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A Pseudo-group of Motions of a Certain Pseudo-Riemannian Space

O pseudogrupie ruchów pewnej przestrzeni pseudoriemannowskiej

Псевдогруппа изометрии некоторого псевдориманского пространства

In this paper we are going to obtain the pseudogroup of motions of pseudo-Riemannian manifold $R_+^3 := \{(x^1, x^2, x^3) \mid x^3 > 0\}$ with the fundamental pseudometric form

$$ds\Big|_{(x^1, x^2, x^3)} = \frac{1}{(x^3)^2}\left[(dx^3)^2 - (dx^2)^2 - (dx^1)^2\right].$$

There are two different manners for finding this pseudogroup. The following theorem [1] yields us the first way:

Given a Riemannian manifold M with a finite number of connected components, then the group G of isometries of M is a Lie transformation group with respect to the compact-open topology in M. Thus G is a Lie group and the Lie algebra of G is naturally isomorphic with the Lie algebra of all complete Killing vector fields.

We shall find the Killing vector fields and their Lie algebra. This algebra is isomophic with algebra of group G which acts on the manifold R as a pseudogroup of transformations. Group G is generated by one-parameter groups which corresponds to Killing vector fields.

In the second part of this paper we shall describe a model of manifold under consideration. We shall base on the ring of antiquaternions [2], [4]. This manner gives us a precise action of unimodular group as a pseudogroup of transformations.

Finally we shall show isomorphism of these spaces. Let us recall the definition of a pseudogroup of transformations [1];

A pseudogroup of transformations on a topological space S is a set Γ of transformations satisfying the following axioms:

(i) Each $f \in \Gamma$ is a homeomorphism of an open set (called the domain of f) of S onto another open set (called the range of f) of S,

(ii) If $f \in \Gamma$ then the restriction of f to an arbitrary open subset of the domain of f is in Γ,

(iii) Let $U = \bigcup_i U_i$ where each U_i is an open set of S. A homeomorphism f of U onto an open set of S belongs to Γ if the restriction of f to U_i is in Γ for every i,

(iv) For every open set U of S, the identity transformation of U is in Γ,

(v) If $f \in \Gamma$, then $f^{-1} \in \Gamma$,

(vi) If $f \in \Gamma$ is a homeomorphism of U onto V and $f' \in \Gamma$ is a homeomorphism of U' onto V' and if $V \cap U'$ is non-empty, then the homeomorphism $f' \cdot f$ of $f^{-1}(V \cap U')$ onto $f'(V \cap U')$ is in Γ.

The Killing vector fields. Let $\mathcal{L}$ denote the Lie derivative. A non-zero vector field F on M such that $\mathcal{L}_F g = 0$ is called a Killing vector field or an infinitesimal motion. Let $x = (x^1, x^2, x^3)$ be the canonical coordinates on R^3_+, and let $F^i (i = 1, 2, 3)$ be the components of a vector field F with respect to x. F is a Killing vector field if and only if the components F^i of F satisfy the following equations:

$$\frac{\partial F^1}{\partial x^1} - \frac{1}{x^3} F^3 = 0, \qquad \frac{\partial F^1}{\partial x^2} + \frac{\partial F^2}{\partial x^1} = 0,$$

$$\frac{\partial F^2}{\partial x^2} - \frac{1}{x^3} F^3 = 0, \qquad \frac{\partial F^1}{\partial x^3} - \frac{\partial F^3}{\partial x^1} = 0, \tag{1}$$

$$\frac{\partial F^3}{\partial x^3} - \frac{1}{x^3} F^3 = 0, \qquad \frac{\partial F^2}{\partial x^3} - \frac{\partial F^3}{\partial x^2} = 0.$$

Theorem 1. *The following vector fields:*

$$F_1(x^1, x^2, x^3) = \frac{1}{2}((x^1)^2 - (x^2)^2 + (x^3)^2)\frac{\partial}{\partial x^1} + x^1 x^2 \frac{\partial}{\partial x^2} + x^1 x^3 \frac{\partial}{\partial x^3},$$

$$F_2(x^1, x^2, x^3) = x^1 x^2 \frac{\partial}{\partial x^1} + \frac{1}{2}(-(x^1)^2 + (x^2)^2 + (x^3)^2)\frac{\partial}{\partial x^2} + x^2 x^3 \frac{\partial}{\partial x^3},$$

$$F_3(x^1, x^2, x^3) = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3},$$

$$F_4(x^1, x^2, x^3) = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}, \qquad F_5(x^1, x^2, x^3) = \frac{\partial}{\partial x^1},$$

$$F_6(x^1, x^2, x^3) = \frac{\partial}{\partial x^2},$$

satisfy the system (1) *and they are linearly independent in the space of all vector fields. Each Killing vector field has the form* $H = \sum_i \alpha^i F_i$ $(i = 1, .., 6)$.

A proof may be obtained by a straightforward verification. Let us denote by [,] the canonical Lie bracket of vector fields.

Proposition 2. $\{\text{lin}\ F_1, F_2, F_3, F_4, F_5, F_6, [\ ,\ ]\}$ *is a 6 domensional Lie algebra.*

Proof. Considering that the form [,] is bilinear, we can compute the structure constants only. They are equal $C^4_{16} = C^1_{24} = C^6_{45} = 1, C^4_{13} = C^2_{23} = C^2_{14} = C^3_{15} = C^4_{25} = = C^3_{26} = C^5_{35} = C^6_{36} = C^5_{46} = -1$.

For any vector field F_i $(i = 1, 2, 3, 4, 5, 6)$ there exists a maximal integral curve $\gamma_m(\cdot)$. For any t_0 let us consider the transformation $E_{t_0 F_1}$ with domain $\{x \in R^3_* \mid (t_0 x^3)^2 \neq \neq (t_0 x^1 - 2)^2 + (t_0 x^2)^2 - 8\}$ defined by $E_{t_0 F_1} m = \gamma_m(t_0)$. We obatin the following transformations which correspond to the vector fields F_i:

$$E_{tF_1}(x^1, x^2, x^3) = \Big(\frac{2t((x^3)^2 - (x^2)^2 - (x^1)^2) + 4x^1}{-((x^3)^2 - (x^2)^2 - (x^1)^2)t^2 - 4tx^1 + 4},$$

$$\frac{4x^2}{-((x^3)^2 - (x^2)^2 - (x^1)^2)t^2 - 4tx^1 + 4}, \frac{4x^3}{-((x^3)^2 - (x^2)^2 - (x^1)^2)t^2 - 4tx^1 + 4}\Big),$$

$$E_{tF_2}(x^1, x^2, x^3) = \Big(\frac{4x^1}{-((x^3)^2 - (x^2)^2 - (x^1)^2)t^2 - 4tx^2 + 4},$$

$$\frac{2t((x^3)^2 - (x^2)^2 - (x^1)^2) + 4x^2}{-((x^3)^2 - (x^2)^2 - (x^1)^2)t^2 - 4tx^2 + 4}, \frac{4x^3}{-((x^3)^2 - (x^2)^2 - (x^1)^2)t^2 - 4tx^2 + 4}\Big),$$

$$E_{tF_3}(x^1, x^2, x^3) = (e^t x^1, e^t x^2, e^t x^3),$$

$$E_{tF_4}(x^1, x^2, x^3) = (x^1 \cos t + x^2 \sin t, -x^1 \sin t + x^2 \cos t, x^3),$$

$$E_{tF_5}(x^1, x^2, x^3) = (x^1 + t, x^2, x^3),\ E_{tF_6}(x^1, x^2, x^3) = (x^1, x^2 + t, x^3).$$

Theorem 3. *The sets* $\{E_{tF_\alpha} \mid t \in R\}$, $\alpha = 1, 2, 3, 4, 5, 6$ *are pseudogroups of transformations on* R^3_*.

Proof. F_3, F_4, F_5, F_6 are complete vector fields so E_{tF_α} for each $\alpha = 3, 4, 5, 6$ is a transformation group. Now we consider the case $\alpha = 1$; the proof for $\alpha = 2$ is analogous. Let us examine the particular points of definition 1.

(i) The domain and range of $E_{t_0 F_1}$ are respectively $\{x \in R^3_* \mid (t_0 x^3)^2 \neq (x^1 t_0 - - 2)^2 + (t_0 x^2)^2 - 8\}$ and $\{x \in R^3_* \mid (t_0 x^3)^2 \neq (t_0 x^1 + 2)^2 + (t_0 x^2)^2 - 8\}$. These sets are open for any $t_0 \in R$.

(iv) $E_{0F_1} = id_{R^3_*}$

(v) The inverse of $E_{t_0 F_1}$ is $E_{-t_0 F_1}$, the domain of $E_{-t_0 F_1}$ is equal to the range of $E_{t_0 F_1}$ and conversely.
(vi) $E_{t_0 F_1} \cdot E_{t_1 F_1} = E_{(t_0 + t_1) F_1}$. The others particular points are trivial.

A model of the pseudo-Riemannian manifold. Let K denotes a ring of antiquaternions. Every antiquaternion q can be represented in the form

$$q = q_1 + q_2 i + q_3 j + q_4 k$$

where the q_i $(i = 1, 2, 3, 4)$ are real numbers and the unites i, j, k have multiplicative rule defined by the table

	i	j	k
i	-1	k	j
j	$-k$	1	$-i$
k	j	i	1

Let us now describe a model of our pseudo-Riemannian manifold.

Itroduce the following equivalence relation $\sim$ in the set of antiquaternions K : h is in relation to k iff there exists a $t \in R$ such that $k = e^{it} h e^{-it}$. Since the relation $\sim$ is obviously an equivalence, we may consider the quotient space $\bar{N}$: $K/\sim$. Let us remark that any antiquaternion h can be uniquely expressed in the form $h' + h''j$, where h' and h'' are complex numbers. If $z \in C$, then we have: $zj = j\bar{z}$, where $\bar{z}$ denotes the complex-conjugate of z. Thus we have for any real t

$$e^{it}(h + h''j)\, e^{-it} = h' + h'' \cdot e^{2it} j.$$

Hence it follows that the equivalence class of the element h with respect to the relation $\sim$ is uniquely determined by the following three numbers

$$x^1([h]) = \mathrm{re}\, h', \; x^2([h]) = \mathrm{im}\, h', \; x^3([h]) = |h''|.$$

Thus $\bar{N}$ is a 3-dimensional manifold with a boundary. Denote the interior of $\bar{N}$ by N. It may be verified that the mapping $x : N \to R^3_+$, $x : [h] \to (\mathrm{re}\, h', \mathrm{im}\, h', |h''|)$ is a homeomorphism. Thus it defines a coordinate system. In the sequel we shall write $x^\alpha(\cdot)$ instead of $x^\alpha([\;])$, $\alpha = 1, 2, 3$.

Let us introduce the action of SL (2, C) on the manifold N. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a non-singular complex matrix. We consider a mapping of K to itself, $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ defined as follows

$$T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} h := (ah + b)(ch + d)^{-1} \tag{2}$$

Observe that

$$T_{\binom{a\ b}{c\ d}}(e^{it}he^{-it}) = (ae^{it}he^{-it}+b)(ce^{it}he^{-it}+d)^{-1} = (e^{it}ahe^{-it}+e^{it}be^{-it})\cdot$$

$$\cdot(e^{it}che^{-it}+e^{it}de^{-it})^{-1} = e^{it}(ah+b)(ch+d)^{-1}\cdot e^{-it} = e^{it}T_{\binom{a\ b}{c\ d}}(h)e^{-it}\,.$$

Hence T induces a mapping T^* of the set N itself.

Represent the antiquaternion h in the form $h' + h''\cdot j$, where $h', h'' \in C$. Then we compute the following expresion of T in the above introduced coordinates $T_{\binom{a\ b}{c\ d}}(h) =$

$$= (ah+b)(ch+d)^{-1} = [a(h'+h''j)+b]\cdot[c(h'+h''j)+d]^{-1} = [a(h'+h''j)+$$

$$+b][\bar c\bar h' + \bar d - ch''j][\bar c\bar h' + \bar d - ch''j]^{-1}[c(h'+h''j)+d]^{-1} = [a(h'+h''j)+$$

$$+b][\bar c\bar h' + \bar d - ch''j]\,|\,\bar c\bar h' + d - ch''j\,|^{-2} = [a\bar c h'\bar h' + a\bar d h' - ach'h''j + ah''j\bar c\bar h' +$$

$$+ ah''j\bar d + ah''jch''j + b\bar c\bar h' - bch''j + b\bar d]\,|\,\bar c\bar h' + \bar d - ch''j\,|^{-2} = (a\bar c\,|\,h'\,|^2 + a\bar d h' -$$

$$- a\bar c\,|\,h''\,|^2 - b\bar c\bar h' + bd + (ad-bc)\cdot h''j)\,|\,ch+d\,|^{-2} = (a\bar c\,|\,h'\,|^2 - a\bar c\,|\,h''\,|^2 +$$

$$+ a\bar d h' + b\bar c\bar h' + b\bar d)\cdot|\,ch+d\,|^{-2} + h''\,|\,ch+d\,|^{-2}\cdot j\,.$$ This result may be written in coordinates as follows: $x\cdot T^*\cdot x^{-1} : R^3_+ \to R^3_+$, $x\cdot T^*\cdot x^{-1}(x^1, x^2, x^3) = (\tilde x^1, \tilde x^2, \tilde x^3)$, where x^{-1} denotes the inverse mapping of the mapping x.

$$\tilde x^1 = \frac{1}{m}\left\{\mathrm{re}(a\bar c)[(x^1)^2+(x^2)^2-(x^3)^2] + \mathrm{re}(a\bar d + b\bar c)x^1 + \mathrm{re}[(a\bar d - b\bar c)i]\,x^2 + \mathrm{re}(b\bar d)\right\}$$

$$\tilde x^2 = \frac{1}{m}\left\{\mathrm{im}(a\bar c)[(x^1)^2+(x^2)^2-(x^3)^2] + \mathrm{im}(a\bar d + b\bar c)x^1 + \mathrm{im}[(a\bar d - b\bar c)i]x^2 + \mathrm{im}(b\bar d)\right\}$$

$$\tilde x^3 = \frac{1}{m}\,x^3\,, \tag{3}$$

where $m = |\,c\,|^2[(x^1)^2+(x^2)^2-(x^3)^2] + (c\bar d + \bar c d)x^1 + [(c\bar d - \bar c d)i]x^2 + |\,d\,|^2$.

Theorem 4. *The unimodular group* SL (2, *C*) *acts on the manifold N as a pseudogroup of transformations.*

Proof. The group action is given by formula (2). The domain of $T_{\binom{a\ b}{c\ d}}$ is an open set $\left\{[h]\;|\;a\bar c\,|\,h'\,|^2 - a\bar c\,|\,h''\,|^2 + a\bar d h' + b\bar c\bar h' + b\bar d \neq 0\right\}$ and the range of $T_{\binom{a\ b}{c\ d}}$ is an open set $\left\{[h]\;|\;-d\bar c\,|\,h'\,|^2 + d\bar c\,|\,h''\,|^2 + \bar a d h' + b\bar c\bar h' - b\bar a \neq 0\right\}$. Thus we checked on the condition (i) of the definition 1.

(v) The inverse transformation of the $T_{\binom{a\ b}{c\ d}}$ is $T_{\binom{d\ -b}{c\ +a}}$; the range $T_{\binom{a\ b}{c\ d}}$ is equal to the domain $T_{\binom{d\ -b}{-c\ \ a}}$ and vice versa.

(vi) A straightforward computation shows that

$$T_{\binom{a\ b}{c\ d}} T_{\binom{a_1\ b_1}{c_1\ d_1}} = T_{\binom{a\ b}{c\ d}\binom{a_1\ b_1}{c_1\ d_1}}$$

The others conditions are satisfied in an obvious way. Denote the pseudogroup generated by E_{tF_α} $(\alpha = 1, ..., 6)$ by G.

Theorem 5. *The manifold R_+^3 is diffeomorphic with the manifold N; the pseudogroup G of transformations is locally isomorphic with pseudogroup*

$$\mathcal{F} = \left\{ T_{\binom{a\ b}{c\ d}} \,\Big|\, \binom{a\ b}{c\ d} \in SL(2, C) \right\}.$$

Proof. It is sufficient to observe that there is 1:1 correspondence between one-parameter pseudogroups generated by Killing vector fields and one-parameter subpseudogroup of G i.e.

$$E_{tF_1} = T_{\begin{pmatrix} 1 & 0 \\ 0{,}5\,t & 1 \end{pmatrix}} \qquad E_{tF_2} = T_{\begin{pmatrix} 1 & 0 \\ 0{,}5\,ti & 1 \end{pmatrix}} \qquad E_{tF_3} = T_{\begin{pmatrix} e^{0{,}5\,t} & 0 \\ 0 & e^{0{,}5\,t} \end{pmatrix}}$$

$$E_{tF_4} = T_{\begin{pmatrix} e^{-0{,}5\,ti} & 0 \\ 0 & e^{0{,}5\,ti} \end{pmatrix}} \qquad E_{tF_5} = T_{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}} \qquad E_{tF_6} = T_{\begin{pmatrix} 1 & ti \\ 0 & 1 \end{pmatrix}}.$$

One can verify these equalities by some straightforward calculations. Note that Lie algebra $SL(2, C) = \left\{ \binom{a\ b}{c\ d} \mid a + d = 0 \right\}$ of the group $SL(2, C)$ is isomorphic with Lie algebra from the proposition 2. Let $SU(1, 1) = \left\{ \binom{a\ \ b}{\bar{b}\ \ \bar{a}} \mid a\bar{a} - b\bar{b} = 1 \right\}$ and let m denote the point in N with coordinates $(0, 0, 1)$.

Theorem 6. *A stationary subpseudogroup of transformations $S \subset \mathcal{F}$ is a pseudogroup which consists of the transformations of the form $T_{\binom{a\ b}{c\ d}}$, such that $\binom{a\ b}{c\ d} \in SU(1, 1)$.*

Proof. Solve the equation $(aj + b)(cj + d)^{-1} = j$ or put the numbers $(0, 0, 1)$ to formula (3). We obtain $c = \bar{b}$ and $d = \bar{a}$.

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STRESZCZENIE

W pracy zbadano pseudogrupę izometrii pseudoriemanowskiej rozmaitości R_+^3. Działanie tej pseudogrupy otrzymano jako trajektorie pól Killinga na R_+^3. W drugiej części opierając się na pierścieniu antykwaternionów znaleziono model rozważanej rozmaitości i pokazano, że grupa SL (2, C) działa na tę rozmaitość jako pseudogrupa transformacji. Pokazano lokalny izomorfizm tych przestrzeni.

РЕЗЮМЕ

В данной работе рассматривается псевдогруппа изометрии псевдоримановского многообразия R_+^3. Действие этой группы получено как траекторию Киллинговых полей на R_+^3. Во второй части работы, опираясь на кольцо антикватернионов, найдено модель рассматриваемого многообразия и доказано, что группа SL (2, C) действует на это многообразие как псевдогруппа преобразования. Показано локальный изоморфизм этих пространств.