ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XXXIII, 5 SECTIO A 1979

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On a Class of Bazilevič Functions

O pewnej klasie funkcji Bazylewicza

О некотором классе функций Базилевича

1. INTRODUCTORY REMARKS

Let f, $f(z) = z + a_2z^2 + \ldots$, be a function analytic in the unit disk Δ such that

(1.1) $$z^{-1}f(z)f'(z) \neq 0 \quad \text{in} \quad \Delta.$$

Not so long ago P.T. Mocanu [6] considered a class $S(\alpha)$ of analytic functions f that satisfy (1.1) and the condition

(1.2) $$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0$$

for $\alpha \in \langle 0,1 \rangle$ and z in Δ.

It was shown that $S(\alpha)$ is a class of univalent functions which, moreover, map Δ onto domains starlike w.r.t. the origin. It is easy to notice that (1.2) is obtained by forming a "linear combination" of two conditions for starlikenes and convexity, respectively.

Recently it was shown [3] that this condition may be replaced by a much general one which, however, implies univalence and starlikeness of f.

One can easily check that if $-\frac{1}{2} \leq \beta < 1$, then the condition

$$\text{(1.3)} \qquad \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \quad \text{in } \Delta$$

implies univalence of f. Let

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > -\frac{1}{2}.$$

A simple calculation gives

$$\frac{1}{3}\left(2\frac{zf''(z)}{f'(z)} + 3\right) = \frac{zg'(z)}{g(z)}$$

where $g \in S^*$.

Then

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > 0.$$

This shows that f is close - to - convex.

Hence f is univalent. Moreover, f maps Δ onto a domain convex in at least one direction.

This remark raises a natural question, if the following condition

$$\text{(1.4)} \qquad \operatorname{Re}\left\{(1-\alpha)(1-\beta)\frac{zf'(z)}{f(z)} + \alpha\left(1-\beta + \frac{zf''(z)}{f'(z)}\right)\right\} > 0$$

for real α and $-\frac{1}{2} \leq \beta < 1$ guarantees univalence of f.

It will be shown in this paper that functions f subject to

(1.4) for some values of α are indeed univalent and some extremal properties of f will be investigated.
Many results ealier obtained by several authors follow from ours as special cases.

2. A CLASS OF BAZILEVIČ FUNCTIONS

We denote by $\bar{F}(\alpha,\beta)$ the class of all analytic functions that satisfy (1.1) and (1.4) in Δ.

First we prove

THEOREM 2.1. Suppose f is in $\bar{F}(\alpha,\beta)$ and

$$(2.1)\qquad \begin{cases} -\frac{1}{2}\leq\beta<0 \ \text{ and } \ 0<\alpha<1-\beta^{-1} \\ \text{or} \\ 0<\beta<1 \ \text{ and } \ \alpha<1-\beta^{-1} \ \text{ or } \ \alpha>0 \\ \text{or} \\ \beta=0 \ \text{ and } \ \alpha>0. \end{cases}$$

Then f is univalent in Δ and it has the form

$$(2.2)\qquad f(z)=\left[m\int_0^z \zeta^{-1}\left(\frac{\zeta}{g(\zeta)}\right)^{\beta} g^m(\zeta)\,d\zeta\right]^{\frac{1}{m}} = z+\ldots$$

where $g(z)=z+\ldots$ is a starlike and univalent function in Δ and $m=1+\alpha^{-1}(1-\alpha)(1-\beta)$, $\alpha\neq 0$.

P r o o f. If g satisfies the conditions stated above then $\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\}>0$ in Δ and (1.4) may have form

$$(1-\alpha)(1-\beta)\frac{zf'(z)}{f(z)}+\alpha\left(1-\beta+\frac{zf''(z)}{f'(z)}\right)=$$
$$=(1-\beta)\frac{zg'(z)}{g(z)}$$

Some easy and straightforward computations show that this equation has a formal solution of the form (2.2). The integral involved here converges provided $m > 0$. It gives (2.1).

For $\beta \in \langle 0,1)$, $F(\alpha,\beta)$ is a subclass of the class of analytic functions introduced by Mocanu.

We may consider this problem for $\beta \in \langle -\frac{1}{2}, 0)$.

Then, by $\left|\frac{1}{2}\arg\frac{z}{g(z)}\right| \leq \frac{\pi}{2}$

we have $\left|\beta \arg\frac{z}{g(z)}\right| \leq |\beta|\,\pi < \frac{\pi}{2}$.

Hence $\mathrm{Re}\left\{\left(\frac{z}{g(z)}\right)^{\beta}\right\} > 0$ in Δ.

Now, for $t \geq 0$ the family $\{f(z,t)\}$, $z \in \Delta$,

$$f(z,t) = \left[m \int_0^z \left[t\,\frac{\zeta g'(\zeta)}{g(\zeta)} + \left(\frac{\zeta}{g(\zeta)}\right)^{\beta}\right] g^m(\zeta)\,\zeta^{-1}\,d\zeta\right]^{1/m}$$

is a subordination chain over the interval $t \geq 0$ in the sense of Pommerenke [7].

Hence, by Pommerenke's theorem [7] $f(z,t)$ is analytic and univalent in Δ for each $t \geq 0$.

It shows also, that $F(\alpha,\beta)$ is a subclass of Bazilevič functions defined in [1].

REMARK 1. For any real α and $-\frac{1}{2} \leq \beta < 1$ the identity function belongs to $F(\alpha,\beta)$ so that $F(\alpha,\beta)$ is not empty. Since our method of proof cannot be applied to values of β, α other than given by (2.1) the question of univalence of f, $f \in F(\alpha,\beta)$ for those values of α remains open.

REMARK 2. It seems plausible that each f of the class $F(\alpha,\beta)$ is close-to-convex. However, we were not able to prove it.

3. SOME EXTREMAL PROBLEMS WITHIN THE CLASS $F(\alpha,\beta)$

We shall consider here some distortion problems and we shall give bounds for initial Taylor coefficients of f.
Let Γ denote the gamma function of Euler and $F(a,b,c;z)$ be the analytic functions for z in Δ defined by

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}du$$

where $\operatorname{Re} b>0$, $\operatorname{Re}(c-b)>0$.

Put

$$K(\alpha,\beta,r) = r\left[F\left(\frac{2(1-\beta)}{\alpha}, m, m+1, r\right)\right]^{\frac{1}{m}}$$

$$f_\theta(\alpha,\beta,z) = \left[m\int_0^z \zeta^{m-1}(1-e^{i\theta}\zeta)^{\frac{-2(1-\beta)}{\alpha}} d\zeta\right]^{\frac{1}{m}}$$

where $m = 1 + \frac{(1-\alpha)(1-\beta)}{\alpha}$, $\alpha \neq 0$, $0 \leq \theta < 2\pi$.
We start with

THEOREM 3.1. <u>If</u> f <u>satisfies conditions of Th.2.1 and</u> $|z| = r$ <u>then</u>

(3.1) $$-K(\alpha,\beta,-r) \leq |f(z)| \leq K(\alpha,\beta,r) \quad \text{for } \alpha>0,$$

<u>or</u>

(3.2) $$K(\alpha,\beta,r) \leq |f(z)| \leq -K(\alpha,\beta,-r) \quad \text{for } \alpha<0$$

<u>This result is sharp. Equality occurs for the function</u> $f_\theta(\alpha,\beta,z)$ <u>with suitably chosen</u> θ.

P r o o f. In view of Th. 2.1 we have

$$(*)\qquad f(z) = \left[m \int_0^z \zeta^{m-1} \left(\frac{g(\zeta)}{\zeta}\right)^{\frac{1-\beta}{\alpha}} d\zeta\right]^{\frac{1}{m}}.$$

Since g is univalent normalized starlike function we have

$$(**)\qquad \frac{|\zeta|}{(1+|\zeta|)^2} \leqslant |g(\zeta)| \leqslant \frac{|\zeta|}{(1-|\zeta|)^2}.$$

Suppose now z_0 is a point on the circumference $|z| = r$ such that

$$|f(z_0)| = \min_{0 \leqslant \theta \leqslant 2\pi} |f(re^{i\theta})|$$

and γ denotes the pre-image under f of the segment $[0, f(z_0)]$. Consider first case $\alpha > 0$.

In view of (*) and (**) we get

$$|f(z_0)|^m = m \int_\gamma \zeta^{m-1} \left|\frac{g(\zeta)}{\zeta}\right|^{\frac{1-\beta}{\alpha}} |d\zeta| \geqslant$$

$$\geqslant m \int_0^r t^{m-1} (1+t)^{\frac{-2(1-\beta)}{\alpha}} dt =$$

$$= mr^m \int_0^1 u^{m-1} (1+ru)^{\frac{-2(1-\beta)}{\alpha}} du.$$

Hence $|f(z)| \geqslant |f(z_0)| \geqslant -K(\alpha, \beta, -r)$.

The proof of (3.1) for the upper bound of $|f(z)|$ is a similar one. The proof of (3.2) is analogous.

COROLLARY 1. If f satisfies conditions of Th 2.1 then

$$|a_2| \leqslant \frac{2(1-\beta)}{|(1-\alpha)(1-\beta) + 2\alpha|}$$

P r o o f. It is sufficient to assume a_2 to be real. Consider first case $\alpha > 0$. We find:

$$K(\alpha, \beta, r) = r + \frac{2(1-\beta)}{(1-\alpha)(1-\beta)+2\alpha} r^2 + O(r^3)$$

and $|f(r)| = r + a_2 r^2 + O(r^3)$,
and in view of Th. 3.1 (3.1) we get:

$$a_2 \leqslant \frac{2(1-\beta)}{(1-\alpha)(1-\beta)+2\alpha}.$$

If $\alpha < 0$ we reason in a similar manner but we make use of Th. 3.1 (3.2).

COROLLARY 2. If f satisfies conditions of Th. 2.1, then

$$\bigcap_{f \in F(\alpha,\beta)} F(\Delta) = \{w, |w| < d(\alpha, \beta)\}$$

where

$$d(\alpha,\beta) = \begin{cases} \left[F\left(\frac{2(1-\beta)}{\alpha} m, m+1, -1\right)\right]^{\frac{1}{m}} & \text{for } \alpha > 0 \\ \left[F\frac{2(1-\beta)}{\alpha} m, m+1, 1\right]^{\frac{1}{m}} & \text{for } \alpha < 0. \end{cases}$$

P r o o f: It is sufficient to notice that $-K(\alpha, \beta, -r)$ and $K(\alpha, \beta, r)$ are an increasing function of r and then let r tend to 1 in the l.h.s. of (3.1) and of (3.2).
Let $L(r)$, $A(r)$ denote the length of the curve C, $C = f(re^{i\Theta})$ $0 \leqslant \Theta < 2\pi$ and the area of the region bounded by C, respectively.

THEOREM 3.2. If $f \in F(\alpha, \beta)$, $\alpha \neq 0$, $T = T(\alpha, \beta) = \frac{|(1-\alpha)(1-\beta)| + |\alpha\beta| + 1 - \beta}{|\alpha|}$,

$M(r) = \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})|$, then

(i) $$2M(r) \leq L(r) \leq 2\pi T M(r).$$

If f satisfies the conditions of Th. 2.1, then

(ii) $$2\pi A(r)^{1/2} \leq L(r) \leq \frac{8T}{r}\left[\pi A(r) \log \frac{1}{1-r^2}\right]^{1/2}$$

P r o o f. Suppose g is a univalent starlike function in Δ. Then the condition (1.4) takes the form

$$(1-\alpha)(1-\beta)\frac{zf'(z)}{f(z)} + \alpha\left(1-\beta+\frac{zf''(z)}{f'(z)}\right) = (1-\beta)\frac{zg'(z)}{g(z)}.$$

Hence, we have

$$g(z) = z\left(\frac{f(z)}{z}\right)^{1-\alpha}(f'(z))^{\frac{\alpha}{1-\beta}}.$$

Solving this with respect to f' we obtain a formal representation

$$zf'(z) = [g(z)]^{\frac{1-\beta}{\alpha}} z^{\beta} [f(z)]^{\frac{-(1-\alpha)(1-\beta)}{\alpha}}.$$

Thus $(re^{i\theta} = z)$

$$L(r) = \int_{|z|=r} |f'(z)|\,|dz| = \int_0^{2\pi} zf'(z)e^{-i\arg zf'(z)}d\theta.$$

The integral on the r.h.s. is now computed in a standard way to yield.

$$L(r) \leq 2M(r)\pi\frac{|(1-\alpha)(1-\beta)| + |\alpha\beta| + 1-\beta}{|\alpha|}$$

It gives us (i).

To prove (ii) observe that in view of the univalence of f we have $M(r) \leqslant 4r^{-1}M(r^2)$.

Hence

$$L(r) \leqslant 2\pi T M(r) \leqslant 8\,\frac{\pi T}{r}\,M(r^2) \leqslant \frac{8\pi}{r}\,T \sum_{n=1}^{\infty} |a_n| r^{2n} =$$

$$= \frac{8\pi}{r}\,T \sum_{n=1}^{\infty} (n^{1/2}|a_n| r^n)(n^{-1/2} r^n).$$

Making now use of the area theorem and the Schwarz inequality we ultimately find

$$L(r) \leqslant \frac{8\pi}{r}\,T \left[\pi^{-1} A(r) \log \frac{1}{1 - r^2}\right]^{1/2}$$

The rest follows from the fact that disk has the minimum of the area among domains bounded by a curve of a given length.

4. COEFFICIENT BOUNDS

We have already obtained the best upper bound for $|a_2|$ within the class $F(\alpha,\beta)$, by making use of the integral representation (Th. 2.1). We want now to show that upper bounds for initial Taylor coefficients of f can be obtained directly from the definition of the class $F(\alpha,\beta)$.

THEOREM 4.1. <u>If</u> μ <u>is a complex number and</u> $A_k = (1-\alpha)(1-\beta) + k\alpha$, $k = 2,3,4$ <u>then</u>

$$\sup_{f \in F(\alpha,\beta)} |a_3 - \mu a_2^2| = \frac{1-\beta}{A_3}\max(1,|v|)$$

<u>where</u>

$$v = \frac{4\mu A_3(1-\beta) - 2A_4(1-\beta) - A_2^2}{A_2^2}$$

The result is best possible.

P r o o f. Let us write the condition (1.4) in the form

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \frac{\alpha}{1-\beta}\left(1-\beta+\frac{zf''(z)}{f'(z)}\right) = \frac{1+w(z)}{1-w(z)}$$

where $w(z) = c_1 z + c_2 z^2 + \dots$ is a holomorphic function subject to the Schwarz Lemma conditions. Comparing the Taylor coefficients of both sides in a neighbourhood of the origin one gets

(i) $$c_1 = \frac{A_2}{2(1-\beta)} a_2$$

(ii) $$c_2 = \frac{A_3}{1-\beta} a_3 - \left(\frac{2A_4(1-\beta)+A_2^2}{4(1-\beta)^2}\right) a_2^2 .$$

It is well-known that $|c_1| \leqslant 1$, $|c_2| \leqslant 1-|c_1|^2$.

Thus we have the sharp inequality

$$|c_2 - vc_1^2| \leqslant |c_2| + |v|\,|c_1^2| \leqslant \max(1,|v|).$$

Making now use of (i) and (ii) after some computations we get the result.

COROLLARY 1. For f in $F(\alpha,\beta)$ there hold the following sharp inequalities:

$$|a_2| \leqslant \frac{2(1-\beta)}{|A_2|}$$

$$|a_3| \leqslant \frac{1-\beta}{|A_3|} \max\left(1, \frac{|2A_4(1-\beta)+A_2^2|}{A_2^2}\right)$$

The extremal functions satisfy the equations

$$\left(\frac{f(z)}{z}\right)^{1-\alpha} \cdot (f'(z))^{\frac{\alpha}{1-\beta}} = \frac{1}{(1-z)^2}$$

or

$$\left(\frac{f(z)}{z}\right)^{1-\alpha}\cdot\left(f'(z)\right)^{\frac{\alpha}{1-\beta}} = \frac{1}{1-z^2}$$

respectively.

One may obtain the sharp estimate for $|a_4|$ in a similar manner but by making use of a lemma of J. Szynal and S. Wajler [9,p. 1153]. The computations are simple but lengthy so we give here the final result without proof.

THEOREM 4.2. *If* $f\in F(\alpha,\beta)$, $A_k = (1-\alpha)(1-\beta)+k\alpha$, $k = 2,\dots,8$, *then*

$$|a_4| \leq \frac{2(1-\beta)}{3|A_4|}\,\phi\left(\left|\frac{3A_6(1-\beta)}{A_2A_3}+2\right|,\ \left|\frac{4A_8(1-\beta)^2}{A_2^3}+1+\frac{4A_4(1-\beta)^2}{A_2^2}\right|\right)$$

where

$$\phi(p,q) = \begin{cases} q & \text{if } (p,q)\in D_1 \\ \frac{2}{3}(p+1)\sqrt{\dfrac{p+1}{3(p+1-q)}} & \text{if } (p,q)\in D_2 \\ 1 & \text{if } (p,q)\in D_3 \end{cases}$$

and

$$D_1 = \left\{(p,q) : q\geq \tfrac{2}{3}(p+1) \text{ and } p\geq 1\right\}$$

$$D_2 = \left\{(p,q) : (p+1)-\tfrac{4}{27}(p+1)^3 \leq q < \tfrac{2}{3}(p+1)\right\}$$

$$D_3 = \left\{(p,q) : p\leq \tfrac{1}{2} \text{ and } q<1\right\}\cup\left\{(p,q) : \tfrac{1}{2}<p<\tfrac{2}{3}\sqrt{3}-1 \text{ and } q<(p+1)-\tfrac{4}{27}(p+1)^3\right\}.$$

The extremal function satisfies the equation

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \frac{\alpha}{1-\beta}\left(1-\beta+\frac{zf''(z)}{f'(z)}\right) = \frac{1+z\varepsilon}{1-z\varepsilon}, \quad |\varepsilon| = 1.$$

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STRESZCZENIE

W pracy rozważa się rodzinę $F(\alpha,\beta)$ funkcji holomorficznych w kole jednostkowym spełniających w tym kole warunki $z^{-1}f(z)f'(z) \neq 0$ oraz $\mathrm{Re}\{(1-\alpha)(1-\beta)\frac{zf'(z)}{f(z)} + \alpha(1-\beta + \frac{zf''(z)}{f'(z)})\} > 0$ dla rzeczywistego α i $\beta \in \langle -\frac{1}{2}, 1)$. Rodzina ta stanowi uogólnienie klasy funkcji α-wypukłych wprowadzonej przez P. Mocanu w 1969 roku. Uzyskano twierdzenie typu: strukturalnego, o zniekształceniu, o pokryciu oraz oszacowania funkcjonału Gołuzina i modułu współczynników a_2, a_3.

Резюме

Пусть $F(\alpha,\beta)$ обозначает класс функции $f(z)$ голоморфных, и таких, что $z^{-1}f(z)f'(z) \neq 0$ и таких что, $Re\{(1-\alpha)(1-\beta)\frac{zf'(z)}{f(z)} + \alpha(1-\beta + \frac{zf''(z)}{f'(z)})\} > 0$ для действительного α и $\beta \in \langle -\frac{1}{2}, 1)$.

Этот класс это обобщение класса введенного Мокану в 1969 году. В работе доказывается структурная формула и теореме искажения. Далее даны оценки функционала Голузина и модулей коэффициентов a_2, a_3.

[illegible]