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# Coefficient Estimates for Powers of Univalent Functions and Their Inverses

**Oszacowania współczynników potęg funkcji jednolistnych i ich funkcji odwrotnych**

**Оценки коэффициентов степени однолистных функций и их обратных функций**

**0. Introduction. Notations.** Let $S$ be the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots = z + \sum_{k=2}^{\infty} a_k(f)\, z^k \tag{0.1}$$

regular and univalent in the unit disk $D$. The inverse $F$ of $f$ is a function regular and univalent in $|w| < 0.25$ which has the form

$$F(w) = w + A_2 w^2 + A_3 w^3 + \ldots = w + \sum_{k=2}^{\infty} A_k(F)\, w^k \, . \tag{0.2}$$

In what follows we denote this class of inverses by $S^{-1}$.

We introduce the matrices $(b_k^{(n)})$, $(A_k^{(n)})$, $\pm n$, $k \in \mathbf{N}$ ( = the set of all positive integers), defined as follows

$$\left[\frac{z}{f(z)}\right]^n = 1 + b_1^{(n)} z + b_2^{(n)} z^2 + \ldots = 1 + \sum_{k=1}^{\infty} b_k^{(n)}(f)\, z^k \, , \tag{0.3}$$

$$\left[\frac{F(w)}{w}\right]^n = 1 + A_1^{(n)} w + A_2^{(n)} w^2 + \ldots = 1 + \sum_{k=1}^{\infty} A_k^{(n)}(F)\, w^k \, . \tag{0.4}$$

Let $\Sigma$ be the associated class of functions $g$ univalent in the outside $D^*$ of the unit disk $D$, i.e. $g(\xi) = (f(z))^{-1}$, $\xi = z^{-1}$ and

$$g(\xi) = \xi + b_0 + b_1 \xi^{-1} + \ldots = \xi + \sum_{k=0}^{\infty} b_k(g)\, \xi^{-k} \, , \tag{0.5}$$

If $\Sigma^{-1}$ is the corresponding class of inverses $G = g^{-1}$, then obviously

$$G(u) = u + B_0 + B_1 u^{-1} + \ldots = u + \sum_{k=0}^{\infty} B_k(G)\, u^{-k} \tag{0.6}$$

and

$$\left[\frac{z}{f(z)}\right]^n = \left[\frac{g(\xi)}{\xi}\right]^n, \quad \left[\frac{F(w)}{w}\right]^n = \left[\frac{u}{G(u)}\right]^n, \quad u = w^{-1}. \tag{0.7}$$

This implies that the relevant matrices for the class $\Sigma$ analogous to (0.3) and (0.4) are essentially the same as those for the class $S$. It follows from (0.2) and (0.4) that

$$A_n(F) = A^{(1)}_{n-1}(F), \tag{0.8}$$

whereas from (0.4), (0.6) and (0.7) we obtain

$$B_n(G) = A^{(-1)}_{n+1}(F). \tag{0.9}$$

There is a simple relation (1.1) between $b_k^{(n)}$ and $A_k^{(n)}$. In Sect. 1 we shall obtain estimates for $A_k^{(n)}$ and $b_k^{(n)}$ originating in Baernstein inequality. Due to the relation (0.8) we obtain from our estimates a simple proof of Loewner's inequality:

$$|A_n| \leq \frac{1}{n+1}\binom{2n}{n} \tag{0.10}$$

independent of Loewner's theory. An analogous estimate

$$|B_n| \leq \frac{1}{n+1}\binom{2n}{n} \tag{0.11}$$

first obtained by Netanyahu [7] with variational methods implies an estimate of $A_k^{(-1)}$. The problem: To determine sharp bounds for the coefficients $b_k^{(n)}, A_k^{(n)}$, is much more general than the familiar problems of estimates for $a_n$ and $b_n$. In fact, $b_k^{(1)}$ are essentially the coefficients of $g \in \Sigma$, whereas $b_k^{(-1)}$ are coefficients of $f \in S$.

It is easily seen that the Koebe function $\tilde{f}$:

$$\tilde{f}(z) = z(1+z)^{-2}, \tag{0.12}$$

resp. its inverse

$$F(w) = (2w)^{-1}(1 - 2w - \sqrt{1-4w}) = \sum_{n=1}^{\infty} \frac{1}{n+1}\binom{2n}{n} w^n \tag{0.13}$$

are not always extremal for $b_k^{(n)}$, or $A_k^{(n)}$. In fact, $[z \cdot (\tilde{f}(z))^{-1}]^n = (1+z)^{2n}$, $n \in \mathbb{N}$, is a polynomial and consequently $b_k^{(n)}(\tilde{f}) = 0$ for $k > 2n$ which obviously is not a maximal value. Nevertheless, $\tilde{f}$ shows to be maximal for $b_k^{(n)}$ with $1 \leqslant k \leqslant n+1$, cf. Theorem 3.

In what follows we use a tilde to indicate objects (functions, coefficients, sets) associated with the Koebe function (0.12).

The notation $G(z) \ll H(z)$ means as usual that all the Taylor (or Laurent) coefficients of $H$ at 0, or $\infty$, are non-negative and majorize corresponding coefficients of $G$ in absolute value.

This paper is a continuation of [4], where the relation between $A_n$, $B_n$ and $b_k^{(n)}$ was established and some applications have been given. The author is much indebted to Professors R. J. Libera and E. Złotkiewicz for helpful discussions and criticism.

The results of this paper were presented and the Conference on Complex Analysis in Halle, GDR, in September, 1980, cf. [3].

1. Estimates of $A_k^{(n)}$ and $b_k^{(n)}$. There is a simple relation between $A_k^{(n)}$ and $b_k^{(n)}$. If $C_r = \{z \in \mathbf{C} : |z| = r\}$, $0 < r < 0.25$ and $n \neq k$ then from (0.3) we obtain on integrating by parts

$$b_k^{(n)}(f) = (2\pi i)^{-1} \int_{C_r} [z \cdot (f(z))^{-1}]^n z^{-k-1}\, dz =$$

$$= (2\pi i)^{-1} \int_{C_r} w^{-n} [F(w)]^{n-k-1} F'(w)\, dw =$$

$$= [n \cdot (n-k)^{-1}] \cdot (2\pi i)^{-1} \int_{C_r} [F(w) \cdot w^{-1}]^{n-k} w^{-k-1}\, dw, \text{ i.e.}$$

$$b_k^{(n)}(f) = \frac{n}{n-k} A_k^{(n-k)}(F). \tag{1.1}$$

From (0.8) and (1.1) we obtain with $k = n-1$

$$A_n = A_{n-1}^{(1)} = n^{-1} \cdot b_{n-1}^{(n)} = (2\pi i n)^{-1} \int_{C_r} [f(z)]^{-n}\, dz, \tag{1.2}$$

whereas (0.9) and (1.1) yield for $k = n+1$.

$$B_n = A_{n+1}^{(-1)} = -n^{-1} b_{n+1}^{(n)} = -(2\pi i n)^{-1} \int_{C_r} [f(z)]^{-n} z^{-2}\, dz. \tag{1.3}$$

or

$$B_n = -(2\pi i n)^{-1} \int_{C_R} [g(\xi)]^n\, d\xi,\ R > 1. \tag{1.4}$$

The formulas (1.2) and (1.4) are particular cases of the Teixeira's formulas for coefficients in the expansion of $\psi(z)$ into a series of powers of $\theta(z)$, cf. § 7.3 in [8]. E.g.

(1.3) corresponds to the case $\psi(z) \equiv z$, $\theta(z) \equiv f(z)$. They are very convenient in evaluating the coefficients of inverse functions, cf. e.g. [4].

As a matter of example consider the class $\Sigma'$ of functions $g \in \Sigma$ with hydrodynamical normalization:

$$g(\xi) = \xi + b_1/\xi + b_2/\xi_2 + \dots , \; |\xi| > 1 .$$

Then we have: $|g(\xi)| \leqslant |\xi| + |\xi|^{-1}$, cf. [5], and using this and (1.4) we obtain at once $|B_n| \leqslant 2^n/n$.

We now consider the case $k = n$. In view of the identity

$$\frac{d}{dw}\left[w^{-n} \log \frac{F(w)}{w}\right] = w^{-n} \frac{F'(w)}{F(w)} - w^{-n-1} - n w^{-n-1} \log \frac{F(w)}{w}$$

we obtain on integration

$$b_n^{(n)}(f) = (2\pi i)^{-1} \int_{C_r} [f(z)]^{-n} \frac{dz}{z} = (2\pi i)^{-1} \int_{C_r} w^{-n} \frac{F'(w)}{F(w)} \, dw =$$

$$= \frac{n}{2\pi i} \int_{C_r} w^{-n-1} \log \frac{F(w)}{w} \, dw .$$

Thus, putting

$$\log \frac{F(w)}{w} = \gamma_1 w + \gamma_2 w^2 + \dots = \sum_{k=1}^{\infty} \gamma_k(F)\, w^k \tag{1.5}$$

we obtain

$$b_n^{(n)}(f) = n\gamma_n(F) . \tag{1.6}$$

We now prove

**Lemma 1.** *We have for fixed $n \in \mathbb{N}$ and $k$, $f$ ranging over $\mathbb{N}$ and $S$, resp., the following sharp estimate*

$$\left|b_k^{(n)}(f)\right| \leqslant \binom{2n}{n} = b_n^{(n)}(\tilde{f}) . \tag{1.7}$$

**Proof.** It follows from (0.3) that

$$|b_k^{(n)}(f)| = (2\pi)^{-1} \left| \int_{C_r} \left[\frac{z}{f(z)}\right]^n z^{-k-1} \, dz \right| \leqslant \frac{r^{n-k}}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{-n} \, d\theta .$$

As shown by Baernstein [1], the last expression is maximal for $f = \tilde{f}$. Using this and making $r \to 1$, we obtain

$$|b_k^{(n)}(f)| \leqslant (2\pi)^{-1}\int_0^{2\pi} |\widetilde{f}(e^{i\theta})|^{-n} d\theta = \frac{2^{2n}}{2\pi}\int_0^{2\pi} |\cos\theta/2|^{2n}\, d\theta = \binom{2n}{n}$$

and this proves (1.7). As an immediate consequence we obtain

**Theorem 1.** *We have for any* $F \in S^{-1}$

$$\log\frac{F(w)}{w} \ll \log\frac{\widetilde{F}(w)}{w} = \sum_{n=1}^{\infty} n^{-1}\binom{2n}{n} w^n . \tag{1.8}$$

**Proof.** It follows from (1.6) and (1.7) that

$$\max_{F\in S^{-1}} |\gamma_n(F)| = n^{-1} \max_{f\in S} |b_n^{(n)}(f)| = n^{-1}\binom{2n}{n} = n^{-1}\, b_n^{(n)}(\widetilde{f}) = \gamma_n(\widetilde{F})$$

and this proves (1.8).

**Corollary 1.** *We obtain from* (1.8) *after multiplication by* $\alpha$ *and a subsequent exponentiation*

$$\left[\frac{F(w)}{w}\right]^{\alpha} \ll \left[\frac{\widetilde{F}(w)}{w}\right]^{\alpha} \tag{1.9}$$

*for all positive* $\alpha$. *In particular*

$$|A_k^{(n)}(F)| \leqslant A_k^{(n)}(\widetilde{F}) = \widetilde{A}_k^{(n)} ;\ n, k \in \mathrm{N} . \tag{1.10}$$

If $k \leqslant n$, then by (1.1), (1.7) and (1.10) the coefficients $b_k^{(n)}(f)$ are maximal in absolute value for $f = \widetilde{f}$. This implies an improvement of (1.7) for $k \leqslant n$:

**Corollary 2.** *We have for fixed* $n, k \in \mathrm{N}$, $1 \leqslant k \leqslant n$, *and* $f$ *ranging over* $S$ *a sharp estimate*

$$|b_k^{(n)}(f)| \leqslant b_k^{(n)}(\widetilde{f}) = \binom{2n}{k} . \tag{1.11}$$

We show later (Theorem 3) that this improved estimate also holds for $k = n + 1$. From (1.2) and (1.11) with $n = k - 1$ we obtain

**Corollary 3,** [6]. *We have a sharp estimate*

$$|A_n(F)| \leqslant n^{-1}\binom{2n}{n-1} = (n+1)^{-1}\binom{2n}{n} = A_n(\widetilde{F}) = \widetilde{A}_n . \tag{1.12}$$

An inductive proof of inequality (1.12), also based Baerstein's inequality, was given in [2], as pointed out to the author by R. J. Libera.

Evidently $b_k^{(\alpha)}, A_k^{(\alpha)}$ make sense for all real $\alpha$ and the equality (1.1) the remains true. Hence, in view of (1.9), we obtain

$$|b_k^{(\alpha)}(f)| \leqslant b_k^{(\alpha)}(\widetilde{f}) = \binom{2\alpha}{k}, \alpha \geqslant k, \tag{1.13}$$

where $b_k^{(\alpha)}(f)$ are defined by (0.3) with $n = \alpha$.

**Corollary 4**, [6]. *Putting $\alpha = 0.5$ in (1.9) we obtain*

$$F_1(w) \ll [\widetilde{F}(w^2)]^{0.5} = w + \sum_{n=2}^{\infty} n^{-1} \binom{2n-2}{n-1} w^{2n-1} \tag{1.14}$$

*for any $F_1 = f_1^{-1}$, where $f_1 \in S$ is an odd univalent function. An analogous formula for $f$ with $k$-fold symmetry may be obtained similarly.*

The following problem arises in connection with (1.13): For a given positive $\alpha$ find the maximal integer $m(\alpha)$ such that (1.13) holds for any $k \leqslant m(\alpha)$ and any $f \in S$.

**2. The estimate of $A_k^{(-1)}$ and its consequences.** If $g \in \Sigma$ and $f \in S$ is its associated function, then, by (0.7) with $n = 1$, we have $G(u) = g^{-1}(u) = 1/F(u^{-1})$, where $F = f^{-1}$. This means that the coefficients of $G$ and $1/F$ coincide. Thus we may consider

$$H(w) = 1/F(w) = w^{-1} - B_0 - B_1 w - B_2 w^2 - \dots, F \in S^{-1}, \tag{2.1}$$

instead of $G \in \Sigma^{-1}$. The reason for changing the signs of $B_n$ is evident because of formula (2.3)

From (0.2) and (2.1) we obtain

$$A_{n+1} = B_0 A_n + B_1 A_{n-1} + B_2 A_{n-2} + \dots + B_{n-1} A_1, A_1 = 1. \tag{2.2}$$

On the other hand, for $F = \widetilde{F}$ and $\widetilde{H} = 1/\widetilde{F}$ we have

$$\widetilde{H}(w) = w^{-1} - 2 - w - \widetilde{A}_2 w^2 - \widetilde{A}_3 w^3 - \dots \tag{2.3}$$

where $\widetilde{A}_n$ are defined in (1.12). Consequently

$$\widetilde{A}_{n+1} = 2\widetilde{A}_n + \widetilde{A}_1 \widetilde{A}_{n-1} + \widetilde{A}_2 \widetilde{A}_{n-2} + \dots + \widetilde{A}_{n-1} \widetilde{A}_1, \widetilde{A}_1 = 1. \tag{2.4}$$

If we take the Netanyahu estimate (0.11) for granted, then by (2.2), (2.4) and an obvious induction we readily obtain Loewner's estimate. On the other hand, it follows from (1.7) that $\log w/F(w) \ll \log \widetilde{F}(w)/w$ and by exponentiation we obtain $w/F(w) \ll \widetilde{F}(w)/w$. This is equivalent to the inequality

$$|B_n| \leqslant \widetilde{A}_{n+2} \tag{2.5}$$

which is not sharp. Therefore it appears plausible that a straight-forward derivation of bounds for $|B_n|$ while taking (0.10) for granted leads to the estimate (2.5) only.

In view of (0.9) and (0.11), we have

$$|B_n(G)| = |A_{n+1}^{(-1)}(\widetilde{F})| \leq (n+1)^{-1}\binom{2n}{n} = A_{n-1}^{(1)}(\widetilde{F}) = \widetilde{A}_n .$$

Hence, we obtain sharp estimate

$$|A_{n+2}^{(-1)}(F)| \leq A_n^{(1)}(\widetilde{F}) \tag{2.6}$$

which may be restated as

Theorem 2. *If $F \in S^{-1}$, then*

$$w^{-1} - [F(w)]^{-1} + a_2 \ll \widetilde{F}(w) . \tag{2.7}$$

On the other hand, we obtain from (1.3) and (2.6) $|B_n| = n^{-1}|b_{n+1}^{(n)}(f)| \leq$ $\leq (n+1)^{-1}\binom{2n}{n} = n^{-1}\binom{2n}{n+1}$, i.e.

$$|b_{n+1}^{(n)}(f)| \leq \binom{2n}{n+1} = b_{n+1}^{(n)}(\widetilde{f}) . \tag{2.8}$$

This leads to the following extension of Corollary 2:

Theorem 3. *We have for fixed n, $k \in \mathbf{N}$, $1 \leq k \leq n+1$, and f ranging over S, a sharp estimate*

$$|b_k^{(n)}(f)| \leq b_k^{(n)}(\widetilde{f}) = \binom{2n}{k} . \tag{2.9}$$

Again we can state a similar problem as before: For a given $n \in \mathbf{N}$ find the maximal integer $m_n$ such that (2.9) holds for any $f \in S$ and any $k \leq m_n$. Obviously $n+1 \leq m_n \leq$ $\leq 2n$. It is easily verified that for $g_0(\xi) = \xi(1+\xi^{-3})^{2/3} - 2^{2/3}$ and an associated $f_0 \in S$ we have $|b_4^{(2)}(f_0)| = (4/3)\cdot 2^{2/3} > 1 = b_4^{(2)}(\widetilde{f})$ so that $m_2 = 3$.

The fact that the coefficients of both $F \in S^{-1}$ and $1/F = H$ are maximal for $F = \widetilde{F}$ has an interesting consequence.

Let us consider the function

$$q(w) = H(w) - F(w) = w^{-1} - B_0 - (A_1 + B_1)w - (A_2 + B_2)w^2 - \ldots$$

which maps conformally the domain $f(\mathrm{D})$ onto $\hat{\mathbf{C}} \setminus [-2i; 2i]$. Obviously the coefficients of $q$ are maximal in absolute value for $\widetilde{q} = \widetilde{H} - \widetilde{F}$. Conversely, any function $q$ with $q(0) = \infty$ mapping conformally a domain $f(\mathrm{D})$ ($f \in S$) onto the outside of a segment of length 4 bisected by the origin is a rotation of a function of the form $H - F$. Hence, we may assume without loss on generality that the segment coincides with the imaginary axis. On replacing $w$ by $u^{-1}$ we may express these considerations in the form of

Theorem 4. *Let $\Omega$ be a simply connected domain in the extended plane $\hat{\mathbf{C}}$ whose complementary set $\Gamma$ is a continuum which contains the origin and has the transfinite diameter $d(\Gamma) = 1$. For any conformal mapping p of $\Omega$ onto $\hat{\mathbf{C}} \setminus [-2i; 2i]$ of the form $p(u) = u + \alpha_0 + \alpha_1 u^{-1} + \alpha_2 u^{-2} + \ldots$ we have sharp estimates*

$$|\alpha_0| \leq 2\,;\, |\alpha_n| \leq 2\tilde{A}_n = 2\cdot(n+1)^{-1}\binom{2n}{n},\ n = 1, 2, \ldots$$

The equality is attained in each case by

$$\tilde{p}(u) = -\tilde{F}(u^{-1}) + [\tilde{F}(u^{-1})]^{-1} = \sqrt{u(u-4)}\ ,$$

and the extremal domain $\tilde{\Omega} = \hat{\mathbb{C}} \setminus [0\,;4]$.

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## STRESZCZENIE

Niech $S$ będzie klasą unormowanych w zwykły sposób funkcji $f$ jednolistnych w kole jednostkowym i niech $b_k^{(n)}(f)$ będzie $k$-tym współczynnikiem taylorowskim funkcji $[z/f(z)]^n$ w punkcie $z = 0$. Otrzymujemy dokładne oszacowania:

a) $|b_k^{(n)}(f)| \leq \binom{2n}{n}$ ; $n, k \in \mathbb{N}, f \in S$;

b) $|b_k^{(n)}(f)| \leq \binom{2n}{k}$ ; $n \in \mathbb{N}, k = 1, 2, \ldots, n+1, f \in S$.

Ostatnia nierówność przy $k = n-1$, wzgl. $k = n+1$ jest równoważna z oszacowaniem $n$-tego współczynnika dla funkcji odwrotnej $f^{-1}$, wzgl. $g^{-1}$, gdzie $g(z^{-1}) = 1/f(z)$.

## РЕЗЮМЕ

Пусть $S$ обозначает класс нормированных однолистных в единичном круге функций и пусть $b_k^{(n)}(f)$. Будет $k$-тый коэффициент функции $[z/f(z)]^n$ в начале координат. Полученные точные оценки:

а) $|b_k^{(n)}(f)| \leq \binom{2n}{n}$; $n, k \in \mathrm{N}, f \in S$;

б) $|b_k^{(n)}(f)| \leq \binom{2n}{k}$; $n \in \mathrm{N}, k = 1, 2, \ldots, n+1, f \in S$.

Последнее неравенство для $k = n - 1$, или $k = n + 1$ эквивалентно оценком $n$-того коэффициента для обратных функций $f^{-1}$ или $g^{-1}$, где $g(z^{-1}) = 1/f(z)$.