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Zakład Zastosowań Matematyki
Akademia Rolnicza, Lublin

Mirosława WESOŁOWSKA

Some Aspects of the Theory of Experimental Desings

Pewne aspekty teorii układów eksperymentalnych

Некоторые аспекты теории экспериментальных схем

1. Introduction. The incomplete bloc design was introduced by Yates [27]. The designs aroused interest because of their usefulness in practice. The properties of known designs were investigated and the possibility of existence of new designs was considered by many authors such as Cochran and Cox [7], Federer [9], Kempthore [12], Chakrabarti [6], Finney [10].

Many statisticians are interested in the construction of new experimental designs and their properties.

The contruction of some useful designs, their analysis and application is given by Caliński [3]. The designs with unequal number of experimental units in blocks are based on the known BIB designs.

Other designs which are called diagonal ones are presented by Nawrocki [14]. He considers incomplete block designs in which one special treatment appears the same number of times in each block. The analysis made by the projection operators method of those designs is simple and not dependent on the dividing of the treatments among the blocks. Moreover, the number of replication is not dependent on the number of treatments.

The general theory of the incomplete block designs for a fixted and mixed model is given by C.R. Rao [20] and for a fixed model by W. Oktaba [17].

Some of particular cases of the incomplete block designs are the inter and intra-group balanced incomplete block designs. They are given by L. C. A. Corsten [8].

Another approach to the general theory of the incomplete block designs is presented by Tocher [26]. The properties of the designs are investigated by means of the coveriance matrix of the adjusted treatment means. This approach is continued by Rees [22], Caliński [3], Caliński and Ceranka [4], Ceranka [5].

The general theory of orthogonality and connectednes in presented in this paper. Moreover the problem of balancing in the incomplete block designs is considered here.

2. Notation and general model of the incomplete block designs. We shall consider the matrix of the form

$$\underset{n1}{y} = \underset{np}{X}\ \underset{p1}{\beta} + \underset{n1}{e} \tag{2.1}$$

where y is the $n \times 1$ random observation vector, X is the $n \times p$ design matrix of the rank $r < p$ and β is the $p \times 1$ vector of fixted parameter. The $n \times 1$ vector e of random errors is distributed with $\epsilon(e) = 0$ and $\epsilon(ee) = \sigma^2 I_n$ (I_n denotes a $n \times n$ unit matrix).

The $p \times 1$ vector of parameters is partitioned such that

$$\beta' = [\beta_1' \vdots \beta_2' \vdots \dots \vdots \beta_s'], \tag{2.2}$$

where β_i ($i = 1, 2, \dots, s$) is the $b_i \times 1$ subvector of parameters of the i-th group and $\sum_{i=1}^{s} b_i = p$. According to (2.2) the matrix X is partitioned such that the (2.1) is

$$y = X_1\beta_1 + X_2\beta_2 + \dots + X_s\beta_s + e \tag{2.3}$$

The least square estimators of every group of parameters are given by solution of the normal equation

$$X'X\tilde{\beta} = X'y \tag{2.4}$$

which according to (2.2) and (2.3) are:

$$\begin{bmatrix} X_1'X_1 & X_1'X_2 \dots X_1'X_s \\ \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \\ X_s'X_1 & X_s'X_2 \dots X_s'X_s \end{bmatrix} \cdot \begin{bmatrix} \tilde{\beta}_1 \\ \vdots \\ \tilde{\beta}_s \end{bmatrix} = \begin{bmatrix} X_1'y \\ \dots \\ X_s'y \end{bmatrix}, \tag{2.5}$$

where the $p \times p$ matrix of $X'X$ is of non full rank.

Hence the unique estimators of the unknown parameters can be obtained under restrictions in the form of:

$$H\beta = 0 \tag{2.6}$$

whereas the ununique estimators which we can obtain using $(X'X)^-$ a generalized inverse of matrix $X'X$ such that $(X'X)(X'X)^-(X'X) = X'X$ as

$$\tilde{\beta} = (X'X)^- X'y \tag{2.7}$$

The incomplete block desings constitute a special group of the experimental desings.

These are the desings in which the number of different treatments v is greater than a number of k experimental units per block ($k < v$).

Let us consider the general model of the incomplete block designs where v is a number of treatments, b is a number of blocks, k_j is a number of experimental units in the j'th block ($j = 1, 2, \dots, b$), r_i is a number of the replication for the i'th treatment ($i = 1, 2, \dots, v$). The arrangement of the treatments is such that every pair of treatments (i, i') occurs together in exactly $\lambda_{ii'}$ blocks for each $i, i' = 1, 2, \dots, v$. If $i = i'$, then $\lambda_{ii'} = r_i$.

If the n_{ij} denotes a number of occurence of the i'th treatment in the j'th block ($i = 1, \dots, v; j = 1, \dots, b$) then the following relations are true

(2.8) $$k_j = \sum_{i=1}^{v} n_{ij} \quad \text{and} \quad r_i = \sum_{j=1}^{b} n_{ij}$$

The number of the all experimental units is equal to

(2.9) $$n = \sum_{i=1}^{v} \sum_{j=1}^{b} n_{ij} = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$$

Definition 2.1. The $v \times b$ matrix

(2.10) $$N = \underset{vb}{N} = [n_{ij}]$$

is called the incidence matrix of design.

It should be noted that

(2.11) $$\underset{1v}{E}\,\underset{vb}{N} = [k_1, k_2, \dots, k_b] = \underset{1b}{E}\,\underset{bb}{K}$$

and

(2.12) $$(\underset{vb}{N}\,\underset{b1}{E})' = \underset{1b}{E}'\,\underset{bv}{N}' = [r_1, r_2, \dots, r_v] = \underset{1v}{E}\,\underset{vv}{R}$$

where $\underset{pq}{E}$ is a $p \times q$ matrix of ones, and

(2.13) $$\underset{bb}{K} = \operatorname{diag}(k_1, k_2, \dots, k_b), \quad \underset{vv}{R} = \operatorname{diag}(r_1, r_2, \dots, r_v)$$

If y_{ijh} is the h observation for the i'th treatment in the j'th block, then

$$B_j = \sum_{i=1}^{v} \sum_{h=1}^{n_{ij}} y_{ijh}$$

for each $j = 1, 2, \dots, b$ is a sum of the results for in the j'th block,

$$T_i = \sum_{j=1}^{b} \sum_{h=1}^{n_{ij}} y_{ijh}$$

for each $i = 1, 2, \dots, v$ is a sum of the results for the i'th treatment and

$$Y = \sum_{i=1}^{v} \sum_{j=1}^{b} \sum_{h=1}^{n_{ij}} y_{ijh} \qquad \text{is a grand total.}$$

The vectors of the block totals and the treatment totals will be used respectively

$$(2.14) \qquad \underset{1b}{B'} = [B_1, B_2, \dots, B_b] \quad \text{and} \quad \underset{1v}{T'} = [T_1, T_2, \dots, T_v],$$

such that

$$(2.15) \qquad Y = \underset{1b}{E}\,\underset{b1}{B} = \underset{1v}{E}\,\underset{v1}{T} \quad \text{and} \quad Y = \underset{1n}{E}\,\underset{n1}{y}, \quad \underset{b1}{B} = X_1' y, \quad \underset{v1}{T} = X_2' y$$

In this notation, the general linear model of observations (2.3) of the incomplete block experiment has a form

$$(2.16) \qquad y = [E \vdots X_1 \vdots X_2] \cdot [\mu \vdots \alpha' \vdots \tau']' + e$$

where y is the $n \times 1$ observation vector, $X = [E \vdots X_1 \vdots X_2]$ is the $n \times (1 + b + v)$ design matrix, partitioned into the $n \times 1$ vector E of unit elements, the $n \times b$ matrix X_1, the $n \times v$ matrix X_2, where μ is the general means, α is the $b \times 1$ vector of block parameters, τ is the $v \times 1$ vector of treatment parameters, and where the $n \times 1$ vector e of random errors is distributed with $\epsilon(e) = 0$ and $\epsilon(ee) = \sigma^2 I_n$.

Then $\epsilon(y) = X, \beta = E\mu + X_1\alpha + X_2\tau$ and $\Sigma_y = \sigma^2 I_n$, where Σ_y denotes a covariance matrix of y.

Using (2.15) we get the normal equation:

$$(2.17) \qquad \begin{bmatrix} nI_1 & \underset{1b}{E}K & \underset{1v}{E}R \\ \underset{b1}{KE} & K & N' \\ \underset{v1}{RE} & N & R \end{bmatrix} \cdot \begin{bmatrix} \tilde{\mu} \\ \tilde{\alpha} \\ \tilde{\tau} \end{bmatrix} = \begin{bmatrix} Y \\ B \\ T \end{bmatrix}$$

It is easy to obtan from (2.17) (see Chakrabarti [6]) the equation for treatment parameters

$$(2.18) \qquad Q = C\tilde{\tau}$$

where

$$(2.19) \qquad \underset{1v}{Q''} = [Q_1, Q_2, \dots, Q_v] = (T - NK^{-1}B)'$$

is the vector of the adjusted treatment such that

(2.20) $$\epsilon(Q) = C\tau, \quad \Sigma_Q = C\sigma^2 \text{ and } \underset{1\nu}{E} \underset{\nu 1}{Q} = 0,$$

and the $\nu \times \nu$ matrix C is

(2.21) $$C = R - NK^{-1}N'$$

By this means the reduced normal equations for the block parameters may be obtained too

(2.22) $$P = D\tilde{\alpha}$$

where the $b \times 1$ vector P of the adjusted block totals is equal to

(2.23) $$P' = [P_1, P_2, \dots, P_b] = (B - N'R^{-1}T)'$$

The following conditions are true

(2.24) $$\epsilon(P) = D\alpha, \quad \Sigma_P = D\sigma^2 \text{ and } \underset{1b}{E} \underset{b1}{P} = 0,$$

where

(2.25) $$D = K - N'R^{-1}N$$

is the $b \times b$ matrix.

The solutions of equations (2.18) and (2.22) are respectively

(2.26) $$\tilde{\tau} = C^{-}Q \quad \text{and} \quad \tilde{\alpha} = D^{-}P$$

where C^{-} and D^{-} are the generalized inverses of matrix C and of D.

Using the condition

(2.27) $$\underset{1b}{E} K\alpha = \underset{1\nu}{E} R\tau = 0$$

we may obtain unique solutions of normal equations (2.18) and (2.22).

3. Connected experimental designs. Estimability of parametric functions. The definition of connected experimental design was given by B. V. Shah. The problem of connectedness is associated with an estimability of parametric functions. The definition and a necessary and sufficient condition for an estimability of the parametric function was presented by Zyskind [28], Chakrabarti [6], and Graybill [11].

In this part of the paper a necessary and sufficient condition for an estimability of a linear function of the treatment effects for the incomplete block designs is given.

The following theorem will be used in a further consideration.

Theorem 3.1. *A necessary and sufficient condition of a linear estimability of a parametric function $L'\beta$ assuming model* (2.1) *is*

$$L' = L'(X'X)^{-}(X'X) \tag{3.1}$$

Definition 3.1. The linear parametric function $L'\beta$ in the model (2.1) for which

$$\underset{1p}{E}\ \underset{p1}{L} = 0 \tag{3.2}$$

is called a contrast.

In the incomplete block designs, the block function $L_1'\alpha$ and the treatment function $L_2'\tau$ may be considered separately.
Then:

Definition 3.2. The linear parametric function $L_2'\tau$ is called a treatment contrast if $\underset{1v}{E}\ \underset{v1}{L_2} = 0$.

The block contrast can be defined similarly.
Thus, we shall prove the following theorem.

Theorem 3.2. *A necessary and sufficient condition for a treatment parametric function $L_2'\tau$ to be linearly estimable is that*

$$L_2' = L_2' C^{-} C \tag{3.3}$$

Proof. An L' may be written in the form $L' = [0 \vdots 0 \vdots L_2']$. In order to obtain a generalized inverses matrix $(X'X)^{-}$ of known matrix from from thr normal equation (2.17) we shall use a formula (see Bhimasankaram [2])

$$\begin{bmatrix} A & B \\ B' & D \end{bmatrix}^{-} = \begin{bmatrix} A^{-} + LM^{-}L' & -LM^{-} \\ -M^{-}L' & M^{-} \end{bmatrix} \tag{3.4}$$

where $M = D - B'A^{-}B$, $L = A^{-}B$ and the all generalized inverse of matrices satisfy a condition

$$AA^{-}A = A \tag{3.5}$$

Let now A^{-} from (3.4) will be equal

$$A^{-} = \begin{bmatrix} 0 & 0 \\ 0 & K^{-1} \end{bmatrix}$$

Then it is easy to veryfy that $M = C$, then $M^- = C^-$ and

$$LM^- = \left[\begin{array}{c} 0 \\ \hline K^{-1}N'C^- \end{array}\right], \quad A^- + LM^-L' = \left[\begin{array}{c:c} 0 & 0 \\ \hline 0 & K^{-1} + K^{-1}N'C^-NK^{-1} \end{array}\right]$$

Hence

$$(3.6) \qquad (X'X)^- = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hline 0 & K^{-1} + K^{-1}N'C^-NK^{-1} & -K^{-1}N'C^- \\ \hline 0 & -C^-NK^{-1} & C^- \end{array}\right]$$

Substituting the maztix $(X'X)$ from (2.17) and (3.6) into (3.1), we obtain

$$[0 \vdots 0 \vdots L_2'] = [0 \vdots 0 \vdots L_2'C^-C]$$

or

$$L_2' = L_2'C^-C$$

and the theorem is proved.

The theorem 3.2 is valid for treatment contrasts, i.e. when the additional condition $\underset{1v}{E}\ L_2 = 0$ is satysfied.

The estimability condition concerning a linear function of block effects may be stated similarly.

In the experimental design with the matrix X of non full rank, the number of linearly independent estimable parametric functions $L'\beta$ is equal to the rank of the matrix X.

We may also consider the linearly independent parametric functions, which are estimable, and in particular the nonestimable parametric contrasts.

The latter one for the treatment parameters can be obtained from

$$(3.7) \qquad \left[\begin{array}{c} C \\ \hline \underset{1v}{E} \end{array}\right] L_2 = 0$$

because vectors L_2 of the unestimable functions belong to the orthocomplement of the subspace generated by the columns of C, while all vectors L_2 of the estimable functions are the linear combinations of the column vectors of C, i.e. the vector L_2 belongs to the space generated by the columns of C (see formula (3.3)).

The number of nonestimable contrast or linearly independent vectors L_2 satisfying the relation (3.7) is equal to a number of solutions of linear homogeneous equations (3.7).

That is:

$$q = \nu - r\begin{bmatrix} C \\ -- \\ E \\ 1\nu \end{bmatrix}, \tag{3.8}$$

where

$$r = \begin{bmatrix} C \\ -- \\ E \end{bmatrix} = r(CC') + 1 = r(C) + 1 \tag{3.9}$$

Therefore

$$q = \nu - r(C) - 1 \tag{3.10}$$

We now note, that if $r(C) = \nu - 1$, then all contrast will be estimable, because $q = 0$ and if $r(C) = \nu - m$, we have

$$q = m - 1 \tag{3.11}$$

unestimable contrasts.

The estimability of the block contrasts may be considered in the same way.

The above consideration leads to the following theorems.

Theorem 3.3. *If $r(C) = \nu - 1$, then a necessary and sufficient condition for every treatment parameter function $L_2'\tau$ to be estimable is to be a contrast.*

However, with respect to (see Chakrabarti [6]) that

$$r(X) = r(X'X) = b + r(C) = \nu + r(D) \tag{3.12}$$

if $r(C) = \nu - 1$, then $r(D) = b - 1$, and $r(X'X) = \nu + b - 1$, thus, we have the following theorem:

Theorem 3.5. *A necessary and sufficient condition for every block contrast and treatment contrast to be estimable is that $r(C) = \nu - 1$.*

The general definition we use is as follows.

Definition 3.3. The design with every block contrasts and treatment contrasts beign estimable is connected.

So the design is connected, if $r(C) = \nu - 1$.

For example BIB design may be given. It is easy to verify that this is a connected design. The rank of $\nu \times \nu$ matrix C

(3.13) $$C = \lambda_v/k\,(I_v - 1/v \underset{v,v}{E})$$

is equal to v-1, because the matrix $(I_v - 1/v \underset{v,v}{E})$, I_v and $1/v \underset{v,v}{E}$ are idempotent.

It may be easy to show also, that the design with the complete confounding of ABC interaction in threefactor experiments is not connected. Let m be the number of blocks in the complete replication. Then, the rank of the matrix C of the treatment normal equation form

(3.14) $$\underset{v,v}{C} = \operatorname{diag}\,(rI_k - r/k \underset{k,k}{E}, \ldots, rI_k - r/k \underset{k,k}{E})$$

is equal to v-m, where $v = mk$ and $b = mr$.

4. Balanced design. The term balance (with respect to treatment effects) is widely used in the literature and its meaning in relation to the usual block treatment experiment is the following (see B. V. Shah [25]).

Definition 4.1. The design all the treatment contrasts of which $(\tau_i - \tau_j)$ are estimated with the same variance is said to be balanced.

They are the designs, where a comparison of every pair of treatments is accomplished with the same precision.

The conditions of balance in the general incomplete block design and some conclusions were given by V. R. Rao [21].

Theorem 4.1. *A necessary and sufficient condition for a design to be balanced is that matrix C of the adjusted normal equations for estimates of treatment effects has v-1 equal latent roots other zero.*

Corollaries:

1) *If the design is balanced, then the matrix C has the form*

(4.1) $$C = a\,[I_v - 1/v \underset{v,v}{E}],$$

where a is v-1 – multiple eigenvalue of C and then

(4.2) $$\tilde{\tau} = 1/a\, Q$$

2) *In a balanced design with equal block sizes k, the replicate numbers must be equal.*

3) *If all the treatments are replicated the same number of times and the blocks are of the same size the only balanced design is Balance Incomplete Block Design, if such a design exists.*

From this if appears that it is a necessary and sufficient condition for the design to be balanced that all diagonal elements of matrix C are equal and the remaining elements also are equal.

In the balanced incomplete block design the latter condition is that every two treatments occur together in λ blocks. The elements matrix of C,

$$C = rI_v - 1/k\, NN' \tag{4.3}$$

are

$$c_{ii} = r - 1/k \sum_{j=1}^{b} n_{ij}^2 \quad \text{and} \quad c_{ii'} = -1/k \sum_{j=1}^{b} n_{ij} n_{i'j} \tag{4.4}$$

Since n_{ij} is equal 0 or 1, we have $\sum_{j=1}^{b} n_{ij}^2 = r$ and $c_{ii'}$ will be constant when $\sum_{j=1}^{b} n_{ij} n_{i'j} = \text{const}$, that is when for every $i, i' = 1, 2, \ldots, v$ $(i \neq i')$ treatments τ_i and $\tau_{i'}$ occur together in the same number of blocks, denoted by λ.

We notice that balanced lattices, lattice squares and Youden squares belong to the incomplete blocks which are balanced.

The corollary 1 was generalized by B. V. Shah for factorial experiments in the incomplete block design.

5. **Orthogonal design.** The 'orthogonality' of designs is closely associated with the simplicity and effectiveness of the statistical methods. We know, that sums of squares for individual hipothesis are independent in the orthogonal designs.

It is frequently (see Oktaba [15], Ahrens [1]) accepted that the orthogonal design is a design with equal or proportional number of the observations in all the subclasses of the suitable cross classification, or the same number of subclasses and equal number of observations in the hierachical classification.

Graybill [11] gas given the following definition of the design orthogonality.

Definition 5.1. If $X_1' X_2 = 0$ in the full rank model of the experimental design $y = X_1 \beta_1 + X_2 \beta_2 + e$, then a vector of the parameter β_1 will be said to be orthogonal to β_2.

This definition was generalized by Kempthorne [12] and Oktaba [16] in a model with an optional number group of parameters. Consider the model (2.3).

The estimator of β parameter are obtained from the normal equation (2.5) where the matrix $S = X'X$ is of full rank, then

$$\tilde{\beta} = S^{-1} X' y \tag{5.1}$$

and the covariance matrix of $\tilde{\beta}$ is equal

$$\Sigma_{\tilde{\beta}} = S^{-1} \sigma^2 \tag{5.2}$$

Consequently, $X_i'X_j = 0$ for every $i, j = 1, 2, \ldots, s$ and $i \neq j$ means, that the estimators of the individual parameters groups are uncorrelated, because the S^{-1} matrix is

$$(5.3) \qquad S^{-1} = \begin{bmatrix} S^{11} & 0 & \ldots & 0 \\ 0 & S^{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & S^{ss} \end{bmatrix}$$

The above condition is especially valid in the analysis of variance and in the verification of hypotheses, since it permits to divide the total sum of square for the regression into a sum of squares for the individual source of variance corresponding to the separate groups of parameters. Then the following condition is performed. When the hypothesis $H_0 : \beta_i = 0$ $(i = 1, 2, \ldots, s)$ is true, the sum of squares $nS^2_{\beta_i} = y'A_iy$ is distributed as $X^2{}_{\nu_i}\sigma^2$ with the digrees of freedom $\nu_i = r(A_i)$.

In a non full rank model this problem is more complicated. The matrices $X_i'X_j$ for $i \neq j$ are not in general zero-matrix. The unique solution of normal equation (2.4), where $r(S) = r < p$ may be obtained after putting the restriction in the separate group of parameters. That is why the above definition has to be changed and the following one is formed.

Definition 5.2. In the non full rank model (2.3) vectors of each group parameter are orthogonal if in the matrix of normal equation $U'U$ under the parameter restriction, the submatrices $U_i'U_j$ are zero-matrix for every $i, j = 1, 2, \ldots, s$, that is $U_i'U_j = 0$.

On the other hand, the following definition of orthogonality is given by B. V. Shah [24].

Definition 5.3. The experimental design with s parameter groups (the parameters may not be linearly independent with each group), is called orthogonal, if

$$(5.4) \qquad \Sigma_{\tilde{\beta}_i, \tilde{\beta}_j} = 0 \quad (i \neq j; i, j = 1, 2, \ldots, s)$$

From above definition follows, that if $\tilde{\beta}_i$ and $\tilde{\beta}_j$ are normaly distributed, then these estimators are stochastic independent.

The orthogonality of design is closely associated with an imposing restriction, since the form of the matrix $U'U$ and estimators of all groups of parameter depend on it.

Let us partition the vector β as in (2.2). The experimental design is an orthogonal one according to the definition 5.3, when the $\Sigma_{\tilde{\beta}}$ is:

$$(5.5) \qquad \Sigma_{\tilde{\beta}} = \begin{matrix} \Sigma_{\tilde{\beta}_1} & & & \\ & \Sigma_{\tilde{\beta}_2} & & 0 \\ 0 & & \ddots & \\ & & & \Sigma_{\tilde{\beta}_s} \end{matrix}$$

Using a well known formula given by Plackett [18], the design will be orthogonal if under the restriction $\underset{1p}{H}\ \underset{p1}{\beta}=0$ and the assumption $\Sigma_y=\sigma^2 I_n$, the covariance matrix β

(5.6) $$\Sigma_{\tilde{\beta}}=(X'X+H'H)^{-1}X'X(X'X+H'H)^{-1}\sigma^2$$

is a block-diagonal as in (5.5).

Theorem 5.1. *If an experimental design with the model* (2.3) *under the restriction* $H\beta=0$ *is orthogonal according to definition* 5.2, *then it is also orthogonal according to definition* 5.3 *and the other way round.*

Proving theorem 5.1 we will use the following theorem (see C. R. Rao [19]).

Theorem 5.2. *Let the rank of the* $p\times m$ *matrix* B *be* m, *the rank of the* $n\times q$ *matrix* C *be* n *and the matrix* A *be of order* $m\times n$. *Then* $BAC=0$ *if and only if* $A=0$.

Proof of the theorem 5.1. Let the vector β and the corresponding matrix X be divided as in (2.3). The β_1 is a general mean and the vectors β_i for $i=1,2,\dots,s$ are of order $b_i\times 1$ (then $b_1=1$). The restriction $H\beta=0$ are imposed on the $s-1$ groups of parameters and allow for the finding of one or a few parameters from each group, as a linear combination of independent parameters in the same group. After such a reparametrization, the model (2.1) is

(5.7) $$y=U\beta^*+e,$$

where matrix U is of order $n\times t$, β^* is $t\times 1$ vector and t is the number of independent parameters in all the groups together. And further on:

(5.8) $$U=[U_1\ \vdots\ U_2\ \vdots \dots \vdots\ U_s]\quad\text{and}\quad \beta^{*\prime}=[\beta_1^{*\prime}\ \vdots\ \beta_2^{*\prime}\ \vdots \dots \vdots\ \beta_s^{*\prime}]$$

and each β_i^* is of $q_i\times 1$ vector ($i=1,2,\dots,s$), then $t=\sum_{i=1}^{s} q_i$ and $q_1=1$.

The normal equations now are

(5.9) $$U'U\tilde{\beta}^*=U'y$$

and with respect to (5.8)

(5.10) $$U'U=\begin{bmatrix} U_1'U_1 & U_1'U_2 & \dots & U_1'U_s\\ U_2'U_1 & U_2'U_2 & \dots & U_2'U_s\\ \dots & \dots & \dots & \dots\\ U_s'U_1 & U_s'U_2 & \dots & U_s'U_s\end{bmatrix}\quad\text{and}\quad U'y=\begin{bmatrix}U_1'y\\ U_2'y\\ \dots\\ U_s'y\end{bmatrix}$$

In definition 5.2 is stated that the design with the model (2.1) is orthogonal one, if

(5.11) $$U_i'U_j = 0 \quad \text{for every} \quad i, j = 1, 2, \dots, s \quad \text{and} \quad i \neq j$$

The matrix $U'U$ is of full rank, therefore

(5.12) $$\tilde{\beta}^* = (U'U)^{-1}U'y$$

and by (5.11) we obtain

(5.13) $$\tilde{\beta}_i^* = (U_i'U_i)^{-1}U_i'y \quad \text{for every} \quad i = 1, 2, \dots, s$$

Let now P_i be a matrix of order $a_i \times q_i$, where $a_i = b_i - q_i$ $(i = 1, 2, \dots, s)$ such that

(5.14) $$\beta_i^{**} = P_i \beta_i^*$$

and β_i^{**} is a $a_i \times 1$ vector of dependent parameters from i'th group. Then

(5.15) $$\tilde{\beta}_i = \begin{bmatrix} \tilde{\beta}_i^* \\ \hline \tilde{\beta}_i^{**} \end{bmatrix} = \begin{bmatrix} I_{q_i} \\ \hline P_i \end{bmatrix} \cdot \tilde{\beta}_i^*$$

and from (5.13) we obtain

(5.16) $$\tilde{\beta}_i = \begin{bmatrix} I_{q_i} \\ \hline P_i \end{bmatrix} (U_i'U_i)^{-1}U_i'y, \quad i = 1, 2, \dots, s$$

Hence, the covariance matrix is equal to

$$\Sigma_{\tilde{\beta}_i, \tilde{\beta}_j} = \Sigma \left[\begin{bmatrix} I_{q_i} \\ \hline P_i \end{bmatrix} (U_i'U_i)^{-1}U_i'y, \begin{bmatrix} I_{q_j} \\ \hline P_j \end{bmatrix} (U_j'U_j)^{-1}U_j'y \right] =$$

$$= \frac{I_{q_i}}{P_i}(U_i'U_i)^{-1}U_i'U_j(U_j'U_j)^{-1}[I_{q_j} \vdots P_j]\sigma^2$$

Using theorem 5.2, we get $\Sigma_{\tilde{\beta}_i, \tilde{\beta}_j} = 0$ since $U_i'U_j = 0$, that means the condition of orthogonality in the definition 5.3 is satisfied.

Now, we assume the condition (5.4). It is well known, that the normal equation (5.9) from the model (5.7) under the restriction $H\beta = 0$, has the solution (5.12).

Let the matrix $(U'U)^{-1} = Z$, then

(5.17) $$\tilde{\beta}^* = ZU'y$$

and according to (5.8) the solution of the normal equation (5.9) is

$$\begin{bmatrix} \tilde{\beta}_1^* \\ \tilde{\beta}_2^* \\ \cdots \\ \tilde{\beta}_s^* \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \dots & Z_{1s} \\ Z_{21} & Z_{22} & \dots & Z_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ Z_{s1} & Z_{s2} & \dots & Z_{ss} \end{bmatrix} \cdot \begin{bmatrix} U_1' \\ U_2' \\ \cdots \\ U_s' \end{bmatrix} y \tag{5.18}$$

Further we denote

$$Z_i = [Z_{i1}, Z_{i2}, \dots, Z_{is}] \quad \text{for } i = 1, 2, \dots, s$$

For every $i = 1, 2, \dots, s$ estimators $\tilde{\beta}_i^*$ are then given by

$$\tilde{\beta}_i^* = Z_i U' y \tag{5.19}$$

Substituting (5.19) into (5.15) we have

$$\tilde{\beta}_i = \begin{bmatrix} l_{q_i} \\ \hline P_i \end{bmatrix} Z_i U' y \tag{5.20}$$

Using (5.20) in condition (5.4) we obtain

$$\Sigma_{\tilde{\beta}_i, \tilde{\beta}_j} = \begin{bmatrix} l_{q_i} \\ \hline P_i \end{bmatrix} Z_i U' U Z_j' [l_{q_j} \vdots P_j] \sigma^2 = 0 \tag{5.21}$$

where

$$Z_i U' U Z_j' = \left[\sum_{k=1}^{s} \sum_{r=1}^{s} Z_{ir} U_r' \right] U_k Z_{jk}' \tag{5.22}$$

According to the theorem 5.2, covariance matrix $\tilde{\beta}_i$, $\tilde{\beta}_j$ is equal to zero if and only if $Z_i U' U Z_j' = 0$ or

$$\left[\sum_{k=1}^{s} \sum_{r=1}^{s} Z_{ir} U_r' \right] U_k Z_{jk}' = 0 \tag{5.23}$$

Since the matrix Z an inverse of matrix $U'U$, the following relation is true

$$\left[\sum_{r=1}^{s} Z_{ir} U_r' \right] U_k = \begin{cases} 1 \text{ when } k = i \\ 0 \text{ when } k \neq i \end{cases} \tag{5.24}$$

From (5.24) and (5.23) it is easy to see that Z_{jk}' for $k = i$ must be equal to zero for every fixed $i, j = 1, 2, \dots, s$ and $i \neq j$. Because matrix Z is symetrical, $Z_{ij} = 0$ also. Hence, the matrix $Z = (U'U)^{-1}$ is a block-diagonal one and the $Z^{-1} = U'U$ is a block-diagonal one too. Therefore, $U_i'U_j = 0, (i, j = 1, 2, \dots, s; i \neq j)$ and the proof is concluded.

6. **Orthogonality condition of some experimental designs.** From the definition 5.2 we can clearly see, that the orthogonality of experimental designs depend on the form of restriction, which are imposed on parameters. Hence, it easy to give the orthogonality condition for various designs. We will consider some of them.

A. One way classification. The mathematical model is

(6.1) $$\underset{n1}{y} = \underset{n1}{J_n}\,\underset{11}{\mu} + \underset{na}{X_1}\,\underset{a1}{\alpha} + \underset{n1}{e},$$

where y is a random vector of observations with the expected value $\epsilon(y) = X\beta = J_n\mu + X_1\alpha$ and $\Sigma_y = I_n\sigma^2$ is an orthogonal design if the used restrictions $\underset{1,a+1}{H}\ \underset{a+1,1}{\beta} = 0$ are

(6.2) $$[0 \vdots J_n'X_1] \cdot \begin{bmatrix} \mu \\ \hline \alpha \end{bmatrix} = 0$$

Then the covariance matrix of $\tilde{\beta}$ is

(6.3) $$\Sigma_{\tilde{\beta}} = \begin{bmatrix} \Sigma_{\tilde{\mu}} & 0 \\ 0 & \Sigma_{\tilde{\alpha}} \end{bmatrix} = \begin{bmatrix} 1/n\,\sigma^2 & 0 \\ 0 & [(X_1'X_1)^{-1} - 1/n \underset{a,a}{E}\,]\,\sigma^2 \end{bmatrix}$$

and estimator vectors of both groups of parameters are uncorrelated.

One way classification with the equal number of observations in sub-classes is a particular case of general model in one way classification.

B. Two stage nested classification is also the orthogonal design under the weighted restriction. Now the model is

(6.4) $$\underset{n1}{y} = \underset{n1}{J_n}\,\mu + \underset{na}{X_1}\,\underset{a1}{\alpha} + \underset{ns}{X_2}\,\underset{s1}{\gamma(\alpha)} + \underset{n1}{e}$$

where $s = \sum_{i=1}^{a} b_i$ is a number all class of classificstion B within a class of classification A, and b_i is a number class of classification B within i'-class of A. Further, if y is a random vector with $\epsilon(y) = J_n\mu + X_1\alpha + X_2\gamma(\alpha)$ and $\Sigma_y = \sigma^2 I_n$ then the orthogonality condition is that the weighted restriction $H\beta = 0$ will be

(6.5) $$\underset{a+1,p}{H} = \begin{bmatrix} 0 & J_n'X_1 & 0 \\ 0 & 0 & X_1'X_2 \end{bmatrix} \text{ and } \beta' = [\,\mu \vdots \alpha' \vdots \gamma(\alpha)'\,]$$

and $p = 1 + a + \sum_{i=1}^{a} b_i$. The covariance matrix of $\tilde{\beta}$ is then a block-diagonal one, so that

(6.6) $$\Sigma_{\tilde{\mu}} = 1/n\,\sigma^2, \qquad \Sigma_{\tilde{\alpha}} = [(X_1'X_1)^{-1} - 1/n\,E_{a,a}]\,\sigma^2,$$

$$\Sigma_{\gamma(\tilde{\alpha})} = [(X_2'X_2)^{-1} - (X_2'X_2)^{-1}X_2'X_1(X_1'X_1)^{-1}X_1'X_2(X_2'X_2)^{-1}]\,\sigma^2,$$

and vectors of estimators of undividual groups of parameters are uncorrelated.

In a particular case, the weighted restriction may be replaced by the unweighted one when the design of the two stage nested classification have equal number class of classification B within each of class classification A, and equal number of observations in all subclasses.

The above consideration may be generalized for the optional nested classification.

C. Two way cross-classification. When the design in the two stage nested classification is the orthogonal one always under the suitable weighted restriction, the design in two way cross classification will be orthogonal, if in all subclasses there are equal or proportional numbers of observations.

The orthogonality conditions in a proportional case are given by Mikos [13].

The incomplete block designs are a particular case of two way cross classification. From definition 5.3 it follows that the incomplete block design with the model (2.16) is orthogonal if

(6.7) $$\Sigma_{\tilde{\tau},\tilde{\alpha}} = 0$$

Hence, and from (2.26) is

(6.8) $$\Sigma_{\tilde{\tau},\tilde{\alpha}} = \Sigma_{C^-Q,\,D^-P} = C^-\Sigma_{Q,P}(D^-)' = 0$$

and then the orthogonality condition is

(6.9) $$\Sigma_{Q,P} = 0$$

that is the vectors of adjusted treatment sums and adjusted block sums are uncorrelated.

From (6.9), (2.19) and (2.23) it follows, that

(6.10) $$NK^{-1}D = CR^{-1}N = 0$$

(see Oktaba [17]).

As an example, a design with complete confounding of ABC interaction in three-factors experiments may be given, where $v = 8$ combinations – treatments, $b = 6$ blocks, $k = 4$ experimental units in each block, $r = 3$ replications of every treatment and $m = 2$ blocks composed of a full replication is considered.

It is easy to verify, that the relations given below are true $v = mk$, $b = mr$ and $n = v \cdot r = bk = 24$.

Because the matrix N is

$$N = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

from the formula (3.14) matrix C may be obtained and the number of block matrices $(rI_k - r/k\, E_{k,k})$ is exact M.

Consequently, matrix C here is

$$C_{8,8} = 1/4 \begin{bmatrix} 9 & -3 & -3 & -3 & 0 & 0 & 0 & 0 \\ -3 & 9 & -3 & -3 & 0 & 0 & 0 & 0 \\ -3 & -3 & 9 & -3 & 0 & 0 & 0 & 0 \\ -3 & -3 & -3 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 & -3 & 9 & -3 & -3 \\ 0 & 0 & 0 & 0 & -3 & -3 & 9 & -3 \\ 0 & 0 & 0 & 0 & -3 & -3 & -3 & 9 \end{bmatrix}$$

Matrix R is equal $R = rI_8$. Hence, $CR^{-1}N = 0$ and the design is an orthogonal one.

Let us see, that if the incomplete block design is a connected one, then the orthogonality condition (6.10) is reduced to

$$\frac{n_{ij}}{r_i} = \text{const, for every } i = 1, 2, \ldots, v \tag{6.11}$$

and

$$\frac{n_{ij}}{k_j} = \text{const, for every } j = 1, 2, \ldots, b, \tag{6.12}$$

that is the number of determined appearance of treatments in blocks may be proportional to the replication and proportional to a number of experimental units in blocks also. Hence, an n_{ij} must be different from zero.

Therefore, the incomplete block design may be an orthogonal one, when it is a connected design.

But if an incomplete block design is a balanced one then it satisfies the condition (4.1), and it is an orthogonal one when

(6.13) $$a[R^{-1}N - 1/\nu \underset{\nu,\nu}{E} R^{-1}N] = 0$$

or

(6.14) $$\frac{n_{kj}}{r_k} = \frac{1}{\nu - 1} \sum_{\substack{i=1 \\ i \neq k}}^{\nu} \frac{n_{ij}}{r_i} \text{ for every } j = 1, 2, \ldots, b \text{ and } k = 1, 2, \ldots, \nu.$$

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STRESZCZENIE

W pracy rozważane są zagadnienia ortogonalności, zwartości i zrównoważenia układów eksperymentalnych ze szczególnym uwzględnieniem układów o blokach niekompletnych. Wykazano równoważność dwóch definicji ortogalności i podano wynikające z nich warunki ortogalności dla pewnych układów eksperymentalnych. Podano również warunki estymowalności funkcji parametrycznych dla obiektów oraz określono liczbę nieestymowalnych kontrastów obiektowych.

РЕЗЮМЕ

В настоящей работе затрагиваются вопросы ортогональности, связности и сбалансирования экспериментальных схем с особенным учётом схем неполных блоков. Доказывается эквивалентность двух определений ортогональности и приводятся вытекающие из них условия ортогональности для определения экспериментальных схем. Приводятся также условия оценки параметрических функций для объектов и определяется число недопускающих оценку контрастов между объектами.