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Curvature Tensors of Conjugate Connections on a Manifold

Tensory krzywizny koneksji sprzężonych na rozmaitości

Тензоры кривизны сопряженных связностей на многообразие

Following Norden [3], Wedernikow [5] I will recall the notions concerning conjugate connections. Suppose that two linear connections Γ and $\hat{\Gamma}$ and the non-singular tensor π of type (0,2) are given on an n-dimensional manifold M.

DEFINITION 1 [3], [5]. The connections Γ and $\hat{\Gamma}$ are said to be conjugate with respect to the tensor π of type (0,2) if and only if the following condition is satisfied along every curve γ on M: if an arbitrary vector $\bar{w}$ is parallel displaced along γ in the sense of the connection Γ, then the covector: $\pi_{\bar{w}} : \begin{array}{c} TM \longrightarrow R \\ \bar{v} \longmapsto \pi(\bar{v},\bar{w}) \end{array}$ is parallel displaced along γ in the sense of the connection $\hat{\Gamma}$.
The following theorem characterizes these connections:

THEOREM 1 [3]. The necessary and sufficient condition for the connections Γ and $\hat{\Gamma}$ to be conjugate with respect to the tensor π of type (0,2) is that their local coordinates Γ^i_{jk} and $\hat{\Gamma}^i_{jk}$ be related in the following way:

(1) $$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} + \tilde{\pi}^{ip} \nabla_j \pi_{pk}$$

where ∇ denotes the covariant differentiation operator with respect to Γ and $\tilde{\pi}$ is the inverse tensor to π.

Now, I'll compute the curvature tensor $\hat{R}$ of the conjugate connection $\hat{\Gamma}$ with the given connection Γ with respect to the tensor π of type (0,2) and will give some relations of this tensor with the curvature tensor R of the given connection Γ. Let R denote the curvature tensor of the given connection Γ on the differentiable manifold M, then:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where X, Y, Z are the vector fields on M.
The vector fields X, Y define at each point $p \in M$ a linear operator, the curvature operator, $R(X_p, Y_p)$ on $T_p(M)$ by the prescription:

$$R(X_p,Y_p)(Z_p) = R(X_p,Y_p)Z_p$$

Each linear operator of vector fields (or tangent vectors at p) may be extended in the unique way to a differentiation of the algebra of vector fields (or tangent vectors at p) [2]. In particular the linear operator $R(X,Y)$ (or $R(X_p,Y_p)$) may be extended to a differentiation of tensor fields. And for any tensor T we have:

$$R(X_p,Y_p)(T_p) = [(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})T\ (p)$$

where X, Y are the vector fields such, that their values at p are X_p and Y_p respectively. The value of this differentiation on the tensor π of type (0,2) is: [2]

$$R(e_j,e_k)(\pi)(e_p,e_l) =$$

$$(2)\quad = (\nabla_{e_j}\nabla_{e_k} - \nabla_{e_k}\nabla_{e_j} - \nabla_{[e_j,e_k]})\pi(e_p,e_l) =$$

$$= \nabla_j\nabla_k\pi_{pl} - \nabla_k\nabla_j\pi_{pl} = - R^m_{jkp}\pi_{ml} - R^m_{jkl}\pi_{pm} .$$

Now we compute the coordinates of the curvature tensor $\hat{R}$ of the conjugate connection $\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} + T^i_{jk}$ where:

$$(3)\qquad T^i_{jk} = \tilde{\pi}^{ip}\nabla_j\pi_{pk}$$

We know that:

$$(4)\qquad \nabla_s T^i_{jk} = \partial_s T^i_{jk} - \Gamma^p_{sj}T^i_{pk} - \Gamma^p_{sk}T^i_{sp} + \Gamma^i_{sp}T^p_{jk}$$

and

$$(5)\qquad 0 = \nabla_p(\tilde{\pi}^{st}\pi_{tu}) = \nabla_p\tilde{\pi}^{st}\pi_{tu} + \tilde{\pi}^{st}\nabla_p\pi_{tu}$$

From (5) we have:

$$(6)\qquad \nabla_p\tilde{\pi}^{sr} = - \tilde{\pi}^{st}\tilde{\pi}^{ur}\nabla_p\pi_{tu}$$

The coordinates of $\hat{R}$ are:

$$\hat{R}^i_{jkl} = \partial_j\hat{\Gamma}^i_{kl} - \partial_k\hat{\Gamma}^i_{jl} + \hat{\Gamma}^i_{jm}\hat{\Gamma}^m_{kl} - \hat{\Gamma}^i_{km}\hat{\Gamma}^m_{jl} =$$

$$= \partial_j\Gamma^i_{kl} - \partial_k\Gamma^i_{jl} + \Gamma^i_{jm}\Gamma^m_{kl} - \Gamma^i_{km}\Gamma^m_{jl} + \partial_j T^i_{kl} -$$

$$- \partial_k T^i_{jl} + \Gamma^i_{jm}T^m_{kl} - \Gamma^i_{km}T^m_{jl} + T^i_{jm}\Gamma^m_{kl} - T^i_{km}\Gamma^m_{jl} +$$

$$+ T^i_{jm}T^m_{kl} - T^i_{km}T^m_{jl} = R^i_{jkl} + \nabla_j T^i_{kl} + \Gamma^m_{jk}T^i_{ml} - \nabla_k T^i_{jl} -$$

$$- \Gamma^m_{kj}T^i_{ml} + T^i_{jm}T^m_{kl} - T^i_{km}T^m_{jl} = R^i_{jkl} + \nabla_j T^i_{kl} - \nabla_k T^i_{kl} -$$

$$+ T^{i}_{jm}T^{m}_{kl} - T^{i}_{km}T^{m}_{jl}$$

Substituting (3) into this relation, we have:

$$\begin{aligned}\hat{R}^{i}_{jkl} = R^{i}_{jkl} &+ \nabla_j \tilde{\pi}^{ip} \nabla_k \pi_{pl} + \tilde{\pi}^{ip} \nabla_j \nabla_k \pi_{pl} - \\ &- \nabla_k \tilde{\pi}^{ip} \nabla_j \pi_{pl} - \tilde{\pi}^{ip} \nabla_k \nabla_j \pi_{pl} + \\ &+ \tilde{\pi}^{ip} \tilde{\pi}^{mr} \nabla_j \pi_{pm} \nabla_k \pi_{rl} - \tilde{\pi}^{ip} \tilde{\pi}^{mr} \nabla_k \pi_{pm} \nabla_j \pi_{rl}\end{aligned}$$

Substituting (6) into above, we get:

$$\begin{aligned}\hat{R}^{i}_{jkl} &= R^{i}_{jkl} + \tilde{\pi}^{ip}(\nabla_j \nabla_k \pi_{pl} - \nabla_k \nabla_j \pi_{pl}) - \\ &- \tilde{\pi}^{im} \tilde{\pi}^{rp} \nabla_j \pi_{mr} \nabla_k \pi_{pl} + \tilde{\pi}^{im} \tilde{\pi}^{rp} \nabla_k \pi_{mr} \nabla_j \pi_{pl} + \\ &+ \tilde{\pi}^{ip} \tilde{\pi}^{mr}(\nabla_j \pi_{pm} \nabla_k \pi_{rl} - \nabla_k \pi_{pm} \nabla_j \pi_{rl}) = \\ &= R^{i}_{jkl} + \tilde{\pi}^{ip}(\nabla_j \nabla_k \pi_{pl} - \nabla_k \nabla_j \pi_{pl}) - \\ &- \tilde{\pi}^{im} \tilde{\pi}^{rp}(\nabla_j \pi_{mr} \nabla_k \pi_{pl} - \nabla_k \pi_{mr} \nabla_j \pi_{pl}) + \\ &+ \tilde{\pi}^{ip} \tilde{\pi}^{mr}(\nabla_j \pi_{pm} \nabla_k \pi_{rl} - \nabla_k \pi_{pm} \nabla_j \pi_{rl})\end{aligned}$$

Interchanging indices $m \to p \to r \to m$ we obtain:

$$\hat{R}^{i}_{jkl} = R^{i}_{jkl} + \tilde{\pi}^{ip}(\nabla_j \nabla_k \pi_{pl} - \nabla_k \nabla_j \pi_{pl})$$

Finally, having used (2), we get:

$$\begin{aligned}\hat{R}^{i}_{jkl} &= R^{i}_{jkl} - \tilde{\pi}^{ip}(R^{m}_{jkp} \pi_{ml} + R^{m}_{jkl} \pi_{pm}) = R^{i}_{jkl} - R^{i}_{jkl} - \\ &- \tilde{\pi}^{ip} \pi_{ml} R^{m}_{jkp} = - \tilde{\pi}^{ip} \pi_{ml} R^{m}_{jkp}\end{aligned}$$

We have the following:

THEOREM 2. The curvature tensor $\hat{R}$ of the conjugate connection $\hat{\Gamma}$ is related with the curvature tensor R of the given connection in the following way:

(7) $$\hat{R}^i_{jkl} = - \tilde{\pi}^{ip}\pi_{ml}R^m_{jkp}$$

REMARK. Non-singular, symmetric tensor π of type (0,2) on a manifold M determines two tensors of type (2,2), so called Obata's operators:

$$\Omega = \frac{1}{2}(I\otimes I - \tilde{\pi}\otimes\pi) \qquad \Omega^{sh}_{ir} = \frac{1}{2}(\delta^s_i\delta^h_r - \tilde{\pi}^{sh}\pi_{ir})$$

$$^*\Omega = \frac{1}{2}(I\otimes I + \tilde{\pi}\otimes\pi) \qquad ^*\Omega^{sh}_{ir} = \frac{1}{2}(\delta^s_i\delta^h_r + \tilde{\pi}^{sh}\pi_{ir})$$

with the properties:

$$\Omega + {}^*\Omega = I\otimes I; \quad \Omega^2 = \Omega, \quad {}^*\Omega^2 = {}^*\Omega, \quad {}^*\Omega\,\Omega = \Omega\,{}^*\Omega = 0$$

One may regard these operators as linear transformations of tensors of type (p,q) where $p,q \geqslant 1$ and for the tensor R of type (1,3) we have:

$$\Omega^{sh}_{ir}R^i_{jkh} = R^s_{jkr} - \pi_{ir}\tilde{\pi}^{sh}R^i_{jkh}$$

Now, the Theorem 2 can be written in the form:

(8) $$\hat{R} = R - 2\,{}^*\Omega R$$

Furthermore, it is easy to see that:

1. The tensor $\hat{R} - R$ belongs to the kernel of Ω
2. The tensor $\hat{R} + R$ belongs to the kernel of ${}^*\Omega$

Now, we will give some properties of the conjugate connections taking into considerations the curvature tensor. To this end we recall some definitions.

DEFINITION 2 [4]. A connection Γ is said to be flat on a manifold M if the curvature tensor R vanishes.

A curvature tensor R determines two following tensors: The Ricci tensor $R_{kl} := R^{i}_{ikl}$ and $V_{jk} := R^{i}_{jki}$.

DEFINITION 3 [4]. A connection Γ is said to preserve a volume on a manifold M if the tensor V_{ij} vanishes.

DEFINITION 4 [4]. The curvature tensor R of a connection Γ on M is said to be recurrent of the first order if there exists a covector p_k such, that:

(9) $$\nabla_r R^{i}_{jkl} = p_r R^{i}_{jkl}$$

Now, we can state:

THEOREM 3. <u>The conjugate connection</u> $\hat{\Gamma}$ <u>is flat iff the connection</u> Γ <u>is flat</u>.

P r o o f. Let $R^{i}_{jkl} = 0$, then, from (7) $\hat{R}^{i}_{jkl} = 0$. Conversely, if $\hat{R}^{i}_{jkl} = 0$, then $\tilde{\pi}^{ip}\pi_{ml}R^{m}_{jkp} = 0$ or $R^{m}_{jkl} = 0$.

Q.E.D.

THEOREM 4.

1. $\hat{V}_{jk} = -V_{jk}$
2. <u>The Ricci tensor of the conjugate connection is</u>:

$$\hat{R}_{kl} = -\tilde{\pi}^{ip}\pi_{ml}R^{m}_{ikp}$$

COROLLARY. The conjugate connection $\hat{\Gamma}$ preserves a volume on M if and only if the connection Γ does.

Now, we'll compute a covariant derivative of the curvature tensor $\hat{R}$ of the conjugate connection $\hat{\Gamma}$ with respect to the conjugate connection $\hat{\Gamma}$ i.e. $\hat{\nabla}\hat{R}$. To this end we need some calculations. First, we'll find $\hat{\nabla}_s \pi_{tu}$

$$\hat{\nabla}_s \pi_{tu} = \partial_s \pi_{tu} - \hat{\Gamma}^r_{st}\pi_{ru} - \hat{\Gamma}^r_{su}\pi_{tr} = \nabla_s \pi_{tu} -$$

$$- \tilde{\pi}^{rq}\pi_{ru}\nabla_s \pi_{qt} - \tilde{\pi}^{rq}\pi_{tr}\nabla_s \pi_{qu} =$$

$$= \nabla_s \pi_{tu} - \nabla_s \pi_{ut} - \nabla_s \pi_{tu} = - \nabla_s \pi_{ut}$$

(10) $$\hat{\nabla}_s \pi_{tu} = - \nabla_s \pi_{ut}$$

Now, we'll get $\hat{\nabla}_s R^m_{jkp}$

(11) $$\hat{\nabla}_s R^m_{jkp} = \nabla_s R^m_{jkp} - \tilde{\pi}^{rq}\nabla_s \pi_{qj}R^m_{rkp} - $$

$$- \tilde{\pi}^{rq}\nabla_s \pi_{qk}R^m_{jrp} - \tilde{\pi}^{rq}\nabla_s \pi_{qp}R^m_{jkr} +$$

$$+ \tilde{\pi}^{mn}\nabla_s \pi_{nq}R^q_{jkp}$$

Now, we can write:

$$\hat{\nabla}_s R^i_{jkl} = - \hat{\nabla}_s \tilde{\pi}^{ip}\pi_{ml}R^m_{jkp} - \tilde{\pi}^{ip}\hat{\nabla}_s \pi_{ml}R^m_{jkp} -$$

$$- \tilde{\pi}^{ip}\pi_{ml}\hat{\nabla}_s R^m_{jkp}$$

Having used (6), (10) and (11), we have:

(12) $$\hat{\nabla}_s \hat{R}^i_{jkl} = - \tilde{\pi}^{ip}\pi_{ml}\nabla_s R^m_{jkp} +$$

$$+ \tilde{\pi}^{ip}\tilde{\pi}^{rq}\pi_{mi}(\nabla_s \pi_{qj}R^m_{rkp} + \nabla_s \pi_{qk}R^m_{jrp})$$

We have the following:

THEOREM 5. Suppose, that on a manifold M with a linear connection Γ there is given a non-singular tensor π of type (0,2) satisfying:

(13) $$\nabla_k \pi_{ij} = p_k \pi_{ij}$$

where p_k is any covector on M, then the curvature tensor $\hat{R}$ of the conjugate connection $\hat{\Gamma}$ is recurrent of the first order if and only if the tensor R of the given connection Γ is recurrent of the first order.

P r o o f. Suppose, that the curvature tensor R of the connection Γ is recurrent of the first order i.e. it satisfies (9).

Then, from (12) and (13):

$$\hat{\nabla}_s \hat{R}^i_{jkl} = - \tilde{\pi}^{ip} \pi_{ml} \nabla_s R^m_{jkp} +$$

$$+ \tilde{\pi}^{ip} \tilde{\pi}^{rq} \pi_{ml} (\nabla_s \pi_{qj} R^m_{rkp} + \nabla_s \pi_{qk} R^m_{jrp}) =$$

$$= - \tilde{\pi}^{ip} \pi_{ml} q_s R^m_{jkp} + \tilde{\pi}^{ip} \tilde{\pi}^{rt} \pi_{ml} p_s \pi_{tj} R^m_{rkp} +$$

$$+ \tilde{\pi}^{ip} \tilde{\pi}^{rt} \pi_{ml} p_s \pi_{tk} R^m_{jrp} = - q_s \tilde{\pi}^{ip} \pi_{ml} R^m_{jkp} +$$

$$+ p_s \tilde{\pi}^{ip} \pi_{ml} R^m_{jkp} + p_s \tilde{\pi}^{ip} \pi_{ml} R^m_{jkp} =$$

$$= - (- 2p_s + q_s) \tilde{\pi}^{ip} \pi_{ml} R^m_{jkp} = \rho_s R^i_{jkl}$$

$$\rho_s := q_s - 2p_s$$

what means that the curvature tensor $\hat{R}$ of the conjugate connection $\hat{\Gamma}$ is recurrent of the first order as well.

Now, conversely, let $\hat{\nabla}_s \hat{R}^i_{jkl} = r_s \hat{R}^i_{jkl}$.

Then, using (7) and (12) we have:

$$- r_s \tilde{\pi}^{ip} \pi_{ml} R^m_{jkp} = - \tilde{\pi}^{ip} \pi_{ml} \nabla_s R^m_{jkp} + 2p_s \tilde{\pi}^{ip} \pi_{ml} R^m_{jkp}$$

Hence

$$\nabla_s R^i_{jkl} = q_s R^i_{jkl}$$

where

$$q_s = 2p_s + r_s$$

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STRESZCZENIE

W pracy tej zajmujemy się tensorami krzywiznowymi koneksji liniowych sprzężonych według definicji Nordena i Wedernikowa. W twierdzeniu 2 obliczony jest tensor krzywiznowy $\hat{R}$ koneksji $\hat{\Gamma}$ sprzężonej z daną koneksją liniową Γ względem danego tensora π typu (0,2) i wyrażony jest za pomocą tensora krzywiznowego R koneksji danej Γ oraz tensora π następująco: $\hat{R} = R - 2{}^*\Omega R$ gdzie ${}^*\Omega$ jest operatorem Obaty.

Następnie podane są warunki konieczne i dostateczne na to, aby

koneksje sprzężone Γ i $\hat{\Gamma}$ były płaskie, zachowując objętość i rekurentne.

Резюме

В данной работе занимаемся тензорами кривизны сопряженных линейных связностей определенных Норденом и Ведерниковым. В теореме 2 вычислено тензор кривизны $\hat{R}$ связности $\hat{\Gamma}$ сопряженной с данной связностей Г относительно тензора П типа 0,2 и выражено его при помощи тензора кривизны Р связности Г и тензора П следующим образом: $\hat{R} = R - 2\Omega^{*}R$, где Ω^{*} -оператор Обаты. Кроме того представлено необходимые и достаточные условия для того, чтобы сопряженные связности Γ и $\hat{\Gamma}$ были плоские, рекурентные и сохраняющие объем.