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Fixed Point Theorems for Continuous Mappings on Complete, Normed in Probability Spaces

Twierdzenia o punkcie stałym dla ciągłych odwzorowań na przestrzeniach zupełnych, unormowanych według prawdopodobieństwa

Теоремы о неподвижной точке для непрерывных преобразований на полных, нормированных по вероятностей пространствах

1. Let Δ^* denote the set of the all distribution functions F with $F(0) = 0$. The $H \in \Delta^*$ is defined by

$$H(x) = \begin{cases} 0, & \text{if } x \leqslant 0 \\ 1, & \text{if } x > 0 \,. \end{cases}$$

By a t-norm we mean a function $T\colon \langle 0, 1\rangle \times \langle 0, 1\rangle \to \langle 0, 1\rangle$ defined as follows.

Definition 1. $T\colon \langle 0, 1\rangle \times \langle 0, 1\rangle \to \langle 0, 1\rangle$ and satisfies the following conditions:

1. $T(a, b) = T(b, a)$ for all $a, b \in \langle 0, 1\rangle$
2. $T(a, T(b, c)) = T(T(a, b), c)$ for all $a, b, c \in \langle 0, 1\rangle$
3. $(a \leqslant c \wedge b \leqslant d) \Rightarrow T(a, b) \leqslant T(c, d)$ for all $a, b, c, d \in \langle 0, 1\rangle$
4. $T(a, 1) = a$ for every $a \in (0, 1)$
5. $\sup\limits_{a<1} T(a, a) = 1$.

Definition 2. By a Menger space (shortly a M-space) we mean an ordered triple $(S, \mathcal{F}, T)$, where S is an abstract set, $\mathcal{F}$ is a function defined on $S \times S$ such that $\mathcal{F} : S \times S \to \Delta^*$ with $\mathcal{F}(p, q) = F_{pq}$ and the functions F_{pq} are assumed to satisfy the following conditions:

I. $F_{pq} = H$ if and only if $p = q$,

II. $F_{pq} = F_{qp}$ for all $p, q \in S$,

III. $F_{pq}(x + y) \geq T(F_{pr}(x), F_{rq}(y))$

for all triples p, q and r in S and all $x > 0$ and $y > 0$, and T is a t-norm.

Definition 3. An ordered triple $(S, \mathcal{F}, T)$ is called a space normed in probability (shortly a N-space), if S is a vector space (on $\mathcal{R}$), $\mathcal{F}$ is a function defined on S such that $\mathcal{F} : S \to \Delta^*$ with $\mathcal{F}(p) = F_p$ and the functions F_p are assumed to satisfy the following conditions

(I) $F_p = H$ if and only if $p = 0$

(II) $F_{\alpha p}(x) = F_p(x/|\alpha|)$ for every $p \in S$, $x > 0$, and $0 \neq \alpha \in \mathcal{R}$,

(III) $F_{p+q}(x + y) \geq T(F_p(x), F_q(y))$ for all $p, q \in S$, $x > 0$ and $y > 0$, and T is a t-norm.

N-spaces have been introduced in [3]. It can be shown that $(S, \mathcal{F}^*, T)$ is a M-space if $(S, \mathcal{F}, T)$ is a N-space and $\mathcal{F}^* : S \times S \to \Delta^*$ with $\mathcal{F}^*(p, q) = F_{p-q}$.

Let $\mathcal{U} \subset 2^{S \times S}$ be the class of sets defined as follows:

$$\mathcal{U} = \left\{U(\epsilon, \lambda); \ \epsilon > 0, 0 < \lambda < 1\right\} = \left\{[(p, q) : F_{pq}(\epsilon) > 1 - \lambda]; \ \epsilon > 0, 0 < \lambda < 1\right\}.$$

It has been shown in [2] that $\mathcal{U}$ is a base of neighbourhoods of a Hausdorff uniform structure. This uniform structure generates a metrizable topology $\tau_{\epsilon, \lambda}$ on S [2]. Then

$$p_n \xrightarrow[n \to \infty]{\tau_{\epsilon, \lambda}} p \iff \underset{0 < \epsilon, \lambda < 1}{\forall} \ \underset{n_{\epsilon, \lambda}}{\exists} \ \underset{n \geq n_{\epsilon, \lambda}}{\forall} \ F_{p_n p}(\epsilon) > 1 - \lambda .$$

For uniform structures it can be introduced the concept of completeness. Note that:

a) A sequence $\{p_n, n \geq 1\}$ of a M-space is a Cauchy sequence if and only if for any $0 < \epsilon, \lambda < 1$, there exists a $n_{\epsilon, \lambda}$ such that for all $m, n \geq n_{\epsilon, \lambda}$ $F_{p_m p_n}(\epsilon) > 1 - \lambda$.

b) A M-space $(S, \mathcal{F}, T)$ is complete if and only if every Cauchy sequence converges in S.

It has been shown in [2] that if T is left continuous, then

$$(p_n \xrightarrow[n \to \infty]{\tau_{\epsilon, \lambda}} p \wedge q_n \xrightarrow[n \to \infty]{\tau_{\epsilon, \lambda}} q) \Rightarrow \underset{x \in \mathcal{R}:}{\forall} \ F_{p_n q_n}(x) \xrightarrow[n \to \infty]{} F_{pq}(x),$$

continuous in x.

2. Let $A \subset \mathfrak{X}$ be a compact convex set in a Banach space $(\mathfrak{X}, \| \ \|)$ and let $M : A \to A$ be a continuous mapping. It is known that M has a fixed point.

We will need yet the Brouwer theorem. Let $A \subset \mathcal{R}^n$ be a closed, bounded and convex set in a normed space $(\mathcal{R}^n, \| \ \|)$ and let $M : A \to A$ be a continuous mapping. Then M has a fixed point.

Now let $(S, \mathcal{F}, T)$ be a complete N-space and $A \subset S$ be a compact (in $\tau_{\epsilon, \lambda}$) convex set.

We are searching for conditions under which a continuous in $\tau_{\epsilon,\lambda}$ mapping $M : A \to A$ has a fixed point. We shall see that they depend on the t-norm T.

Definition 4. A set $A \subset S$ is called bounded in the M-space $(S, \mathcal{F}, T)$ if

$$\underset{0<\lambda<1}{\forall} \quad \underset{\epsilon>0}{\exists} \quad \underset{p,q\in A}{\forall} \quad F_{pq}(\epsilon) > 1-\lambda .$$

Lemma 1. *Let $(S, \mathcal{F}, T)$ be a M-space. A set $A \subset S$ is bounded if and only if*

$$\underset{p_0\in S}{\exists} \quad \underset{0<\lambda<1}{\forall} \quad \underset{\epsilon>0}{\exists} \quad \underset{q\in A}{\forall} \quad F_{p_0 q}(\epsilon) > 1-\lambda .$$

Proof. Fix $0<\lambda<1$.
Necessity: This is obvious.
Sufficiency:

$$\underset{p,q\in A}{\forall} \quad F_{pq}(\epsilon) \geqslant T(F_{pp_0}(\epsilon'), F_{p_0 q}(\epsilon')) \geqslant T(1-\lambda', 1-\lambda') > 1-\lambda$$

if $T(1-\lambda', 1-\lambda') > 1-\lambda$, $F_{pp_0}(\epsilon/2) > 1-\lambda'$ for all $p \in S$.

Lemma 2.

$$\left[\underset{p\in S}{\forall} \quad \underset{0<\lambda<1}{\forall} \quad \underset{\epsilon>0}{\exists} \quad \underset{q\in A}{\forall} \quad F_{pq}(\epsilon) > 1-\lambda\right] \Longleftrightarrow$$

$$\Longleftrightarrow \left[\underset{p_0\in S}{\forall} \quad \underset{0<\lambda<1}{\forall} \quad \underset{\epsilon>0}{\exists} \quad \underset{q\in A}{\forall} \quad F_{p_0 q}(\epsilon) > 1-\lambda\right].$$

The proof is obvious.

The probabilistic diameter of $A \subset S$, $A \neq \emptyset$ in the M-space $(S, \mathcal{F}, T)$ has been introduced in [4] as

$$D_A(x) = \sup_{t<x} \inf_{p,q\in A} F_{pq}(t) .$$

There was shown that $D_A(0) = 0$, D_A is left continuous and non-decreasing. It is obvious that

$$[A \text{ is bounded}] \Rightarrow [D_A \in \Delta^*] .$$

If the t-norm is left continuous then $D_A = D_{\bar{A}}$.

Definition 5. A set $\{p_i, 1 \leq i \leq n\} \subset S$ is called a ϵ, λ-system, $0 < \epsilon, \lambda < 1$, for the set $A \subset S$ in the M-space $(S, \mathcal{F}, T)$ iff

$$\forall_{p \in A} \quad \exists_{i(p) \in \{1, 2, \ldots, n\}} \quad F_{pp_{i(p)}}(\epsilon) > 1 - \lambda .$$

Lemma 3. *In a M-space A is bounded if and only if $\bar{A}$ is bounded.*

Proof. Necessity: is obvious.

Sufficiency: Fix $p_0 \in S$, $0 < \lambda < 1$, and take such λ', $0 < \lambda' < 1$, that $T(1-\lambda', 1-\lambda') > 1 - \lambda$. Take $p \in \bar{A}$. Then, for $x > 0$ there exists $p'(p) \in A$ that $F_{pp'(p)}(x) > 1 - \lambda'$. For this λ' there exists $\epsilon' > 0$ so that

$$\forall_{p' \in A} \quad F_{p_0 p'(p)}(\epsilon') > 1 - \lambda' .$$

Then putting $\epsilon = x + \epsilon'$ we get

$$\forall_{p \in A} \quad F_{pp_0}(\epsilon) \geq T(F_{pp'(p)}(x), F_{p'(p)p_0}(\epsilon')) \geq T(1-\lambda', 1-\lambda') > 1 - \lambda .$$

Lemma 4. *In a M-space $(S, \mathcal{F}, T)$ a set $A \subset S$ has for every $0 < \epsilon, \lambda < 1$ a ϵ, λ-system iff $\bar{A}$ has for every $0 < \epsilon, \lambda < 1$ a ϵ, λ-system.*

Proof. The part (sufficiency) is obvious since $A \subset \bar{A}$.

Necessity: suppose that $p_k \to p$ as $k \to \infty$, where $p_k \in A$ and A has for every $0 < \epsilon$, $\lambda < 1$ a ϵ, λ-system. Let us choose λ', $0 < \lambda' < 1$ so that $T(1-\lambda', 1-\lambda') > 1 - \lambda$. If $\{p_i, n \geq i \geq 1\}$ is a $\epsilon/2$, λ'-system for A, then there exists $p_k \in A$ so that $F_{pp_k}(\epsilon/2 > > 1 - \lambda'$. Therefore,

$$F_{pp_{i(p_k)}}(\epsilon) \geq T(F_{pp_k}(\epsilon/2), F_{p_k p_{i(p_k)}}(\epsilon/2)) \geq T(1-\lambda', 1-\lambda') > 1 - \lambda ,$$

which completes the proof.

Theorem 1. *Let $(S, \mathcal{F}, T)$ be a M-space. A compact set $\bar{A} \subset S$ has for every $0 < \epsilon$, $\lambda < 1$ a ϵ, λ-system in A.*

Proof. Obviously $A \subset \bigcup_{p \in A} U_p(\epsilon, \lambda)$, where $U_p(\epsilon, \lambda)$ is a neighbourhood of p. Since $A \subset S$ is compact, so there must be $A \subset \bigcup_{i=1}^{n} U_{p_i}(\epsilon, \lambda)$, $p_i \in A$, $i = 1, 2, \ldots, n$.

Theorem 2. *Let $(S, \mathcal{F}, T)$ be a complete M-space. Then $\bar{A} \subset S$ is compact, if for every $0 < \epsilon, \lambda < 1$ A has a ϵ, λ-system.*

Proof. It is enough to show that every sequence $\{p_n, n \geq 1\} \subset \bar{A}$ contains a convergent subsequence to a point of $\bar{A}$ ($\tau_{\epsilon, \lambda}$ is metrizable).

Take $\epsilon_n \downarrow 0$, $1 > \lambda_n \downarrow 0$ as $n \to \infty$.

For $n = 1$ there exists a ϵ_1, λ_1-system $\{p_{i1}, 1 \leq i \leq k_1\}$ for $\{p_n, n \geq 1\}$ such that a subsequence $\{p_{n1}, n \geq 1\} \subset \{p_n, n \geq 1\}$ belongs to a set $U_{p_{i_1 1}}(\epsilon_1, \lambda_1)$. Suppose that a sequence $\{p_{n(l-1)}, n \geq 1\}$ is defined. We see that for $n = 1$ there exists a ϵ_l, λ_l-system, $\{p_{il}, 1 \leq i \leq k_l\}$ for $\{p_n, n \geq 1\}$ such that a subsequence $\{p_{nl}, n \geq 1\} \subset \{p_{n(l-1)}, n \geq 1\}$ belongs to a set $U_{p_{i_0 l}}(\epsilon_l, \lambda_l)$.

We now show that $\{p_{ll}, l \geq 1\} \subset \{p_n, n \geq 1\} \subset \bar{A}$ is a Cauchy sequence. Let $x > 0$, $0 < \lambda < 1$ be arbitrary numbers. Note that there exists $n_{x,\lambda}$ such that for all $l \geq n_{x,\lambda}$, $2\epsilon_l \leq x$ and $T(1-\lambda_l, 1-\lambda_l) > 1-\lambda$. Suppose that $m \geq l \geq n_{x,\lambda}$. Then $\{p_{n\,m}, n \geq 1\} \subset \{p_{nl}, n \geq 1\} \subset \{p_{nn_{x,\lambda}}, n \geq 1\} \subset U_{p_{i_0 n_{x,\lambda}}}(\epsilon_{n_{x,\lambda}}, \lambda_{n_{x,\lambda}})$. Therefore, for all $l, m \geq n_{x,\lambda}$

$$F_{p_{ll}p_{mm}}(x) \geq F_{p_{ll}p_{mm}}(2\epsilon_{n_{x,\lambda}}) \geq T(F_{p_{ll}p_{i_0 n_{x,\lambda}}}(\epsilon_{n_{x,\lambda}}), F_{p_{i_0 n_{x,\lambda}}p_{mm}}(\epsilon_{n_{x,\lambda}})) \geq$$

$$\geq T(1-\lambda_{n_{x,\lambda}}, 1-\lambda_{n_{x,\lambda}}) > 1-\lambda .$$

Taking into account that $(S, \mathcal{F}, T)$ is complete, we see that there exists a $p \in \bar{A}$ so that $p_{ll} \to p$ as $l \to \infty$.

Lemma 5. *Let $(S, \mathcal{F}, T)$ be a M-space. A compact set $A \subset S$ is closed and bounded.*

Proof. The fact that A is closed is obvious.

We now fix $0 < \lambda < 1$ and choose $0 < \lambda', \lambda'' < 1$ for that $T(1-\lambda', 1-\lambda') > 1-\lambda$ and $T(1-\lambda'', 1-\lambda'') > 1-\lambda'$. By Theorem 1 there exists a ϵ, λ''-system $\{p_i, 1 \leq i \leq n\}$ for A, $\epsilon > 0$. Take $x' > 0$ so that $\inf\limits_{i,j \in \{1,\dots,n\}} F_{p_i p_j}(x') > 1-\lambda'$. Then for all $p, q \in A$

$$F_{pq}(x' + 2\epsilon) \geq T(T(F_{pp_{i(p)}}(\epsilon), F_{p_{i(q)}q}(\epsilon)), F_{p_{i(p)}p_{i(q)}}(x')) \geq$$

$$\geq T(T(1-\lambda'', 1-\lambda''), 1-\lambda') > 1-\lambda ,$$

what ends the proof.

Remark. Note that in a M-space $(S, \mathcal{F}, T)$:

1. $A \subset S$ is closed if and only if $[(p_n \to p \wedge p_n \in A) \Rightarrow p \in A]$,

2. $A \subset S$ is open if and only if $\forall_{p \in A} \; \exists_{0 < \epsilon, \lambda < 1} \; U_p(\epsilon, \lambda) \subset A$,

3. $A \subset S$ is compact if and only if

$$\forall_{\{p_n, n \geq 1\} \subset A} \; \exists_{\{p_{n_k}, k \geq 1\}, p \in A} \; p_{n_k} \to p \text{ as } k \to \infty .$$

3. Now let $(S, \mathcal{F}, T)$ be a N-space with $\dim S = n < \infty$. One can immediately show that such a space is isometric to a N-space of the type $(\mathcal{R}^n, \mathcal{F}, T)$. It is enough to fix a base

$\{b_i, 1 \leq i \leq n\} \subset S$, define to isomorfizm $h : S \to \mathcal{R}^n$, $p = \sum\limits_{i=1}^{n} \lambda^i(p) b_i \to (\lambda^1, \dots, \lambda^n)$,

and define $\mathcal{F} : \mathcal{R}^n \to \Delta^*$ by

$$(\lambda^1, \ldots, \lambda^n) \to F_{(\lambda^1, \ldots, \lambda^n)} = F^*_{h^{-1}(\lambda^1, \ldots, \lambda^n)} .$$

In [3] it is shown that $(\mathcal{S}, \tau_{\epsilon,\lambda})$ is a metrizable topological vector space in the case when $(S, \mathcal{F}, T)$ is a N-space. By this assertion two N-spaces have equivalent topologies $\tau_{\epsilon,\lambda}$ if and only if the convergences to zero are equivalent on these spaces.

Lemma 6. *In a N-spaces $(\mathcal{R}^n, \mathcal{F}, T)$ there exists a base $\{B_k, k \geq 1\}$ at zero such that $B_k = \overline{B}_k$, $B_{k+1} \subset B_k$, B_k, $k \geq 1$, are convex.*

Proof. Let $\epsilon_k \downarrow 0$, $1 > \lambda_k \downarrow 0$. Then $\{U_0(\epsilon_k, \lambda_k), k \geq 1\}$ is a base at zero. $\tau_{\epsilon,\lambda}$ is metrizable so $\{\overline{U_0(\epsilon_k, \lambda_k)}, k \geq 1\}$ is also a base at zero. Put $\epsilon_k' = \epsilon_k/n$ and choose λ_k', $0 < \lambda_k' < 1$ so that $T_{j=1}^n (1 - \lambda_k') > 1 - \lambda_k$. If $p \in \operatorname{conv} U_0(\epsilon_k', \lambda_k')$, where $\operatorname{conv} U_0(\epsilon_k', \lambda_k') =$

$= [p = \sum_{j=1}^n \lambda^{(j)} p_j : \sum_{j=1}^n \lambda^{(j)} = 1; \lambda^{(j)} \geq 0, p_j \in U_0(\epsilon_k', \lambda_k')$ for $j = 1, \ldots, n]$ then

$$F_p(\epsilon_k) = F_{\sum_{j=1}^n \lambda^{(j)} p_j}(\epsilon_k) \geq T_{j=1}^n (F_{p_j}(\epsilon_k')) \geq T_{j=1}^n (1 - \lambda_k') > 1 - \lambda_k .$$

Therefore, $\overline{\operatorname{conv} U_0(\epsilon_k', \lambda_k')} \subset \overline{U_0(\epsilon_k, \lambda_k)}$. We conclude that $[\overline{\operatorname{conv} U_0(\epsilon_k, \lambda_k)}]_{k=1}^{\infty}$ is a base at zero. Because the intersection of convex sets is a convex set, then putting

$$B_1 = \overline{\operatorname{conv} U_0(\epsilon_1, \lambda_1)}, \quad B_k = \overline{\operatorname{conv} U_0(\epsilon_k, \lambda_k)} \cap B_{k-1} ,$$

we obtain the required base at zero.

Theorem 3. *The topology generated by an arbitrary norm $\| \ \|$ on $\mathcal{R}^n$ and $\tau_{\epsilon,\lambda}$ in a N-space $(\mathcal{R}^n, \mathcal{F}, T)$ are equivalent.*

Proof. First we show that $\tau_{\epsilon,\lambda}$ is not stronger than the topology generated by the norm $\| \ \|$. It is enough to show that

$$p_k \xrightarrow{\| \ \|} 0 \text{ as } k \to \infty \Rightarrow p_k \xrightarrow{\tau_{\epsilon,\lambda}} 0 \text{ as } k \to \infty .$$

Take the base $[e_i, 1 \leq i \leq n] \subset R^n$, where $e_i = [\delta_{ij}]_{1 \leq j \leq n}$. Of course, $p_k =$ $= \sum_{i=1}^n \lambda_k^{(i)} e_i \xrightarrow{\| \ \|} 0$ as $n \to \infty \Longleftrightarrow \forall_{i=1,2,\ldots,n} \lambda_k^{(i)} \to 0$ as $k \to \infty$. Then

$$\forall_{x>0} \quad F_{\sum_{i=1}^n \lambda_k^{(i)} e_i}(x) \geq T_{i=1}^n (F_{\lambda_k^{(i)} e_i}(x/n)) = T_{i=1}^n (F_{e_i}(\frac{x}{n|\lambda_k^{(i)}|})) \to 1 \text{ as } k \to \infty ,$$

since $\forall_{i=1,2,\ldots,n} \dfrac{x}{n|\lambda_k^{(i)}|} \to \infty$, as $k \to \infty$, $F_{e_i} \in \Delta^*$ and $\sup_{a<1} T(a,a) = 1$.

We now prove that the topology generated by the norm $\| \ \|$ is not stronger than $\tau_{\epsilon,\lambda}$. We will show that every ball $K_{\| \ \|}(0, r)$ contains a set of the base $\{B_k, k \geq 1\}$ which has appeared in the Lemma 6.

Suppose that

$$\underset{K_{\| \|}(0,r)}{\exists} \quad \underset{k=1,2,\dots}{\forall} \quad C_k = (R^n K_{\| \|}(0,r)) \cap B_k \neq \emptyset.$$

Because $B_k = \overline{B}_k$ in $\tau_{\epsilon,\lambda}$, moreover, $\tau_{\epsilon,\lambda}$ is not stronger than the topology generated by $\| \ \|$, therefore $C_k = \overline{C}_k$ in the norm. Obviously, $C_{k+1} \subset C_k$. But the sets B_k are convex, so that

$$\underset{K_{\| \|}(0,r+r')}{\exists} \quad \underset{k=1,2,\dots}{\forall} \quad C_k \cap K_{\| \|}(0, r+r') = D_k \neq \emptyset.$$

Of course, $D_k = \overline{D}_k$ and D_k are bounded in $(\mathcal{R}^n, \| \ \|)$, $D_{k+1} \subset D_k$. We see that $\{D_k, k \geqslant 1\}$ is a nonincreasing sequence of nonempty compact sets, so that $\emptyset \neq$ $\neq \bigcap_{k=1}^{\infty} D_k \subset \bigcap_{k=1}^{\infty} B_k$ and $0 \notin \bigcap_{k=1}^{\infty} D_k$ as $0 \notin \mathcal{R}^n \setminus K_{\| \|}(0,r)$. Thus

$\bigcap_{k=1}^{\infty} B_k$ contains at least two points, but this is a contradiction to the fact that $\tau_{\epsilon,\lambda}$ is metrizable.

Lemma 7. *In a N-space* $(\mathcal{R}^n, \mathcal{F}, T)$:

$$[K_{\| \|}(0,r) \underset{(\supset)}{\subseteq} U_0(\epsilon,\lambda)] \iff [\underset{\alpha>0}{\forall} \quad K_{\| \|}(0,\alpha r) \underset{(\supset)}{\subseteq} U_0(\alpha\epsilon,\lambda)].$$

Proof. Sufficiency: putting $\alpha = 1$.

Necessity: suppose that $\{p : \| p \| < r\} \subset \{p : F_p(\epsilon) > 1 - \lambda\}$. Then $\{p : \| p \| < \alpha r\} = \{p' = p/\alpha : \| p \| < r\} \subset \{p' = p/\alpha : F_{p'}(\epsilon) > 1 - \lambda\} = \{p : F_p(\alpha\epsilon) > 1 - \lambda\}$.

Lemma 8. $A \subset \mathcal{R}^n$ *is bounded in* $\tau_{\epsilon,\lambda} \iff A$ *is bounded in a norm* $\| \ \|$ *on* $\mathcal{R}$.

Proof. Necessity: Suppose that

$$\underset{0<\lambda<1}{\forall} \quad \underset{\epsilon>0}{\exists} \quad \underset{p \in A}{\forall} \quad p \in U_0(\epsilon,\lambda).$$

Since the topology generated by the norm $\| \ \|$ and $\tau_{\epsilon,\lambda}$ are equivalent, then for a ball $K_{\| \|}(0,r)$ there exists $U_0(\epsilon',\lambda')$ such that $K_{\| \|}(0,r) \supset U_0(\epsilon',\lambda')$. Moreover, there exists $\epsilon > 0$ for which $A \subset U_0(\epsilon,\lambda')$. For $\alpha\epsilon' = \epsilon$, by Lemma 7, we have

$$A \subset U_0(\epsilon,\lambda') \subset K_{\| \|}(0, r\alpha).$$

Therefore, A is bounded in $(\mathcal{R}^n, \| \ \|)$.

Sufficiency: Suppose that $A \subset K_{\| \|}(0,r)$. For $0 < \lambda < 1$ and $\epsilon > 0$ there exists $\alpha > 0$ such that $K_{\| \|}(0,\alpha r) \subset U_0(\epsilon,\lambda)$, as the considered topologies are equivalent. Hence, by Lemma 7, we get

$$A \subset K_{\|\ \|}(0, r) \subset U_0(\epsilon/\alpha, \lambda).$$

Thus A is bounded in $(\mathcal{R}^n, \mathcal{F}, T)$.

Lemma 9. *In a N-space* $(\mathcal{R}^n, \mathcal{F}, T)$:

$$A \subset \mathcal{R}^n \text{ is compact in } \tau_{\epsilon,\lambda} \iff A \text{ is closed and bounded.}$$

Proof. The assertion of Lemma 9 is true in $(\mathcal{R}^n, \|\ \|)$. But the topologies $\tau_{\epsilon,\lambda}$ and $\|\ \|$ are equivalent, therefore, by Lemma 8, we have the equivalence of Lemma 9.

Theorem 4. *If* $A \subset \mathcal{R}^n$ *is convex, closed and bounded in a N-space* $(\mathcal{R}^n, \mathcal{F}, T)$, *and* $f: A \to A$ *is continuous in* $\tau_{\epsilon,\lambda}$, *then* f *has a fixed point.*

Proof. The statement of Theorem 4 is true in $(\mathcal{R}^n, \|\ \|)$. Knowing that the topologies $\tau_{\epsilon,\lambda}$ and $\|\ \|$ are equivalent, and using lemmas 8 and 9, we get the assertion of Theorem 4.

Lemma 10. *Every Cauchy sequence on a M-space is bounded.*

Proof. Fix $0 < \lambda < 1$, and next take $x > 0$ and $0 < \lambda' < 1$, such that $T(1-\lambda', 1-\lambda') > 1-\lambda$. Since $\{p_n, n \geq 1\}$ is a Cauchy sequence, then

$$\exists_{n_{x,\lambda'}} \ \forall_{m \geq n_{x,\lambda'}} \ F_{p_m p_{n_{x,\lambda'}}}(x) > 1-\lambda',$$

and, of course

$$\exists_{\epsilon' > x} \ \forall_{m = 1, 2, \ldots, n_{x,\lambda'^{-1}}} \ F_{p_m p_{n_{x,\lambda'}}}(\epsilon') > 1-\lambda'.$$

Putting $\epsilon = 2\epsilon'$

$$\forall_{m, n = 1, 2, \ldots} \ F_{p_m p_n}(\epsilon) \geq T(F_{p_m p_{n_{x,\lambda'}}}(\epsilon'), F_{p_{n_{x,\lambda'}} p_n}(\epsilon')) \geq$$

$$\geq T(1-\lambda', 1-\lambda') > 1-\lambda.$$

One can immediately state that:

(i) a convergent sequence is a Cauchy sequence,

(ii) a Cauchy sequence with a convergent subsequence converges.

Theorem 5. *A N-space* $(\mathcal{R}^n, \mathcal{F}, T)$ *is complete.*

Proof. Let $\{p_n, n \geq 1\} \subset \mathcal{R}^n$ be a Cauchy sequence. Then by Lemma 10 $\{p_n, n \geq 1\}$ is bounded. Therefore $\overline{\{p_n, n \geq 1\}}$ is compact. From (ii) we conclude that $\{p_n, n \geq 1\}$ has a limit $p \in \mathcal{R}^n$.

Definition 6. A mapping $M: S \to S$ on a M-space $(S, \mathcal{F}, T)$ is said to be compact iff M is continuous and $\overline{M(S)}$ is compact.

Definition 7. A mapping $M: S \to S$ is said to be bounded on a M-space $(S, \mathcal{F}, T)$ iff $M(S)$ is bounded.

Definition 8. A mapping $M: S \to S$ is said to be finite dimensional on a N-space $(S, \mathcal{F}, T)$ iff $\dim M(S) < \infty$.

Notice that in a M-space $(S, \mathcal{F}, T)$:

1. $M_n \to M,\ n \to \infty \iff \underset{p \in S}{\forall}\ \underset{0 < \epsilon, \lambda < 1}{\forall}\ \underset{n_{\epsilon,\lambda}}{\exists}\ \underset{n \geq n_{\epsilon,\lambda}}{\forall}\ F_{M_n p M p}(\epsilon) > 1 - \lambda$,

2. $M_n \rightrightarrows M,\ n \to \infty \iff \underset{0 < \epsilon, \lambda < 1}{\forall}\ \underset{n_{\epsilon,\lambda}}{\exists}\ \underset{n \geq n_{\epsilon,\lambda}}{\forall}\ \underset{p \in S}{\forall}\ F_{M_n p M p}(\epsilon) > 1 - \lambda$,

3. M is continuous $\iff$

$$\underset{p \in S}{\forall}\ \underset{0 < \epsilon, \lambda < 1}{\forall}\ \underset{0 < \delta, \tau < 1}{\exists}\ \underset{q \in S}{\forall}\ (F_{pq}(\delta) > 1 - \tau \Rightarrow F_{MpMq}(\epsilon) > 1 - \lambda),$$

4. M is uniformly continuous $\iff$

$$\underset{0 < \epsilon, \lambda < 1}{\forall}\ \underset{0 < \delta, \tau < 1}{\exists}\ \underset{p, q \in S}{\forall}\ (F_{pq}(\delta) > 1 - \tau \Rightarrow F_{MpMq}(\epsilon) > 1 - \lambda),$$

5. M being continuous on a compact set is uniformly continuous.

Lemma 11. *If $M_n \rightrightarrows M$ as $n \to \infty$ on a M-space $(S, \mathcal{F}, T)$ and M_n are continuous, then M is continuous.*

Proof. Suppose that $p_k \to p$ as $k \to \infty$ and fix $0 < \epsilon, \lambda < 1$. Take $0 < \lambda', \lambda'' < 1$ such that $T(1 - \lambda', 1 - \lambda') > 1 - \lambda$, and $T(1 - \lambda'', 1 - \lambda'') > 1 - \lambda'$. Since $M_n \rightrightarrows M$ as $n \to \infty$, then there exists M_{n_0} such that

$$\underset{p \in S}{\forall}\ F_{MpM_{n_0}p}(\epsilon/3) > 1 - \lambda''.$$

M_{n_0} is continuous, therefore

$$\underset{k_0}{\exists}\ \underset{k \geq k_0}{\forall}\ F_{M_{n_0}p_k M_{n_0}p}(\epsilon/3) > 1 - \lambda'.$$

Thus for $k \geq k_0$

$$F_{Mp_kMp}(\epsilon) \geq T(T(F_{Mp_k M_{n_0}p_k}(\epsilon/3),\ F_{M_{n_0}p\, Mp}(\epsilon/3)), F_{M_{n_0}p_k M_{n_0}p}(\epsilon/3)) \geq$$

$$\geq T(T(1 - \lambda'', 1 - \lambda''), 1 - \lambda') > 1 - \lambda.$$

It means that $M_{p_k} \to Mp$ as $k \to \infty$.

Theorem 6. *Let $(S, \mathcal{F}, T)$ be a complete M-space and $M_n : S \to S$ be a sequence of compact mappings. If $M_n \rightrightarrows M$ as $n \to \infty$, then M is also compact.*

Proof. By Lemma 11 M is continuous. Hence, using Lemma 4 and Theorem 2, it is enough to show that $M(S)$ has for every $0 < \epsilon, \lambda < 1$ ϵ, λ-system. Take now $0 < \lambda' < 1$ such that $T(1 - \lambda', 1 - \lambda') > 1 - \lambda$. Since $M_n \rightrightarrows M$ as $n \to \infty$ and M_n are compact, then

$$\exists_{n_0} \forall_{p \in S} F_{M_{n_0}p\,Mp}(\epsilon/2) > 1 - \lambda' .$$

Let $\{p_i, n \geq i \geq 1\}$ be $a_n\,\epsilon/2, \lambda'$-system for $M_{n_0}(S)$. Then $\{p_i, n \geq i \geq 1\}$ is $a_n\,\epsilon, \lambda$-system for $M(S)$:

$$F_{Mp\,p_i(M_{n_0}p)}(\epsilon) \geq T(F_{Mp\,M_{n_0}p}(\epsilon/2), F_{M_{n_0}p\,p_i(M_{n_0}p)}(\epsilon/2)) \geq T(1-\lambda', 1-\lambda') > 1 - \lambda .$$

Theorem 7. *Let* $(S, \mathcal{F}, \min)$ *be a complete N-space. Then* A *is compact if and only if* $\overline{\text{conv } A}$ *is compact.*

Proof. Sufficiency: This implication is obvious as $A \subset \overline{\text{conv } A}$.

Necessity: fix $0 < \epsilon, \lambda < 1$. Let $\{p_i, 1 \leq i \leq k\} \subset A$ be a $\epsilon/2, \lambda/2$-system for A. We note that conv $[p_1, p_2, ..., p_k]$ is compact. This follows from the fact that it is compact in a norm, and by Theorem 3.

Let $\{w_i, 1 \leq i \leq n\} \subset$ conv $[p_1, p_2, .., p_n]$ be $\epsilon/2, \lambda/2$-sytem for this set. We show now that $w_i, 1 \leq i \leq n$ $\subset$ conv A, where

$$\text{conv } A = [p = \sum_{j=1}^{r} \lambda^{(j)} p_j : p_j \in A, \sum_{j=1}^{r} \lambda^{(j)} = 1, \lambda^{(j)} \geq 0;\ r = 1, 2, ...] ,$$

is ϵ, λ-system for conv A.

For

$$w_{i(p)} = w_{i(\Sigma_{j=1}^{r} \lambda^{(j)} p_{i(p_j)})},\ p \in \text{conv } A,$$

we get

$$F_{p - w_{i(p)}}(\epsilon) = F_{\Sigma_{j=1}^{r} \lambda^{(j)} p_j - w_{i(p)}}(\epsilon) \geq$$

$$\geq \min (F_{\Sigma_{j=1}^{r} \lambda^{(j)} (p_j - p_{i(p_j)})}(\epsilon/2), F_{\Sigma_{j=1}^{r} \lambda^{(j)} p_{i(p_j)} - w_{i(p)}}(\epsilon/2)) \geq$$

$$\geq \min (\min_{j=1}^{r} (F_{\lambda^{(j)}(p_j - p_{i(p_j)})}(\epsilon\lambda^{(j)}/2)), F_{\Sigma_{j=1}^{r} \lambda^{(j)} p_{i(p_j)} - w_{i(p)}}(\epsilon/2)) \geq$$

$$\geq \min (\min_{j=1}^{r} (F_{p_j - p_{i(p_j)}}(\epsilon/2)), 1 - \lambda/2) \geq \min (1 - \lambda/2, 1 - \lambda/2) > 1 - \lambda .$$

Lemma 12. *Let* $(S, \mathcal{F}, \min)$ *be a M-space. Define for* $0 < \lambda < 1$ *the function* $f_\lambda : S \times S \to \mathcal{R}_0^+ (\mathcal{R}_0^+ = \mathcal{R}^+ \cup \{0\}), (p, q) \to \inf [x : F_{pq}(x) > 1 - \lambda]$. *Then*

1. $f_\lambda (p, q) < \epsilon \Longleftrightarrow F_{pq}(\epsilon) > 1 - \lambda$

2. f_λ *is continuous*.

Proof. ad. 1 Sufficiency: F_{pq} is left continuous, therefore $F_{pq} > 1 - \lambda$ on an interval $(\epsilon - \epsilon', \epsilon)$.

Necessity: it follows immediately from the definition.

ad. 2. Suppose that $p_n \to p$, and $q_n \to q$ as $n \to \infty$. Note that

$$f_\lambda(p,q) = \inf\,[x: F_{pq}(x) > 1-\lambda \text{ and } F_{pq} \text{ is continuous in } x\,].$$

By the definition of inf

$$\underset{\epsilon' > 0}{\forall}\ \underset{x' > f_\lambda(p,q)}{\exists}\quad x' - \epsilon'/2 < f_\lambda(p,q)\,.$$

Since the set of all points of continuity of F_{pq} is dense in $\mathbb{R}$, then

$$\underset{x' > x > f_\lambda(p,q)}{\exists}\quad x - \epsilon' < f_\lambda\,(p,q) \text{ and } F_{pq} \text{ is continuous in } x.$$

From 1, we have $F_{pq}(x') > 1-\lambda < F_{pq}(x)$. Therefore,

$$f_\lambda(p,q) \geqslant \inf\,[x: F_{pq}(x) > 1-\lambda \text{ and } F_{pq} \text{ is continuous in } x\,].$$

By the reason we will consider only points of continuity of F_{pq}. But min is continuous, hence in these points $\{F_{p_nq_n}, n \geqslant 1\}$ converges to F_{pq}.

Now we are going to prove:

a) $\limsup_{n\to\infty} f_\lambda\,(p_n, q_n) \leqslant f_\lambda(p,q)$: Take $x > f_\lambda(p,q)$. Since $F_{pq}(x) > 1-\lambda$, then there exists an integer N such that $F_{p_nq_n}(x) > 1-\lambda$, $n > N$. Therefore, by 1., $\limsup_{n\to\infty} f_\lambda\,(p_n, q_n) \leqslant x$. But it was assumed that $x > f_\lambda\,(p,q)$, so we have $\limsup_{n\to\infty} f_\lambda(p_n, q_n) \leqslant f_\lambda(p,q)$.

b) $\liminf_{n\to\infty} f_\lambda(p_n, q_n) \geqslant f_\lambda(p,q)$: Assume that for some sequences $\{p_n, n \geqslant 1\}$ and $\{q_n, n \geqslant 1\}$ such that $p_n \to p$, and $q_n \to q$ as $n \to \infty$ $\lim_{n\to\infty} f_\lambda(p_n, q_n) < f_\lambda(p,q)$. Let us consider now the following cases:

(i) F_{pq} takes at most at one point the value $1-\lambda$. Then there exists $x > 0$ such that $\lim_{n\to\infty} f_\lambda(p_n, q_n) < x < f_\lambda(p,q)$ and $1-\lambda > F_{pq}(x) = \lim_{n\to\infty} F_{p_nq_n}(x) \geqslant 1-\lambda$.

(ii) $F_{pq} \equiv 1-\lambda$ on an interval. Then, there exist $x, y > 0$ such that

$$\underset{N_1}{\exists}\ \underset{n > N_1}{\forall}\quad f_\lambda(p_n, q_n) < y < x < f_\lambda(p,q)\,,\ \text{ and } F_{pq}(y) = F_{pq}(x) = 1-\lambda\,.$$

But $F_{pp_n}((x-y)/2) \to 1$, $F_{qq_n}((x-y)/2) \to 1$, and $F_{p_nq_n}(y) \to F_{pq}(y) = 1-\lambda$ as $n \to \infty$, then there exists $N_2 \geqslant N_1$ such that

$$\underset{n > N_2}{\forall}\quad \min(F_{pp_n}((x-y)/2), F_{qq_n}((x-y)/2), F_{p_nq_n}(y)) = F_{p_nq_n}(y).$$

Hence, for $n \geq N_2$, $1-\lambda = F_{pq}(x) \geq \min(F_{pp_n}((x-y)/2), F_{qq_n}((x-y)/2), F_{p_nq_n}(y)) =$

$= F_{p_nq_n}(y) > 1-\lambda$. Thus in both cases we have a contradiction, what completes the proof of Lemma 12.

Theorem 8. *Let $\emptyset \neq A \subset S$ be a compact convex set in a N-space $(S, \mathcal{F}, \min)$. Then*

$$\underset{\substack{M:\,A\to A\\ M \text{ is continuous}}}{\forall}\quad \underset{0<\epsilon,\lambda<1}{\forall}\quad \underset{\substack{M_{\epsilon,\lambda}:\,A\to A\\ M_{\epsilon,\lambda} \text{ is continuous and}\\ \text{finite dimensional}}}{\exists}\quad \underset{p\in A}{\forall}\; F_{Mp-M_{\epsilon,\lambda}p}(\epsilon) > 1-\lambda .$$

Proof. Let $\{y_i,\ 1 \leq i \leq k\} \subset A$ be $\epsilon, \lambda/2$-system for A. We define for $i = 1, 2, .., k$, $\mu_i : A \to R_0^+, p \to \max[0, \epsilon - f_{\lambda/2}(Mp - y_i)]$. By 1 from Lemma 11, we have

$$\mu_i(p) > 0 \iff f_{\lambda/2}(Mp - y_i) < \epsilon \iff F_{Mp-y_i}(\epsilon) > 1 - \lambda/2 .$$

From the definition of $\{y_i,\ 1 \leq i \leq k\}$ we have

$$\underset{p\in A}{\forall}\quad \underset{i(p)\in\{1,2,\ldots,k\}}{\exists}\quad \mu_{i(p)}(p) > 0 .$$

Since M is continuous, then the functions μ_i are continuous. Therefore, for $i \in \{1, 2, .., k\}$ the functions

$$\lambda_i : A \to \langle 0, 1\rangle, p \to \mu_i(p)/\Sigma_{j=1}^k \mu_j(p)$$

are also continuous and $\Sigma_{i=1}^k \lambda_i(p) = 1$. Define now $M_{\epsilon,\lambda}p = \sum_{i=1}^{k} \lambda_i(p)\, y_i$. Of course, $M_{\epsilon,\lambda}$ is finite dimensional and $M_{\epsilon,\lambda}(A) \subset \operatorname{conv}[y_1, .., y_k] \subset A$, hence $M_{\epsilon,\lambda} : A \to A$. $M_{\epsilon,\lambda}$ is continuous as it is continuous in a norm $\|\ \|$ and $\tau_{\epsilon,\lambda}$ is equivalent to the topology generated by the norm. Take now $p \in A$. Then, using above facts on λ and F, we have

$$F_{M_{\epsilon,\lambda}p-Mp}(\epsilon) = F_{\Sigma_{i=1}^k \lambda_i(p) y_i - Mp}(\epsilon) = F_{\Sigma_{s=1}^r \lambda_{i_s}(p) y_{i_s} - Mp}(\epsilon) =$$

$$= F_{\Sigma_{s=1}^r \lambda_{i_s}(p)(y_{i_s} - Mp)}(\epsilon) \geq \min_{s=1}^r (F_{\lambda_{i_s}(p)(y_{i_s}-Mp)}(\lambda_{i_s}\epsilon)) =$$

$$= \min_{s=1}^r (F_{y_{i_s}-Mp}(\epsilon)) \geq \min_{s=1}^r (1-\lambda/2) > 1-\lambda ,$$

where $\lambda_{i_s}(p) > 0$.

Theorem 9. *Let $\emptyset \neq A \subset S$ be a convex compact set in a M-space $(S, \mathcal{F}, \min)$. Then every continuous mapping $M : A \to A$ has a fixed point.*

Proof. Take $1 > \epsilon_n \searrow 0$, $1 > \lambda_n \searrow 0$. By Theorem 8 there exists a mapping $M_n : A \to A$ which is continuous and finite dimensional, and moreover,

$$\forall_{p \in A} F_{Mp - M_n p}(\epsilon_n) > 1 - \lambda_n .$$

Define $D_n = \overline{\text{conv}\, M_n(A)} \subset A$. Since A is convex, then $M_n(D_n) = M_n(\overline{\text{conv}\, M_n(A)}) \subset \overline{M_n(A)} \subset \overline{\text{conv}\, M_n(A)} = D_n$. Hence, $M_n : D_n \to D_n$, where D_n is compact and convex in a finite dimensional N-space.

Now, by Theorem 4, we get $\forall_{n \in N} \exists_{p_n \in A} M_n p_n = p_n$. But A is compact, therefore there exists a convergent subsequence $p_{n_k} \to p \in A$ as $k \to \infty$. Fix now $x > 0$. If $2\epsilon_{n_{k_0}} \leq x$, then

$$\forall_{k \geq k_0} F_{Mp-p}(x) \geq \min(F_{Mp - M_{n_k} p_{n_k}}(\epsilon_{n_k}), F_{p_{n_k} - p}(x/2)) \geq \min(1 - \lambda_{n_k}, F_{p_{n_k} - p}(x/2)).$$

It is obvious that $1 - \lambda_{n_k} \to 1$ as $k \to \infty$ and $F_{p_{n_k} - p}(x/2) \to 1$, $k \to \infty$, so that

$$\forall_{x > 0} F_{Mp-p}(x) = 1 \iff Mp = p .$$

Theorem 10. *Every compact mapping $M: S \to S$ on a complete N-space $(S, \mathcal{F}, \min)$ has a fixed point.*

Proof. We know that $\overline{M(S)}$ is compact. Hence, by Theorem 7, $A = \overline{\text{conv}\, \overline{M(S)}}$ is compact too, and

$$M(A) \subset \overline{M(S)} \subset \overline{\text{conv}\, \overline{M(S)}} = A .$$

Noting that M satisfies the assumption of Theorem 9, we end the proof.

4. In what follows the t-norm T will be always left continuous.

Definition 9. By the probabilistic distance between two nonempty sets A, B of a M-space $(S, \mathcal{F}, T)$ we mean the function $\text{dist}_{AB}(x) = \sup_{t < x} \sup_{\substack{p \in A \\ q \in B}} F_{pq}(t)$.

We see that $\text{dist}_{AB} \in \Delta^*$.

Lemma 13. $\text{dist}_{\bar{A}\bar{B}} = \text{dist}_{AB}$.

Proof. First we will show that $\text{dist}_{\bar{A}B} = \text{dist}_{AB}$. Of course, $\text{dist}_{\bar{A}B} \geq \text{dist}_{AB}$. Fix now $x > 0$, $\epsilon > 0$. Then there exist $p \in \bar{A}$, $q \in B$ and $t' < x$ for which

$$\text{dist}_{\bar{A}B}(x) < F_{pq}(t') + \epsilon/4 .$$

Since the set of points of continuity of F_{pq} is dense in $\mathcal{R}$, then there exists $t \leq t'$ such that F_{pq} is continuous in t and

$$F_{pq}(t') < F_{pq}(t) + \epsilon/4 .$$

Hence,

$$\mathrm{dist}_{\bar{A}B}(x) < F_{pq}(t) + \epsilon/2 .$$

But $p \in \bar{A}$, therefore there must exist $p_n \to p$ as $n \to \infty$, $p_n \in A$. The point t is a continuity point of the function, and consequently $F_{p_n q}(t) \to F_{pq}(t)$ as $n \to \infty$. Obviously, there exists a n_0 such that $F_{pq}(t) - \epsilon/2 < F_{p_{n_0} q}(t)$. Thus

$$\underset{x>0}{\forall}\ \underset{\epsilon>0}{\forall}\ \underset{\substack{p_{n_0} \in A \\ q \in B}}{\exists}\ \underset{t<x}{\exists}\ \mathrm{dist}_{\bar{A}B}(x) < F_{p_{n_0} q}(t) + \epsilon ,$$

i.e. $\mathrm{dist}_{\bar{A}B} \leqslant \mathrm{dist}_{AB}$. Moreover, we have

$$\mathrm{dist}_{\bar{A}\bar{B}} = \mathrm{dist}_{A\bar{B}} = \mathrm{dist}_{\bar{B}A} = \mathrm{dist}_{BA} = \mathrm{dist}_{AB} .$$

In [1] it has been defined the probabilistic Hausdorff distance between two nonempty bounded sets A, B of a M-space $(S, \mathcal{F}, T)$ in the following way

$$F^H_{AB}(x) = \sup_{t<x} T(\inf_{p \in A} \sup_{q \in B} F_{pq}(t), \inf_{q \in B} \sup_{p \in A} F_{pq}(t)) .$$

It was proved that $F^H_{AB} \in \Delta^*$, $F^H_{\bar{A}\bar{B}} = F^H_{AB}$, and that $(\mathfrak{M}, \mathcal{F}^H, T)$ is again a Menger space, where $\mathfrak{M}$ denotes the class of all nonempty, closed and bounded sets. We will prove now

Theorem 11. *If $(S, \mathcal{F}, T)$ is a complete M-space, then $(\mathfrak{M}, \mathcal{F}^H, T)$ is also a complete M-space.*

Proof. Suppose that for $\{A_n, n \geqslant 1\} \subset \mathfrak{M}$ we have

$$\underset{x>0}{\forall}\ F^H_{A_n A_m}(x) \to 1 \text{ as } m, n \to \infty .$$

We have to show that

$$\underset{A \in \mathfrak{M}}{\exists}\ \underset{x>0}{\forall}\ F^H_{AA_n}(x) \to 1 \text{ as } n \to \infty .$$

Notice that $F^H_{AA_n} \to H$ as $n \to \infty \iff$

$$\underset{x>0}{\forall}\ T(\inf_{p_n \in A_n} \sup_{p \in A} F_{p_n p}(x), \inf_{p \in A} \sup_{p_n \in A_n} F_{p_n p}(x)) \to 1 \text{ as } n \to \infty \iff$$

$$\underset{x>0}{\forall}\ \inf_{p_n \in A_n} \sup_{p \in A} F_{p_n p}(x) \to 1 \text{ as } n \to \infty \wedge \underset{x>0}{\forall}\ \inf_{p \in A} \sup_{p_n \in A_n} F_{p_n p}(x) \to 1 \text{ as } n \to \infty$$

$$\iff \forall_{0<\lambda<1, x>0} \exists_{n_{x,\lambda}} \forall_{n \geq n_{x,\lambda}} \forall_{p \in A} \exists_{p_n \in A_n} F_{p_n p}(x) > 1-\lambda \wedge \quad \text{(i)}$$

$$\forall_{0<\lambda<1, x>0} \exists_{n_{x,\lambda}} \forall_{n \geq n_{x,\lambda}} \forall_{p_n \in A_n} \exists_{p \in A} F_{p_n p}(x) > 1-\lambda . \quad \text{(ii)}$$

By the assumptions, we have

$$F^H_{A_m A_n} \to H \text{ as } m, n \to \infty \iff$$

$$\forall_{x>0} T(\inf_{p_m \in A_m} \sup_{p_n \in A_n} F_{p_m p_n}(x), \inf_{p_n \in A_n} \sup_{p_m \in A_m} F_{p_m p_n}(x)) \to 1 \text{ as } m, n \to \infty$$

$$\iff \forall_{x>0} \inf_{p_m \in A_m} \sup_{p_n \in A_n} F_{p_m p_n}(x) \to 1 \text{ as } n \to \infty \iff$$

$$\forall_{0<\lambda<1, x>0} \exists_{n_{x,\lambda}} \forall_{m, n \geq n_{x,\lambda}} \forall_{p_m \in A_m} \exists_{p_n \in A_n} F_{p_m p_n}(x) > 1-\lambda. \quad \text{(iii)}$$

Define $A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$. Then $\bar{A} = A$. Note that $p \in A \iff$

$p = \lim_{k\to\infty} p_{n_k}$, $p_{n_k} \in A_{n_k}$, n_k increases. Fix $0 < \lambda < 1$, $x > 0$. Ad (i): Take $0 < \lambda' < 1$ such that $T(1-\lambda', 1-\lambda') > 1-\lambda$ and next take p_{n_k} such that $F_{pp_{n_k}}(x/2) > 1-\lambda'$, for $n_k \geq n_{x/2,\lambda'}$. Then we take p_n for p_{n_k} from (iii) and we get

$$\forall_{0<\lambda'<1, x>0} \exists_{n_{x/2,\lambda'}} \forall_{n \geq n_{x/2,\lambda'}} \forall_{p \in A} \exists_{p_n \in A_n} F_{p_n p}(x) > T(F_{p_n p_{n_k}}(x/2),$$

$$F_{p_{n_k} p}(x/2)) \geq T(1-\lambda'; 1-\lambda') > 1-\lambda .$$

Ad (ii): Define for $j = 1, 2, .., x^{(j)} = x/2^{j+1}$, $x^{(0)} = \epsilon/4$, $T(1-\lambda^{(0)}, 1-\lambda^{(0)}) > 1-\lambda$; $T(1-\lambda^{(j)}, 1-\lambda^{(j)}) > 1-\lambda^{(j-1)}$. From (iii), we get

$$\exists_{n_0} \forall_{m_0, n \geq n_0} \forall_{p_n \in A_n} \exists_{p_{m_0} \in A_{m_0}} F_{p_{m_0} p_n}(x^{(0)}) > 1-\lambda^{(0)} ,$$

and

$$\exists_{n_1 > n_0} \forall_{m_0, m_1 \geq n_1} \forall_{p_{m_0} \in A_{m_0}} \exists_{p_{m_1} \in A_{m_1}} F_{p_{m_0} p_{m_1}}(x^{(1)}) > 1-\lambda^{(1)} .$$

Take $m_0 = n_1$ and suppose that n_{j-1}, $j = 2, 3, ...$ is defined. Then

$$\exists_{n_j > n_{j-1}} \forall_{m_j > n_j} \forall_{p_{n_j} \in A_{n_j}} \exists_{p_{m_j} \in A_{m_j}} F_{p_{n_j} p_{m_j}}(x^{(j)}) > 1 - \lambda^{(j)}.$$

Therefore, for every $p_n \in A_n$, $n \geqslant n_0$, there exists a sequence $\{p_{n_j}, j \geqslant 1\}$, n_j increases, $p_{n_j} \in A_{n_j}$ such that

$$F_{p_n p_{n_1}}(x^{(0)}) > 1 - \lambda^{(0)}, F_{p_{n_j} p_{n_{j+1}}}(x^{(j)}) > 1 - \lambda^{(j)}, j = 1, 2, \ldots .$$

We see that $\{p_{n_j}, j \geqslant 1\}$ is a Cauchy sequence. Take now $\epsilon^{(j_0)} \leqslant x$, $\lambda^{(j_0 - 1)} < \lambda$. Then

$$\forall_{l \in N} \forall_{j > j_0} F_{p_{n_j} p_{n_{j+l}}}(x) \geqslant F_{p_{n_j} p_{n_{j+l}}}(x^{(j)}) \geqslant$$

$$\geqslant T_{s=j}^{j+l-1}(F_{p_{n_s} p_{n_{s+1}}}(x^{(s)})) \geqslant T_{s=j}^{j+l-1}(1 - \lambda^{(s)}) > 1 - \lambda^{(j-1)} > 1 - \lambda .$$

Therefore the sequence $\{p_{n_j}, j \geqslant 1\}$ converges to $p \in A \neq \emptyset$.

We have to show yet that A is bounded. Fix $0 < \lambda < 1$, $q \in S$ and take $0 < \lambda', \lambda'' < 1$ such that $T(1 - \lambda', 1 - \lambda') > 1 - \lambda$, $T(1 - \lambda'', 1 - \lambda'') > 1 - \lambda'$. From (iii) we have

$$\forall_{0 < \epsilon, \lambda'' < 1} \exists_{n_{\epsilon, \lambda''}} \forall_{m, n > n_{\epsilon, \lambda''}} \forall_{p_n \in A_n} \forall_{p_m \in A_m} F_{p_n p_m}(\epsilon) > 1 - \lambda'' .$$

Fix now $m \geqslant n_{\epsilon, \lambda''}$. Then for an arbitrary $p \in A$ there exists $p_n \in A_n$, $n \geqslant n_{\epsilon, \lambda''}$ such that $F_{p p_n}(\epsilon) > 1 - \lambda''$. Since A_m is bounded, then

$$\forall_{0 < \lambda'' < 1} \exists_{x' > 0} \forall_{p_m \in A_m} F_{p_m q}(x') > 1 - \lambda'' .$$

Hence, for $x = x' + 2\epsilon$

$$\forall_{p \in A} F_{pq}(x) \geqslant T(F_{p p_n}(\epsilon), T(F_{p_m p_n}(\epsilon), F_{p_m q}(x'))) \geqslant$$

$$\geqslant T(1 - \lambda', T(1 - \lambda'', 1 - \lambda'')) > 1 - \lambda .$$

Now we prove that

$$\forall_{0 < \lambda < 1, x > 0} \exists_{n_0} \forall_{n \geqslant n_0} \forall_{p_n \in A_n} \exists_{p \in A} F_{p p_n}(x) > 1 - \lambda .$$

We see that

$$\forall_{j = 1, 2, \ldots} F_{p_n p}(x) \geqslant T(T(F_{p_n p_{n_1}}(x/4), T_{s=1}^{j}(F_{p_{n_s} p_{n_{s+1}}}(x^{(s)}))), F_{p_{n_{j+1}} p}(x/4)) .$$

Since $F_{p_{n_{j+1}}p}(\epsilon/4) \to 1$, then $F_{p_np}(x) \geq T(F_{p_np_{m_1}}(x/4), T^{\infty}_{s=1}(F_{p_{n_s}p_{n_{s+1}}}(x^{(s)})))$.

5. Let $\mathcal{M}$ denote the class of all compact sets different from $\emptyset$ and let $\mathcal{N}_0$ be the class of all finite sets different from $\emptyset$.

Theorem 12. *If a Menger space is complete, then $\overline{\mathcal{N}_0} = \mathcal{N}$ in $\tau^H_{\epsilon,\lambda}$.*

Proof. Necessity: Take $A \in \mathcal{N}$ and $1 > \epsilon_n \searrow 0$, $1 > \lambda_n \searrow 0$. Let $A_n \subset A$ be ϵ_n, λ_n-system for A. We note that

$$F^H_{A_nA} \to H \text{ as } n \to \infty \Longleftrightarrow \underset{0<\lambda<1,\, x>0}{\forall}\ \underset{n_0}{\exists}\ \underset{n \geq n_0}{\forall}\ \underset{p \in A}{\forall}\ F_{p_np}(x) > 1 - \lambda .$$

It follows from the definition of ϵ, λ-system by taking n_0 such that $\epsilon_{n_0} \leq x$, $\lambda_{n_0} \leq \lambda$.

Sufficiency: Suppose that $A_n \in \mathcal{N}_0$ and $F^H_{A_nA} \to H$ as $n \to \infty$, $A \in \mathcal{M}$. Then

$$\underset{0<\lambda<1,\, x>0}{\forall}\ \underset{n_0}{\exists}\ \underset{n>n_0}{\forall}\ \underset{p \in A}{\forall}\ \underset{p_n \in A_n}{\exists}\ F_{p_np}(x) > 1 - \lambda .$$

Thus A_{n_0} is ϵ, λ-system for A.

Definition 10. By the noncompactness measure of nonempty a bounded set $A \subset S$ we mean the function $\mu_A(x) = \mathrm{dist}^H_{\mathcal{N}_0\{A\}}(x)$.

Lemma 14. $\mu: \mathcal{M} \to \Delta^*$.

Proof. This fact follows from the properties of the distance.

Lemma 15. $\mu_{\overline{A}} = \mu_A$.

Proof. It is enough to note that $F^H_{\overline{A}\overline{B}} = F^H_{AB}$.

Lemma 16. $\mu_A = \mathrm{dist}^H_{\mathcal{N}\{A\}}$.

Proof. By Theorem 12 $\overline{\mathcal{N}_0} = \mathcal{N}$ and by Lemma 13 $\mathrm{dist}_{AB} = \mathrm{dist}_{\overline{A}\overline{B}}$ which prove Lemma 16.

Lemma 17. $A \subset B \Rightarrow \mu_A \geq \mu_B$.

Proof.
$$\mu_A(x) = \sup_{t<x} \sup_{A_0 \in \mathcal{N}_0} F^H_{A_0A}(t) =$$
$$= \sup_{t<x} \sup_{A_0 \in \mathcal{N}_0} \inf_{p \in A} \sup_{p_0 \in A_0} F_{pp_0}(t) \geq$$
$$\geq \sup_{t<x} \sup_{A_0 \in \mathcal{N}_0} \inf_{p \in B} \sup_{p_0 \in A_0} F_{pp_0}(t) = \mu_B(x) .$$

Lemma 18. *A is compact $\Longleftrightarrow \mu_A = H$.*

Proof.
$$A \in \mathcal{N} \Longleftrightarrow A \in \overline{\mathcal{N}_0} \Longleftrightarrow \underset{0<\lambda<1,\, x>0}{\forall}\ \underset{A_0 \in \mathcal{N}_0}{\exists}\ F^H_{A_0A}(x) > 1 - \lambda \Longleftrightarrow$$
$$\underset{x>0}{\forall}\ \sup_{A_0 \in \mathcal{N}_0} F^H_{A_0A}(x) = 1 \Longleftrightarrow \mathrm{dist}^H_{\mathcal{N}_0\{A\}} = \mathrm{dist}^H_{\mathcal{T}\{A\}} = H .$$

Theorem 13. *If the N-space $(S, \mathcal{F}, \min)$ is complete then $\mu_{\mathrm{conv}\,A} = \mu_A$.*

Proof. We know, by Lemma 16, that $\mu_{\mathrm{conv}A} \leqslant \mu_A$. Fix now $x > 0$, $\epsilon' > 0$. We will show that $\mu_A(x) \leqslant \mu_{\mathrm{conv}A}(x) + \epsilon'$. One has

$$\underset{0<\epsilon<x}{\exists}\ \underset{A_0=\{p_i,\ 1\leqslant i\leqslant k\}\in\mathfrak{N}_0}{\exists}\ \underset{p\in A}{\forall}\ \underset{p_{i(p)}\in A_0}{\exists}\ F_{p-p_{i(p)}}(x-\epsilon)\geqslant$$

$$\geqslant \mu_A(x) - \epsilon'.$$

But, by Theorem 3, conv A_0 is compact so that there exists $\epsilon/2$, λ-system $\{w_i,\ 1\leqslant i\leqslant n\}\subset \mathrm{conv}\,A_0$ such that $1-\lambda > \mu_A(x)-\epsilon'$, where

$$\mathrm{conv}\,A = [p = \sum_{j=1}^{r} \lambda^{(j)} p_j:\ \lambda^{(j)}\geqslant 0,\ \sum_{j=1}^{r}\lambda^{(j)} = 1,\ p_j\in A;\ r = 1, 2, \ldots]$$

$$\mu_{\mathrm{conv}\,A}(x) \geqslant \inf_{p\in\mathrm{conv}\,A}\ \sup_{w\in\{w_1, w_2, \ldots, w_n\}} F_{p-w}(x-\epsilon/2) =$$

$$= \min\left(F_{\Sigma_{j=1}^r \lambda^{(j)}(p_j - p_{i(p_j)})}(x-\epsilon),\ F_{\Sigma_{j=1}^r \lambda^{(j)} p_{i(p_j)} - w_i(\Sigma_{j=1}^r \lambda^{(j)} p_{i(p_j)})}(\epsilon/2)\right)\geqslant$$

$$\geqslant \min\left(\min_{j=1}^r \left(F_{\lambda^j(p_j - p_{i(p_j)})}\ \lambda^j(x-\epsilon)\right),\ F_{\Sigma_{j=1}^r \lambda^{(j)} p_{i(p_j)} - w_i(p)}(\epsilon/2)\right)\geqslant$$

$$\geqslant \min\left(\min_{j=1}^r \left(F_{p_j - p_{i(p_j)}}(x-\epsilon),\ 1-\lambda\right)\right)\geqslant \min(\mu_A(x)-\epsilon',\ 1-\lambda) = \mu_A(x)-\epsilon'.$$

Theorem 14. *Let $(S, \mathcal{F}, T)$ be a complete M-space and $\{A_n,\ n\geqslant 1\}\subset\mathfrak{M}$ a nonincreasing sequence with $\mu_{A_n}\to H$ as $n\to\infty$. Then $A = \bigcap_{n=1}^{\infty} A_n \in \mathfrak{N}$.*

Proof. First we show that $A\neq\emptyset$. Take $1>\epsilon_k\searrow 0$ and $1>\lambda_k\searrow 0$. We have

$$\mu_{A_n}\to H \text{ as } n\to\infty \iff \underset{0<\lambda_k<1,\,\epsilon_k>0}{\forall}\ \underset{A^0_{(n_k)}\in\mathfrak{N}_0}{\exists}\ \underset{p_{(n_k)}\in A_{(n_k)}}{\forall}\ F_{p_{n_k}p^0_{(n_k)}}(\epsilon_k) > 1-\lambda_k,$$

what implies that $A^0_{(n_k)}$ is ϵ_k, λ_k-system for A_{n_k}. Take now a sequence $\{p^{(0)}_{n_k},\ k\geqslant 1\}$, $p_{n_k}\in A_{n_k}$. Since $A^0_{(n_1)}$ is finite, then there exists a subsequence $\{p^{(1)}_{n_k},\ k\geqslant 1\}\subset\{p^{(0)}_{n_k},\ k\geqslant 1\}$ and $\bar p^{(0)}_{(n_1)}\in A^0_{(n_1)}$ such that $\{p^{(1)}_{n_k},\ k\geqslant 1\}\subset U_{\bar p^0_{(n_1)}}(\epsilon_1, \lambda_1)$.

Suppose that they are defined $\{p^{(l-1)}_{n_k},\ k\geqslant 1\}\subset\{p^{(l-2)}_{n_k},\ k\geqslant 1\}$ and $\bar p^0_{(n_{l-1})}\in A^0_{(n_{l-1})}$, $l\geqslant 2$. There exist a subsequence $\{p^{(l)}_{n_k},\ k\geqslant 1\}\subset\{p^{(l-1)}_{n_k},\ k\geqslant 1\}$

and $\bar{p}^0_{(n_l)} \in A^0_{(n_l)}$ such that $\{p_{n_k}, k \geqslant 1\} \subset U_{\bar{p}^0_{(n_l)}}(\epsilon_l, \lambda_l)$. We consider now the sequence $\{p^{(l)}_{n_l}, l \geqslant 1\}$. This a Cauchy sequence. Fix $0 < x, \lambda < 1$. We see that

$$\exists_{l_0} \forall_{l \geqslant l_0} \; T(1-\lambda_l, 1-\lambda_l) > 1-\lambda, \;\; 2\epsilon_l \leqslant x .$$

Then

$$\forall_{l \geqslant l_0} \forall_{j = 1, 2, \ldots} F_{p^{(l)}_{n_l} p^{(l+j)}_{l+j}}(x) \geqslant T(F_{p^{(l)}_{n_l} \bar{p}^0_{(n_l)}}(\epsilon), F_{\bar{p}^0_{(n_l)} p^{(l+j)}_{n_{l+j}}}(\epsilon))$$

$$T(1-\lambda_l, 1-\lambda_l) > 1-\lambda .$$

Therefore $\{p^{(l)}_{n_l}, l \geqslant 1\}$ has the limit point $p \in S$. Since $\{p^{(l)}_{n_l}, l \geqslant 1\}$ is a subsequence of $\{p_{n_k}, k \geqslant 1\}$, $p_{n_k} \in A_{n_k}$, and $A_{n_k}\searrow$ are closed, then $p \in \bigcap_{n=1}^{\infty} A_n = A \neq \emptyset$. Of course $A = \bar{A}$. It is also true that A is compact. Since $\mu_{A_n} \leqslant \mu_{A_{n+1}} \leqslant \ldots \leqslant \mu_A$ and $\mu_{A_n} \to H$ as $n \to \infty$, then $\mu_A = H$. From Lemma 18, $A \in \mathfrak{N}$.

Theorem 15. *Let $(S, \mathcal{F}, \min)$ be a complete N-space. If $C \in \mathfrak{M}$ is convex and the mapping $M: C \to C$ is continuous and*

$$\exists_{k \in (0,1)} \forall_{C \supset A \in \mathfrak{M}} \forall_{x > 0} \mu_{M(A)}(x) \geqslant \mu_A(x/k) ,$$

then M has a fixed point.

Proof. Define $C_0 = C$, $C_{n+1} = \overline{\text{conv}\, M(C_n)}$, $n = 0, 1, 2, \ldots$. Of course, $C_n \in \mathfrak{M}$ are convex. We are going to show that $M(C_n) \subset C_n$.

It is obvious that $M(C_0) \subset C_0$. Suppose that $M(C_{n-1}) \subset C_{n-1}$. Then $M(C_n) = M(\overline{\text{conv}\, M(C_{n-1})}) \subset M(\overline{\text{conv}\, C_{n-1}}) = M(C_{n-1}) \subset \overline{\text{conv}\, M(C_{n-1})} = C_n$. Therefore,

$$C_{n+1} = \overline{\text{conv}\, M(C_n)} \subset M(C_n) \subset C_n .$$

We show now that $\mu_{C_n} \to H$ as $n \to \infty$. Note that $\forall_{x>0} \mu_{C_{n+1}}(x) = \mu_{\overline{\text{conv}\, M(C_n)}}(x) = \mu_{\text{conv}\, M(C_n)}(x) \geqslant \mu_{C_n}(x/k) \geqslant \ldots \geqslant \mu_{C_0}(x/k^{n+1}) \to 1$ as $n \to \infty$, since $\mu_{C_0} \in \Delta^*$.

Therefore, $C_\infty = \bigcap_{n=0}^{\infty} C_n \in \mathfrak{N}$. This follows from facts $C_n\searrow$, $C_n \in \mathfrak{M}$, $\mu_{C_n} \to H$ as $n \to \infty$, and from Theorem 14. C_∞ is convex as C_n are convex. Note that

$$\forall_{n=0,1,\dots} \quad M(C_\infty) \subset M(C_n) \subset \operatorname{conv} M(C_n) = C_{n+1} \cdot$$

Hence,

$$M(C_\infty) \subset \bigcap_{n=1}^{\infty} C_n = \bigcap_{n=0}^{\infty} C_n = C_\infty.$$

Thus we can apply here the Theorem 9, and this completes the proof.

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STRESZCZENIE

Praca zawiera twierdzenia o punkcie stałym dla ciągłych odwzorowań na przestrzeniach zupełnych, unormowanych według prawdopodobieństwa. Uzyskane wyniki uogólniają pewne klasyczne twierdzenia o punktach stałych.

РЕЗЮМЕ

Работа содержит теоремы о неподвижной точке для непрерывных преобразований на полных, нормированных по вероятности пространствах. Полученные результаты обобщают некоторые классические теоремы.