ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XXXIII, 3 SECTIO A 1979

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On the Existence of a Linear Connection so as a Given Tensor Field of the Type (1,1) is Parallel with Respect to This Connection

O istnieniu koneksji liniowej takiej, aby dane pole tensorowe typu (1,1) było równoległe względem tej koneksji

О существовании линейной связности такой, чтобы данный тензор π типа (1,1) был параллельный относительно этой связности

This problem is solved by means of conjugate connections. Conjugate connections have been investigated by a number of authors, among others [3]. In this paper some applications of this concept will be introduced. Suppose that two linear connections Γ and $\hat{\Gamma}$ and a non-singular tensor field π of the type (1,1) are given on an n-dimensional manifold M.

DEFINITION 1. The connections Γ and $\hat{\Gamma}$ are said to be conjugate with respect to the tensor field π of the type (1,1) if and only if the following condition is satisfied along every curve γ on M: if an arbitrary covector $\underline{a}$ is parallel displaced along γ in the sense of the connection Γ, then the covector $\pi_{\underline{a}} : \begin{matrix} TM \longrightarrow R \\ \bar{v} \longmapsto \pi(\bar{v},\underline{a}) \end{matrix}$ is parallel displaced along γ in the sense of the connection $\hat{\Gamma}$.

The following theorem characterizes these connections:

THEOREM 1. *The necessary and sufficient condition for the connections Γ and $\hat{\Gamma}$ to be conjugate with respect to the tensor field π of the type (1,1) is that their local coordinates Γ^i_{jk} and $\hat{\Gamma}^i_{jk}$ be related by the relation:*

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} + \tilde{\pi}^i_p \nabla_j \pi^p_k \tag{1}$$

where ∇ denotes the covariant differentiation operator with respect to the connection Γ, and $\tilde{\pi}$ is the inverse to π.

P r o o f. The covector $\underline{a}$ is parallel dispaced along every curve γ on M in the sense of Γ, so its coordinates satisfy the following condition:

$$\partial_i a_k - \Gamma^p_{ik} a_p = 0$$

The covector $\pi_{\underline{a}}$ is parallel displaced along every curve γ on M in the sense of $\hat{\Gamma}$, so its coordinates $\pi^j_i a_j$ satisfy the condition:

$$\partial_k(\pi^j_i a_j) - \hat{\Gamma}^s_{ki} \pi^j_s a_j = 0$$

or

$$\nabla_k \pi^j_i a_j + \pi^j_s \Gamma^s_{ki} a_j - \hat{\Gamma}^s_{ki} \pi^j_s a_j = 0$$

This equality holds for any covector $\underline{a}$, so we have:

$$\nabla_k \pi^j_i + \pi^j_s \Gamma^s_{ki} = \hat{\Gamma}^s_{ki} \pi^j_s$$

hence

$$\hat{\Gamma}^s_{ki} = \Gamma^s_{ki} + \tilde{\pi}^s_j \nabla_k \pi^j_i$$

QED

As an example of the application of the theory of conjugate connections with respect to the tensor field π of the type (1,1) we'll give the following:

THEOREM 2. If a non-singular tensor field π of the type (1,1) on a manifold M with a given connection Γ satisfies the condition:

$$\nabla_k(\pi^m_l \pi^l_s) = 0 \tag{2}$$

then there exists the connection $\hat{\hat{\Gamma}}$ on M such, that: $\hat{\hat{\nabla}}_k \pi^i_j = 0$.

P r o o f. We define the connection $\hat{\hat{\Gamma}}$ on M in the following way:

$$\hat{\hat{\Gamma}} = \frac{1}{2}(\Gamma + \hat{\Gamma}) \tag{3}$$

where $\hat{\Gamma}$ is the conjugate connection with Γ with respect to π. In the local map U, the coordinates of this connection are the following:

$$\hat{\hat{\Gamma}}^i_{jk} = \Gamma^i_{jk} + \frac{1}{2}\tilde{\pi}^i_p \nabla_j \pi^p_k \tag{4}$$

Now, let's compute the value of $\hat{\hat{\nabla}}\pi^i_j$:

$$\hat{\hat{\nabla}}_k \pi^i_j = \partial_k \pi^i_j - \hat{\hat{\Gamma}}^s_{kj}\pi^i_s + \hat{\hat{\Gamma}}^i_{ks}\pi^s_j = \partial_k \pi^i_j - \Gamma^s_{kj}\pi^i_s +$$
$$+ \Gamma^i_{ks}\pi^s_j - \frac{1}{2}\pi^i_s \tilde{\pi}^s_p \nabla_k \pi^p_j + \frac{1}{2}\pi^s_j \tilde{\pi}^i_p \nabla_k \pi^p_s =$$
$$= \nabla_k \pi^i_j - \frac{1}{2}\nabla_k \pi^i_j + \frac{1}{2}\tilde{\pi}^i_p \pi^s_j \nabla_k \pi^p_s$$

Using the condition 2 we have:

$$\hat{\hat{\nabla}}_k \pi^i_j = \frac{1}{2}\nabla_k \pi^i_j - \frac{1}{2}\tilde{\pi}^i_p \pi^p_s \nabla_k \pi^s_j = \frac{1}{2}\nabla_k \pi^i_j - \frac{1}{2}\nabla_k \pi^i_j = 0$$

We'll need the following:

THEOREM 3 [2], [4], [5], [6]. If a tensor field π of the type (1,1) is covariantly constant with respect to a given connection on a manifold M, then there exists an atlas (in the main-nonholonomic) on M such, that the tensor field π has constant coordinates at each map of this atlas.

Now we have:

THEOREM 4. If a non-singular tensor field π of the type (1,1) on a manifold M with a given connection Γ satisfies the condition (2), then there exists an atlas (in the main-nonholonomic) on M such, that the tensor field π has constant coordinates at each map of this atlas.

P r o o f. It is the obvious consequence of the theorems 2 and 3.

Finally, if we proceed similarly to [1] we'll have the following:

THEOREM 5. The curvature tensors $\hat{R}$ and R of the conjugate connections $\hat{\Gamma}$ and Γ respectively, with respect to the tensor field π of the type (1,1) are related in the following way:

$$\hat{R}^{i}_{jkl} = \tilde{\pi}^{i}_{p}\pi^{m}_{l}R^{p}_{jkm} \tag{5}$$

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STRESZCZENIE

W pracy tej zdefiniowane są koneksje liniowe sprzężone względem pola tensorowego typu (1,1) a następnie ich zastosowanie między innymi w podstawowym wyniku pracy tzn. twierdzeniu 2 o istnieniu koneksji liniowej takiej, aby dane pole tensorowe π typu (1,1) było równoległe, przy warunku, że pole π^2 jest równoległe względem danej koneksji oraz w twierdzeniu 4 o istnieniu atlasu w którym pole π ma stałe współrzęd-

ne, przy założeniu że pole π^2 jest równoległo.
W twierdzeniu 5 podany jest wzór na tensor krzywiznowy koneksji sprzężonej.

Резюме

В этой работе определено линейные сопряженные связности относительно тензора типа $(1,1)$, а затем их употребление например в основном результате работы, это значит в теореме 2 о существовании линейной связности такой, чтобы данный тензор π типа $(1,1)$ был параллельный при условии, что π^2 является параллельным относительно данной связности. Другое употребление выступает в теореме 4 о существовании атласа, в котором π имеет постоянные координаты, при условии, что π^2 является параллельным. В теореме 5 представлено формулу тензора кривизны сопряженной связности.