$. Hence there exists $x^* := \lim_{\lambda \to +\infty} x_\lambda$. Now choose $r_1 > 0$ such that $\|x^*\| \leq r_1$. For λ sufficiently large we have $\|x_\lambda - x^*\| \leq r_1$ and then $\|x_\lambda\| \leq \|x_\lambda - x^*\| + \|x^*\| \leq 2r_1$. Then, by the assumption (i), $\Psi(x^*) = \lim_{\lambda \to +\infty} \Psi(x_\lambda)$.

To prove that $x^* = \tilde{x}$ we first observe that since $\Psi(\tilde{x}) < \Psi(x_\lambda)$ $(\lambda > 0)$, $\Phi(\tilde{x}) \geq \Phi(x_\lambda)$ for all $\lambda > 0$ in view of Lemma 2.1 (I). Hence it follows from Corollary 2.3 that for $\lambda > 1$

$$\frac{\Phi(x_1)}{\lambda} \leq \frac{\Phi(x_\lambda)}{\lambda} \leq \frac{\Phi(\tilde{x})}{\lambda},$$

so $\lim_{\lambda \to +\infty} \frac{\Phi(x_\lambda)}{\lambda} = 0$. Using (2.2) and passing to the limit when $\lambda \longrightarrow +\infty$ we obtain that $\Psi(x^*) \leq \Psi(x)$, $x \in X$. Hence $x^* = \hat{x}$ in virtue of Theorem 1.4. The proof is complete.

LEMMA 2.6. If

$$\Psi(x_\lambda)<0 \quad \text{for any} \quad \lambda>0,$$

then there exists $x^0 \in Z$ such that

$$\Phi(x^0) \le \Phi(x) \quad \text{for any} \quad x \in X.$$

Moreover

$$\Phi(x^0) = \lim_{\lambda \to 0} \Phi(x_\lambda).$$

P r o o f. Suppose that for any $\lambda>0$, $\Psi(x_\lambda)<0$. Since the function ψ is nonincreasing (Lemma 2.2), there exists $\lim_{\lambda\to 0} \Psi(x_\lambda)$. Assume that $0<\mu\le\lambda$. Hence it follows from (2.3) that there exists $x^0 := \lim_{\lambda\to 0} x_\lambda$.

We shall prove that $\Phi(x^0) = \lim_{\lambda\to 0} \Phi(x_\lambda)$. Choose $r_1>0$ such that $\|x^0\| \le r_1$. There exists $\lambda_0>0$ such that $\|x_\lambda - x^0\| \le r_1$ for $\lambda \in (0, \lambda_0)$. Hence $\|x_\lambda\| \le \|x_\lambda - x^0\| + \|x^0\| \le 2r_1$. Then, by (i′)

$$\|\Phi(x_\lambda) - \Phi(x^0)\| \le \beta(2r_1)\,\|x_\lambda - x^0\| \quad \text{for} \quad \lambda \in (0, \lambda_0)$$

Hence $\Phi(x^0) = \lim_{\lambda\to 0} \Phi(x_\lambda)$.

To prove that $\Phi(x^0) \le \Phi(x)$, $(x \in X)$ we use the inequality:

$$\Phi(x_\lambda) + \lambda\Psi(x_\lambda) \le \Phi(x) + \lambda\Psi(x), \qquad x \in X \tag{2.4}$$

which follows immediately from (2.2). Observe that

$$\lambda \inf_{x\in X} \Psi(x) \le \lambda\Psi(x_\lambda) < 0,$$

whence $\lim_{\lambda\to 0} \lambda\Psi(x_\lambda) = 0$. Then, passing to the limit when $\lambda \to 0$ in (2.4), we obtain $\Phi(x^0) \le \Phi(x)$, $x \in X$. This ends the proof of Lemma 2.6.

REMARK 2.7. The assumption of Lemma 2.6 may be replaced by the following:

$$\bigvee_{\lambda_0>0} \bigwedge_{\lambda\in(0,\lambda_0)} \Psi(x_\lambda)<0$$

THEOREM 2.8. *Assume that* Ψ *satisfies* (i), (ii) *and that there exists* $x'\in X$ *such that* $\Psi(x')<0$. *If* Φ *satisfies* (i′) *and* (ii′), *then there exists* $\bar{x}\in Z$ *such that*

$$\Phi(\bar{x}) \leqslant \Phi(x) \qquad \textit{for each} \quad x\in Z.$$

P r o o f. Assume first that there exists $\mu_0>0$ such that $\Psi(x_{\mu_0})\geqslant 0$. From Lemma 2.5 it follows that

$$\lim_{\lambda\to+\infty} \Psi(x_\lambda) = \Psi(\tilde{x}) = \inf_{x\in X} \Psi(x)<0.$$

Then there exists $\mu_1>0$ such that $\Psi(x_{\mu_1})<0$. Since Ψ is continuous (Lemma 2.4), it follows that there exists $\mu_2>0$ such that $\Psi(x_{\mu_2}) = 0$. Then we have $x_{\mu_2}\in Z$ and, by (2.2),

$$\frac{1}{\mu_2}\Phi(x_{\mu_2}) = \frac{1}{\mu_2}\Phi(x_{\mu_2}) + \Psi(x_{\mu_2}) \leqslant \frac{1}{\mu_2}\Phi(x) + \Psi(x) \leqslant \frac{1}{\mu_2}\Phi(x), \qquad x\in Z,$$

i.e. $\Phi(x_{\mu_2}) \leqslant \Phi(x)$, $x\in Z$. Putting $\bar{x}:= x_{\mu_2}$ we complete the proof in this case.

Now suppose that for any $\lambda>0$, $\Psi(x_\lambda)<0$. It follows from Lemma 2.6 that there exists $x^0\in Z$, $x^0 = \lim_{\lambda\to 0} x_\lambda$, such that $\Phi(x^0)\leqslant \Phi(x)$, $x\in X$, i.e. the functional Φ attains its minimum at $x^0 \in Z$. Putting $\overline{x} := x^0$ we complete the proof in this case.

Thus Theorem 2.8 follows in every case.

3. UNCONDITIONAL MINIMUM OF THE FUNCTIONAL Φ

In this section we apply the results obtained in section 2 to prove the following theorem:

THEOREM 3.1. The following assertions are equivalent:

(a) The functional Φ is bounded from below and it attains its minimum.

(b) There exists a constant $C > 0$ such that

$$\|x_\lambda\| \leq C \qquad \text{for any } \lambda > 0. \tag{3.1}$$

Moreover, if (a) or (b) holds, then there exists $x^0 := \lim_{\lambda \to 0} x_\lambda$ and

$$\Phi(x^0) = \inf_{x \in X} \Phi(x) .$$

P r o o f. (a) $\Longrightarrow$ (b). Assume that Φ is bounded from below and that there exists $x^* \in X$ such that $\Phi(x^*) \leq \Phi(x)$, $x \in X$. Hence, for any $\lambda > 0$, $\Phi(x^*) \leq \Phi(x_\lambda)$. Therefore, in view of Lemma 2.1 (II),

$$\Psi(x_\lambda) \leq \Psi(x^*) .$$

Set $\Psi_0(x) := \Psi(x) - \Psi(x^*)$, $x \in X$; since $\Psi_0(x^*) = 0$ and $\Psi_0(x_\lambda) \leq 0$, $\lambda > 0$, it follows from (1.5) that

$$\|x^* - x_\lambda\| \leq \gamma_1^{-1}(\Psi(x^*) - d) + \gamma_1^{-1}(\Psi(x_\lambda) - d) \leq 2\gamma_1^{-1}(-d)$$

where $d = \inf_{x \in X} \Psi(x)$. If we choose $C := 2\gamma_1^{-1}(-d) + \|x^*\|$, then

$$\|x_\lambda\| \leq \|x^* - x_\lambda\| + \|x^*\| \leq 2\gamma_1^{-1}(-d) + \|x^*\| = c .$$

(b) $\Rightarrow$ (a). Assume that there exists $C > 0$ such that $\|x_\lambda\| \leq C$ for any $\lambda > 0$. By virtue of (i)

$$\Psi(x_\lambda) - \Psi(0) \leq \beta(C) \cdot C.$$

For $x \in X$ let

$$\Psi_1(x) := \Psi(x) - \Psi(0) - \beta(C) \cdot C - 1 .$$

It is obvious that the functional Ψ_1 satisfies the assumptions (i) and (ii) (with Ψ replaced by Ψ_1). Moreover, by virtue of (2.2) we have for $\lambda > 0$ fixed:

$$\frac{1}{\lambda}\Phi(x_\lambda) + \Psi_1(x_\lambda) \leq \frac{1}{\lambda}\Phi(x) + \Psi(x) - \Psi(0) - \beta(C)C - 1 =$$

$$= \frac{1}{\lambda}\Phi(x) + \Psi_1(x), \qquad x \in X$$

Further, for any $\lambda > 0$,

$$\Psi_1(x_\lambda) = \Psi(x_\lambda) - \Psi(0) - \beta(C)C - 1 < 0 .$$

Therefore Ψ_1 satisfies the assumptions of Lemma 2.6 (with Ψ replaced by Ψ_1). Hence it follows from Lemma 2.6 that there exists $x^0 \in X$ such that $\Phi(x^0) \leq \Phi(x)$, $x \in X$ and $x^0 = \lim_{\lambda \to 0} x_\lambda$.

The proof is complete.

REMARK 3.2. Using Remark 2.7, we may replace (3.1) by the following:

$$\bigvee_{\sigma>0} \bigvee_{\lambda_0>0} \bigwedge_{\lambda\in(0,\lambda_0)} \|x_\lambda\| \leq C,$$

which is equivalent to (3.1).

4. APPLICATION TO THE THEORY OF QUASI-LINEAR EQUATIONS

4.1. Let H denote a real Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$ and $M\subset H$ a linear subset, dense in H. Let $H_1\subset H$ be the completion of M in the norm $\|\cdot\|_1$, where

$$\|x\|_1 := \sqrt{(x,x)_1} \geq \alpha\|x\|, \quad (x,y)_1 := (Ax,y) \quad \text{for } x,y\in M \text{ and } \alpha>0$$

$A: M \longrightarrow H$ is a linear operator, symmetric and strictly positive (cf. [4], N^0 124).

Let $\tilde{\Phi}$ denote a real valued functional defined on $M\times M$. Suppose that $\tilde{\Phi}$ satisfies the following properties:

(a) $\tilde{\Phi}(x,\cdot)$ is a linear functional for any fixed $x\in M$ and

(4.1.1) $$|\tilde{\Phi}(0,h)| \leq K\|h\|_1, \quad h\in M, \ K>0$$

(b) For $h\in M$

(4.1.2) $$\frac{\tilde{\Phi}(h,h)}{\|h\|_1} \longrightarrow +\infty \quad \text{if} \quad \|h\|_1 \longrightarrow +\infty$$

(c) For any fixed $x,f,h\in M$ a function $(t,s)\longmapsto \tilde{\Phi}(x+th+sf,h)$, $t,s\in\mathbb{R}$ possess continuous partial derivatives of the first order.

(d) There exists the derivative $\tilde{\Phi}'$:

$$\tilde{\Phi}'(x,h,f) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\tilde{\Phi}(x + \varepsilon h,f) - \tilde{\Phi}(x,f)\right], \quad x,h,f \in M$$

linear and symmetrical with respect to h and f:

$$(4.1.3) \qquad \tilde{\Phi}'(x,h,f) = \tilde{\Phi}'(x,f,h)$$

and furthermore

$$(4.1.4) \qquad |\tilde{\Phi}(x,h,f)| \leq m \|h\|_1 \|f\|_1, \qquad m>0$$

and

$$(4.1.5) \qquad \tilde{\Phi}'(x,h,h) \geq 0.$$

First observe that, in view of (4.1.4),

$$(4.1.6) \qquad |\tilde{\Phi}(x,h) - \tilde{\Phi}(y,h)| \leq m \|x - y\|_1 \|h\|_1, \quad x,y,h \in M$$

and that from (4.1.5)

$$(4.1.7) \qquad \tilde{\Phi}(x + h,h) - \tilde{\Phi}(x,h) > 0$$

(see [3]).

From (4.1.1) and (4.1.6) it follows that the functional $\tilde{\Phi}$ can be extended from $M \times M$ to $H_1 \times H_1$ in a unique way so that the extended functional (which we shall denote also by $\tilde{\Phi}$) satisfies (a) - (d) (with M replaced by H_1).

Because of (4.1.1) and (4.1.6) we get

$$(4.1.8) \qquad |\tilde{\Phi}(x,h)| \leq (K + m\|x\|_1)\|h\|_1, \qquad x,h \in H_1$$

(cf. [3]). Hence it follows from the Riesz theorem that for $x \in H_1$ there exists a unique element $F(x) \in H_1$ such that

$$(4.1.9) \qquad \tilde{\Phi}(x,h) = (F(x),h)_1 \qquad \text{for } x,h \in H_1 .$$

Using (4.1.6), (4.1.7) and (4.1.2) we obtain

(4.1.10) $$\|F(x) - F(y)\|_1 \leq m\|x - y\|_1,$$

(4.1.11) $$(F(x + h) - F(x), h)_1 \geq 0,$$

(4.1.12) $$\frac{(F(h),h)_1}{\|h\|_1} \longrightarrow +\infty \quad \text{if} \quad \|h\|_1 \longrightarrow +\infty,$$

$x, y, h \in H_1$.

LEMMA 4.1.1. <u>Let</u> $F : H_1 \longrightarrow H_1$ <u>be defined by</u> (4.1.9). <u>Then there exists a functional</u> $\Phi : H_1 \longrightarrow \mathbb{R}$ <u>such that</u> $F(x) = \operatorname{grad} \Phi(x)$, $(x \in H_1)$; <u>moreover, the functional</u> Φ <u>satisfies</u> (i′) <u>and</u> (ii′) (<u>with</u> $X = H_1$).

P r o o f. The existence of the functional Φ follows because of (c) and (4.1.3) (see [5]).

Now we prove (i′). Assume that $\|x_1\|_1 \leq r$, $\|x_2\|_1 \leq r$, $x_1, x_2 \in H_1$. In view of (4.1.8) we have

$$|\Phi(x_1) - \Phi(x_2)| = \left| \int_0^1 \frac{d}{dt} \Phi(x_1 + t(x_2 - x_1))dt \right| =$$

$$= \left| \int_0^1 \tilde{\Phi}(x_1 + t(x_2 - x_1), x_2 - x_1)dt \right| \leq$$

$$\leq \|x_1 - x_2\|_1 \int_0^1 (K + m\|x_1 + t(x_2 - x_1)\|_1)dt \leq$$

$$\leq (K + m(2\|x_1\|_1 + \|x_2\|_1))\|x_1 - x_2\|_1 \leq$$

$$\leq (K + 3mr)\|x_1 - x_2\|_1$$

(we have used the identity $\tilde{\Phi}(x,h) = \Phi'(x,h)$). Setting $\beta'(r) := K + 3mr$, (i′) is proved.

To prove (ii′) we observe that in view of (4.1.11), the operator F is monotone; hence the functional Φ is convex (cf. [5]).

The proof is thus complete.

Let Ψ denote a real valued functional defined on H_1. Assume that there exists $G(x) := \operatorname{grad} \Psi(x)$, $x \in H_1$, and moreover, that G is a bounded operator mapping H_1 into H_1, and

$$(4.1.13) \qquad (G(y) - G(x), y - x)_1 \geq \| y - x \|_1 \, \gamma(\| y - x \|_1), \quad x, y \in H_1$$

LEMMA 4.1.2. *The functional Ψ satisfies the conditions (i) and (ii) (with $X = H_1$).*

P r o o f. To prove (i) suppose that $\| x_1 \|_1 \leq r$, $\| x_2 \|_1 \leq r$, $x_1, x_2 \in H_1$. Since G is bounded on H_1 we therefore obtain

$$|\Psi(x_1) - \Psi(x_2)| = \left| \int_0^1 \frac{d}{dt} \Psi(x_1 + t(x_2 - x_1))dt \right| =$$

$$= \left| \int_0^1 \Psi'(x_1 + t(x_2 - x_1), x_2 - x_1)dt \right| =$$

$$= \left| \int_0^1 (G(x_1 + t(x_2 - x_1)), x_2 - x_1)_1 dt \right| \leq$$

$$\leq \int_0^1 \| G(x_1 + t(x_2 - x_1)) \|_1 \| x_2 - x_1 \|_1 dt \leq$$

$$\leq \| x_1 - x_2 \|_1 \int_0^1 L \| x_1 + t(x_2 - x_1) \|_1 dt \leq$$

$$L(2\|x_1\|_1 + \|x_2\|_1)\|x_1 - x_2\|_1 \leq 3Lr\|x_1 - x_2\|_1 ,$$

where $L>0$ is a constant such that $\|G(x)\|_1 \leq L\|x\|_1$, $x \in H_1$. Putting $\beta(r) := 3Lr$ we complete the proof of (i).

Now we sketch the proof of (ii); we refer the reader for details to the paper [1]. Observe that, in view of (4.1.13), the inequality

$$\Psi(y) - \Psi(x) = \int_0^1 \frac{d}{dt}\Psi(x + t(y - x))dt =$$

$$= \int_0^1 \frac{1}{t}(G(x + t(y - x)) - G(x), t(y - x))_1 dt + (G(x), y - x)_1 \geq$$

$$\geq \int_0^1 \frac{1}{t}\gamma(t\|y - x\|_1)t\|y - x\|_1 dt + (G(x), y - x)_1 =$$

$$= \gamma_1(\|y - x\|_1) + (G(x), y - x)_1$$

holds. Defining $x_0 := sx + (1 - s)y$, $s \in (0,1)$, we have

$$s\Psi(x) - s\Psi(x_0) + (1 - s)\Psi(y) - (1 - s)\Psi(x_0) \geq$$

$$\geq s\gamma_1(\|x - x_0\|_1) + (1 - s)\gamma_1(\|y - x_0\|_1)$$

and consequently

$$s\Psi(x) + (1 - s)\Psi(y) - \Gamma(s, \|y - x\|_1) \geq \Psi(sx + (1-s)y)$$

(we have used the identities: $x - x_0 = (1 - s)(x - y)$, $y - x_0 = s(y - x)$). Putting $y = x + h$, $s = 1 - t$ and using the fact that $\Gamma(1 - t, s) = \Gamma(t, s)$ we obtain (ii), which completes the proof.

LEMMA 4.1.3. <u>Assume that</u> $\tilde{\Phi}$ <u>satisfies</u> (a), (c), (d) <u>and that</u> $\tilde{a} \in H$. <u>If there exists</u> $R>0$ <u>such that</u>

(4.1.14) $\|\tilde{a}\| < \frac{\alpha}{R}\tilde{\Phi}(h,h)$ *if* $\|h\|_1 = R$, $h \in H_1$

then there is $x^0 \in H_1$ *such that*

$$\tilde{\Phi}(x^0,h) + (\tilde{a},h) = 0 \quad \textit{for any} \quad h \in H_1,$$

and $\|x^0\|_1 < R$.

P r o o f. Suppose that for $\tilde{a} \in H$ (4.1.14) is fulfilled. Then for $h \in H_1$, $\|h\|_1 = R$ we have $\|\tilde{a}\| \|h\|_1 < \alpha\tilde{\Phi}(h,h)$. Hence and from the inequality $\|h\|_1 \geq \alpha\|h\|$ it follows that $\|\tilde{a}\| \|h\| < \tilde{\Phi}(h,h)$, i.e. $(-\tilde{a},h) \leq \|-\tilde{a}\| \|h\| < \tilde{\Phi}(h,h)$. Consequently

$$\tilde{\Phi}(h,h) + (\tilde{a},h) > 0, \qquad \|h\|_1 = R, \quad h \in H_1 .$$

Now for $x,h \in H_1$ put

$$\tilde{\Phi}_1(x,h) := \tilde{\Phi}(x,h) + (\tilde{a},h) .$$

Observe that $\tilde{\Phi}_1$ is linear and bounded (in the norm $\|\cdot\|_1$) with respect to the second variable (the boundedness follows from (4.1.1)). In view of the Riesz theorem we then have

$$\tilde{\Phi}_1(x,h) = (F_1(x),h)_1, \qquad x,h \in H_1$$

where $F_1(x)$ is an element of H_1 (cf. (4.1.9)). Further, from (c) and (4.1.3) it follows that F_1 is a potential operator, i.e. there exists a functional $\Phi_1 : H_1 \longrightarrow R$ such that $F_1(x) = \operatorname{grad}\Phi_1(x)$, $x \in H_1$.

Using the previous result we obtain

$$(F_1(h),h)_1 > 0 \quad \text{if} \quad \|h\|_1 = R .$$

Hence it follows that there exists $x^0 \in H_1$, $\|x^0\|_1 < R$, such that

$$\Phi_1(x^0) \leq \Phi_1(x), \qquad x \in H_1 ,$$

i.e. $F_1(x^0) = 0$ (see [5], theorem 9.8). Finally

$$\tilde{\Phi}(x^0,h) + (\tilde{a},h) = 0, \qquad h \in H_1,$$

which completes the proof.

THEOREM 4.1.4. Assume that $\tilde{\Phi}$ satisfies (a) - (d) and that $\tilde{a} \in H$. Then there exists a solution of the equation

$$(4.1.15) \qquad \tilde{\Phi}(x,h) + (\tilde{a},h) = 0 \quad \text{for any} \quad h \in H_1 .$$

Moreover, if G is a potential and bounded operator, satisfying (4.1.13), then for any $\lambda > 0$ there exists a unique element $x_\lambda \in H_1$ such that

$$(4.1.16) \qquad \frac{1}{\lambda}\left[\tilde{\Phi}(x_\lambda,h) + (\tilde{a},h)\right] + (G(x_\lambda),h)_1 = 0 \quad \text{for any} \quad h \in H_1$$

and there exists the limit $\tilde{x}^0 := \lim_{\lambda \to 0} x_\lambda$, which is a solution of (4.1.15).

P r o o f. First observe that in view of (4.1.2) the assumption (4.1.14) of Lemma 4.1.3 is valid for any $\tilde{a} \in H$. Hence it follows from Lemma 4.1.3 that for any $\tilde{a} \in H$ there exists a solution of the equation (4.1.15).

For $x,h \in H_1$ put

$$\tilde{\Phi}_1(x,h) = \tilde{\Phi}(x,h) + (\tilde{a},h).$$

We have $\tilde{\Phi}_1(x,h) = (F_1(x),h)_1$ and $F_1(x) = \operatorname{grad} \Phi_1(x)$ (see the proof of Lemma 4.1.3). Note that Φ_1 satisfies the conditions (i′) and (ii′) (with Φ replaced by Φ_1) ((ii′) follows from (4.1.5)). Further, in view of Lemma 4.1.3, there exists $x^0 \in H_1$ such that

$$\Phi_1(x^0) \leq \Phi_1(x), \qquad x \in H_1 .$$

Hence it follows from Theorem 3.1 that the function $\lambda \longrightarrow x_\lambda$ $\lambda > 0$ defined by the inequality:

$$\frac{1}{\lambda}\Phi_1(x_\lambda) + \Psi(x_\lambda) < \frac{1}{\lambda}\Phi_1(x) + \Psi(x), \qquad x \in H_1,$$

(i.e. by (4.1.16) with $G(x) = \operatorname{grad}\Psi(x)$) has the limit $\tilde{x}^0 := \lim_{\lambda \to 0} x_\lambda$ and

$$\Phi_1(\tilde{x}^0) = \inf_{x \in H_1} \Phi(x),$$

i.e. $F_1(\tilde{x}^0) = 0$. Hence $\tilde{x}^0$ is a solution of the equation (4.1.15), which proves the theorem.

4.2. Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, denote an open set, bounded and simply connected, with the boundary $S := \partial\Omega$ smooth. Set $H := \mathcal{L}^2_\Omega$, $M := \{u \in C^2_\Omega : u|_s = 0\}$,

$$(x,y) := \int_\Omega x(\xi)y(\xi)d\xi, \qquad x,y \in H,$$

$$(u,v)_1 := \int_\Omega \sum_{i=1}^{n} \frac{\partial u}{\partial \xi_i}\,\frac{\partial v}{\partial \xi_i}\,d\xi, \qquad u,v \in M,$$

where $\xi := (\xi_1, \ldots, \xi_n) \in \Omega$.

Observe that M is dense in H (in the norm $\|\cdot\|$) and that

(4.2.1) $$\|x\|_1 \geq \alpha\|x\|, \qquad x \in M, \quad \alpha > 0$$

$(\|x\| := \sqrt{(x,x)}, \quad \|x\|_1 := \sqrt{(x,x)_1})$. It follows from the theory of Friedrichs that there exists H_1 $(M \subset H_1 \subset H)$ - the completion of M in the norm $\|\cdot\|_1$ (see [4], N° 124). It is obvious that the inequality (4.2.1) is valid for any $x \in H_1$.

Let $a_i : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}$, $i = 1,\dots,n$, be continuous functions with the continuous partial derivatives of the first order $a_{ik} := \frac{\partial}{\partial t_k} a_i$, $i,k = 1,\dots,n$, $t := (t_1,\dots,t_n) \in \mathbb{R}^n$.

We assume that

$$(4.2.2) \qquad a_{ik} = a_{ki}, \qquad i,k = 1,\dots,n,$$

$$(4.2.3) \qquad \Big|\sum_{i,k=1}^{n} a_{ik} s_i r_k\Big|^2 \leqslant m\Big[\sum_{i=1}^{n} s_i^2\Big]\Big[\sum_{k=1}^{n} r_k^2\Big] \qquad m > 0,$$

$$(4.2.4) \qquad \sum_{i,k=1}^{n} a_{ik} s_i s_k \geqslant 0,$$

$$(4.2.5) \qquad a_i(s_1,\dots,s_n,\xi) s_i \geqslant \tilde{\alpha}|s_i|^2 - \sum_{k=1}^{n} |\beta_{ik}(\xi)||s_k| - |c_i(\xi)|$$

for $i = 1,\dots,n$ and $\xi \in \Omega$

where $s_i, r_i \in \mathbb{R}$, $i = 1,\dots,n$, $\tilde{\alpha} > 0$, $\beta_{ik} \in \mathcal{L}^2_\Omega$, $c_i \in \mathcal{L}_\Omega$ $i,k = 1,\dots,n$.

We consider the following boundary value problem:

$$(4.2.6) \qquad \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i} a_i(u_{\xi_1},\dots,u_{\xi_n},\xi) + \tilde{a}(\xi) = 0, \qquad \xi \in \Omega,$$

$$u(\xi) = 0 \quad \text{for} \quad \xi \in S,$$

where $u : \Omega \longrightarrow \mathbb{R}$, $u_{\xi_i} := \frac{\partial u}{\partial \xi_i}$, $i = 1,\dots,n$ and $\tilde{a} \in \mathcal{L}^2_\Omega$.

For $u,h \in M$ set

$$(4.2.7) \qquad \tilde{\Phi}(u,h) := \int_\Omega \sum_{i=1}^{n} a_i(u_{\xi_1},\dots,u_{\xi_n},\xi) h_{\xi_i} d\xi .$$

LEMMA 4.2.1. The functional $\tilde{\Phi}$ defined by (4.2.7) satisfies the conditions (a) - (d).

P r o o f. The inequality (4.1.1) follows from (4.2.7) with the aid of Holder's inequality. To prove (4.1.2) observe that by Holder's inequality and (4.2.5) we have

$$\tilde{\Phi}(h,h) = \int_{\Omega}\sum_{i=1}^{n} a_i(h_{\xi_1},\dots,h_{\xi_n},\xi)h_{\xi_i}d\xi \geq$$

$$\geq \tilde{\alpha}\int_{\Omega}\sum_{i=1}^{n} h^2_{\xi_i}d\xi - \int_{\Omega}\sum_{i=1}^{n}(\sum_{k=1}^{n}|\beta_{ik}(\xi)||h_{\xi_k}|)d\xi -$$

$$- \int_{\Omega}\sum_{i=1}^{n}|c_i(\xi)|d\xi \geq \tilde{\alpha}\|h\|_1^2 - b\|h\|_1 - c ,$$

where $b := (\int_{\Omega}\sum_{i=1}^{n}(\sum_{k=1}^{n}|\beta_{ik}(\xi)|)^2 d\xi)^{\frac{1}{2}}$, $c := \int_{\Omega}\sum_{i=1}^{n}|c_i(\xi)|d\xi$ (cf. [5]). Hence $\frac{\tilde{\Phi}(h,h)}{\|h\|_1} \geq \tilde{\alpha}\|h\|_1 - b - \frac{c}{\|h\|_1} \longrightarrow +\infty$ if $\|h\|_1 \longrightarrow +\infty$.

The condition (c) and the existence of $\tilde{\Phi}'$ follow from the continuity of a_i and a_{ik}, $i,k = 1,\dots,n$. Moreover,

$$\tilde{\Phi}'(u,h,f) = \int_{\Omega}\sum_{i,k=1}^{n} a_{ik}(u_{\xi_1},\dots,u_{\xi_n},\xi)h_{\xi_i}f_{\xi_k}d\xi, \quad u,h,f \in M$$

(cf. [3]). Hence it follows from (4.2.2) that (4.1.3) holds. The inequalities (4.1.4) and (4.1.5) follow because of (4.2.3) and (4.2.4) respectively (see [3], p. 138). This completes the proof of the lemma.

It follows from Lemma 4.2.1 that we can apply 4.1 to the functional $\tilde{\Phi}$ (defined by (4.2.7)). Therefore, $\tilde{\Phi}$ can be extended to all of $H_1 \times H_1$ in a unique way so that the extension (which we shall denote also by $\tilde{\Phi}$) satisfies (a) - (d) (with M replaced by H_1) and there exists $F : H_1 \longrightarrow H_1$ such that $\tilde{\Phi}(u,h) = (F(u),h)_1$, $u,h \in H_1$. Moreover, $F(u) = \operatorname{grad} \Phi(u)$, $u \in H_1$, where $\Phi : H_1 \longrightarrow \mathbb{R}$ satisfies (i′) and (ii′) (with $X = H_1$).

DEFINITION 4.2.1. We say that a function $u^0 \in H_1$ is a generalized solution of the problem (4.2.6) if

$$\tilde{\Phi}_1(u^0,h) := \tilde{\Phi}(u^0,h) + (\tilde{a},h) = 0 \qquad \text{for any} \quad h \in H_1$$

REMARK 4.2.2. Note that if $u^0 \in M$ is a solution of (4.2.6), then u^0 is a generalized solution and vice versa, if $u^0 \in C^2_\Omega$ is a generalized solution of (4.2.6), then u^0 is a solution of this problem (see [2], Théoreme 4.3.1).

In view of Theorem 4.1.4 we have

THEOREM 4.2.3. Let the assumptions (4.2.2) - (4.2.5) be fulfilled. Then there exists a generalized solution of the problem (4.2.6).

Moreover, if G is a potential and bounded operator, satisfying (4.1.13), then for any $\lambda > 0$ there exists a unique element $x_\lambda \in H_1$ such that

$$\frac{1}{\lambda}\Phi_1(x_\lambda,h) + (G(x_\lambda),h)_1 = 0, \qquad h \in H_1$$

and there exists the limit $\lim_{\lambda \to 0} x_\lambda$ which is a generalized solution of the problem (4.2.6).

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STRESZCZENIE

Pierwsza część pracy poświęcona jest problemowi osiągania kresu dolnego przez rzeczywisty funkcjonał Φ, określony na rzeczywistej przestrzeni Banacha X i spełniający założenia (i') oraz (ii'). Udowodniono twierdzenia o minimum warunkowym i bezwarunkowym oraz skonstruowano pewien ciąg zbieżny do punktu, w którym Φ osiąga swój kres dolny. Opierając się o uzyskane wyniki, w drugiej części pracy wykazano twierdzenie o istnieniu rozwiązania uogólnionego pewnego równania różniczkowego cząstkowego quasi-liniowego.

Резюме

Первая часть работы посвящена проблеме достижимости нижней грани вещественного функционала Φ, определенного на вещественном банаховом пространстве X и удовлетворяющего условиям /i'/ и /ii'/. Доказаны теоремы об условном и безусловном минимуме и построена некоторая последовательность сходящаяся к точке, в которой Φ достигает своей нижней грани. Основываясь на полученных результатах, во второй части работы установлено теорему о существовании обобщенного решения некоторого квази-линейного уравнения с частными производными.