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Liftings of π-Conjugate Connections

Podniesienie koneksji π-sprzężonych

Возведение π-сопряженных связностей

K. Yano and S. Isihara have investigated complete and horizontal lifts of geometric objects from a differential manifold M to the manifold TM in [4].

In this paper we present some proporties of the pair of linear connections on TM given by complete and horozontal lifts of the pair of π-conjugate connections on M. With the aid of metods given in [4] we transfer some results concerning π-conjugate connections and π-geodesics from M to TM.

The main results are contained in theorems (21) and (44). All our considerations are in the category C^∞.

1. Introduction. Let (M, ∇) be a smooth n-dimensional manifold with a linear connection ∇. We denote by TM the tangent bundle over the manifold M. Let p denote the natural projection $p : TM \to M$. On TM there exists the natural structure of smooth $2n$-dimensional differential manifold induced from M. (see e. g. [4] chapter I. §1.)

We assume that indices $h, i, j, \ldots$ vary over $[1, \ldots, n]$, indices $\bar{h}, \bar{i}, \bar{j}, \ldots$ vary over $[n + 1, \ldots, 2n]$ and indices $H, I, J, \ldots$ vary over $[1, \ldots, n, n + 1, \ldots, 2n]$. The Einstein summation convention will be used with respect to these systems of indices. Let f_* denotes the tangent map of a given mapping f and exp_x the exponential mapping with respect to the given linear connection ∇. The exp_x yields a diffeomorphism of a neighborhood U' of 0 in T_xM onto a neighborhood U of x in M, and t_Z denotes the automorphism of T_xM given by $t_Z(Y) = Y - Z$ for $Y \in T_xM$. p_* being the tangent map of canonical projection $p : TM \to M$ and projection K is denoted as follows: for each $A \in T_Z TM$ and $x = p(Z)$ we set

$$K_{|Z}(A) := (\exp_x \cdot t_Z \cdot \tau)_*(A) \tag{1}$$

where τ denotes the C^∞-map of $p^{-1}(U)$ into T_xM which assings to every $Y \in p^{-1}(U)$ the element $\tau(Y) \in T_x M$ which is obtained by a parallel transport of Y from $y = p(Y)$ to the point x along the unique geodesic arc in U joining x and y.

Let R^{2n} be the Euclidean space of dimension $2n$, and let (U, x) be a local chart on M. We denote by $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ a holonomic field of frames on U determined by x. Let's define X:

$$X : p^{-1}(U) \to R^{2n}/Z \to (x^1, \dots, x^n, Z^1, \dots, Z^n)$$

where $(Z^1, \dots, Z^n)$ are components of $Z \in p^{-1}(U)$ with respect to the frame $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ and $(x^1, \dots, x^n)$ are the coordinates of the point $x = p(Z)$ in (U, x). The pair $(p^{-1}(U), X)$ is called the natural lift of the chart (U, x). The natural lift of the local chart (U, x) is the local chart on TM. The field of holonomic frames with respect to the local hart $(p^{-1}(U), X)$ is called the natural frame. We denote it by $(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{x}_{n+1}, \dots \mathbf{x}_{2n})$.

Let $(\overset{i}{\underset{jk}{\Gamma}})$ be local coefficients of the linear connection ∇; i. e. $\nabla_{\mathbf{x}_i} \mathbf{x}_j = \sum_{k=1}^{n} \overset{k}{\underset{ij}{\Gamma}} \mathbf{x}_k$. Then there are defined [2] the two linear mappings of TTM onto TM. p_* is tangential to p. thus if Ψ maps a neighbourhood of 0 in R into TM so that X is a corresponding 'velocity vector', then $p \cdot \Psi$ describes a curve of its foot points in M and $p_* X$ is just its velocity vektor. Another mapping, K, is defined locally as follows:

$$K(X) = \sum_{i=1}^{n} (X^{i+n} + \overset{i}{\underset{jk}{\Gamma}} X^j Z^k)\mathbf{x}_i, \text{ where } X = \sum_{A=1}^{2n} X^A \mathbf{x}_A . \tag{2}$$

The projections p and K have the followingproperties:

a) for each $Z \in TM$

$$p_* | T_Z(TM) \quad \text{and} \quad K | T_Z(TM)$$

are linear mappings of rank n $n = \dim M$ with values in $T_{p(Z)}M$.

b) for arbitrary $Z \in TM$ there exists the following decomposition into a direct sum

$$T_Z TM = \ker(p_* | T_Z TM) \oplus \ker(K | T_Z TM)$$

where $\dim \ker p_* = \dim \ker K = n$.

The definitions of horizontal, vertical and complete liftings used in thesequel are takenfrom [2] and [4].

A vector field v^H on TM is said to be the horizontal lift of the vector field v on M iff for every $Z \in TM$ with $x = p(Z)$ we have

$$p_*(v^H(Z)) = v(x) \text{ and } K(v^H(Z)) = 0 .$$

A vector field v^V on TM is said to be vertical lift of the vector field v on M, iff for all $Z \in TM$ we have

$$p_*(v^V(Z)) = 0 \text{ and } K(v^V(Z)) = v(x).$$

A vector field v^C on TM is called a complete lift of the vector field v on M, iff we have

$$p_*(v^C(Z)) = v(x) \text{ and } K(v^C(Z)) = (\nabla_Z v)(x).$$

Let T^*TM denote a bundle which is cotangent with respect to the tangent bundle $T(TM)$. If there is given a frame $(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{x}_{n+1}, \dots, \mathbf{x}_{2n})$, then there exists a unique co-frame $dx^1, \dots, dx^n, dx^{n+1}, \dots, dx^{2n}$, such that the value of $dx^A \mid Z$ on $\mathbf{x}_B \mid Z$ is eaqual to the Kronecker δ^A_B.

A covector field $\omega^H = \sum_{A=1}^{2n} \omega^H_A dx^A$ on TM is said to be the horizontal lift of the covector field ω on M iff for all $Z \in TM$ with $x = p(Z)$ and for each vector field v on M we have

$$\omega^H(v^H)|Z = 0 \text{ and } \omega^H(v^V)|Z = \omega(v)|_x .$$

A covector field ω^V on TM is said to be the vertical lift the convector field ω on M iff

$$\omega^V(v^H)|Z = \omega(v)|x \text{ and } \omega^V(v^V)|Z = 0 .$$

A covector field ω^C on TM is said to be the complete lift of the covector field ω iff

$$\omega^C(v^C)|Z = \omega(\nabla_Z v)|x + (\nabla_Z \omega)(v)|x$$

$$\omega^C(v^H)|Z = (\nabla_Z \omega)(v)|x$$

$$\omega^C(v^V)|Z = \omega(v)|x$$

Let f be a smooth real valued function on M. A function f^V on TM defined by $f^V(Z)$: $:= f(p(Z))$, for arbitrary $Z \in TM$ is the vertical lift of the function f.

A function f^C on TM defined on each $Z \in TM$ by the formula $f^C(Z) = \sum_{k=1}^{n} Z^k f|_k$, is the complete lift of the function f.

Let (U, x) be a local chart on M and $(p^{-1}(U), X)$ a local chart on TM. Let $(v^1, \dots, v^n)$ and $(\omega_1, \dots, \omega_n)$ be local coordinates of the vector field v and the covector field ω on M respectively. The vector field v^C on TM has the coordinates

(3) $$[v^K(Z)] := (v^1, \dots, v^n, \sum_{i=1}^{n} Z^i v^1_{|i}, \dots, \sum_{i=1}^{n} Z^i v^n_{|i})$$

for any point Z of the local chart $(p^{-1}(U), X)$. Then the covector $\omega^C(Z)$ has local coordinates

$$\left(\sum_{i=1}^{n} Z^i \omega_{1|i}, \ldots, \sum_{i=1}^{n} Z^i \omega_{n|i}, \omega_1, \ldots, \omega_n\right)$$

Let f be a smooth real-valued function on M and v be a vector field on M. Thus the complete lift of the product fv has the form

$$(fv)^C(Z) = f^C(Z)v^V(Z) + f^V(Z)v^C(Z)\,.$$

Lemma. *If $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is the holonomic frame field on U, the set of $2n$ vector fields $(\mathbf{x}_1^C, \ldots, \mathbf{x}_n^C, \mathbf{x}_{n+1}^V, \ldots, \mathbf{x}_{2n}^V)$ on $p^{-1}(U)$, is a local field of frames on TM.*

The matrix which transfers the linear bazis $(\mathbf{x}_A)_{A=1,\ldots,2n}$ into the $(\mathbf{x}_1^C, \ldots, \mathbf{x}_n^C, \mathbf{x}_1^V, \ldots, \mathbf{x}_n^V)$ has the following form

$$\left(\begin{array}{c|c} \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & 0 \\ \hline * & \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix}\end{array}\right)$$

This matrix is non-singular what comletes the proof of the lemma.

Remark. The module of local vector filds on TM is generated by means of complete and vertical lifts of holonomic vector fields from M.

Let π be a symmetric non-singular tensor field on M of the type (0, 2). A symmetric non-singular tensor field π^C on TM of the type (0, 2) is said to be the complete lift of the tensor field if the equalities

(4.1) $$\pi^C(v^C, u^C)|_Z = (\nabla_Z \pi)(v, u)|_x + \pi(\nabla_Z v, u)|_x + \pi(v, \nabla_Z u)|_x$$

(4.2) $$\pi^C(v^C, u^V)|_Z = \pi(\nabla_Z v, u)|_x$$

(4.3) $$\pi^C(v^V, u^C)|_Z = \pi(v, \nabla_Z u)|_x$$

(4.4) $$\pi^C(v^V, u^V)|_Z = 0\,,$$

hold for arbitrary vector fields u, v on M and for each point $Z \in TM$ with $x = p(Z)$. Let (π_{ij}) be local coordinates of the tensor field π and (π^{ij}) the local coordinates in a chart (u, x) of the inverse tensor field π^{-1}. The local coordinates of π^C in the local chart $(p^{-1}(U), X)$ are the following

(5) $$(\tilde{\pi}_{ij})(Z) := \begin{pmatrix} \sum_{k=1}^{n} Z^k \pi_{ij|k} & \pi_{ij} \\ \pi_{ij} & 0 \end{pmatrix}$$

The local coordinates of $(\pi^{-1})^C$ are

(6) $$(\tilde{\pi}^{ij})(Z) := \begin{pmatrix} 0 & \pi^{ij} \\ \pi^{ij} & \sum_{k=1}^{n} Z^k \pi^{ij}_{\ |k} \end{pmatrix}$$

where we have $\pi_{is}\, \pi^{sj} = \delta^j_i$.

2. **Complete lift of π-conjugate linear connections.** The following theorem is valid: (cf. [4]). If M is a differentiable manifold with a linear connection ∇, then there exists a unique linear connections on TM which satisfies

(7) $$\nabla^C_{v^C} u^C = (\nabla_v u)^C$$

for every vector fields v, u on M. If $(\overset{h}{\underset{ji}{\Gamma}})$ are the local coefficiens of the connection ∇, then the coefficients of the connection ∇^C with respect to the local chart $(p^{-1}(U), X)$ are as follows

(8)
$$\tilde{\Gamma}^h_{ji}(Z) = \overset{h}{\underset{ji}{\Gamma}}(x),\ \tilde{\Gamma}^h_{j\bar{i}}(Z) = 0,\ \tilde{\Gamma}^h_{\bar{j}i}(Z) = 0,\ \tilde{\Gamma}^h_{\bar{j}\bar{i}}(Z) = 0,$$
$$\tilde{\Gamma}^{\bar{h}}_{ji}(Z) = \sum_{k=1}^{n} Z^k \overset{h}{\underset{ji|k}{\Gamma}},\ \tilde{\Gamma}^{\bar{h}}_{\bar{j}i}(Z) = \overset{h}{\underset{ji}{\Gamma}}(x),\ \tilde{\Gamma}^{\bar{h}}_{j\bar{i}}(Z) = \overset{h}{\underset{ji}{\Gamma}}(x),\ \tilde{\Gamma}^{\bar{h}}_{\bar{j}\bar{i}}(Z) = 0.$$

The connections ∇^C is called the complete lift of the linear connection ∇.

For any vector field v on M we define the covector field $\pi\wedge v := \pi(-, v)$. Thus we have $\pi\wedge v(w) = \pi(w, v)$ for arbitrary vector field w on M. So $\pi\wedge$ denote izomorphism of the module of vector fields on M onto the module of covector fields on M. Let $\pi\vee$ denote the mapping which is reciprocal to $\pi\wedge$. We define a mapping $\pi\vee$ as follows: if θ is a covector field on M then $\pi\vee\theta$ is such a vector field that it holds $\pi(\pi\vee\theta, v) := \theta(v)$ for any vector field. The composition $\pi\vee \cdot \pi\wedge$ is the identity map on the module of vector fields on M and the composition $\pi\wedge \cdot \pi\vee$ is the identity map on the module of covector fields on M. The map $\nabla_{(-)}(\pi\wedge v)$ denotes the linear mapping from the module of vector fields on M onto the module of covector fields: i. e.

$$\nabla_{(-)}(\pi\wedge v) : u \to \nabla_u(\pi\wedge v)$$

where $\nabla_u(\pi\wedge v)$ is the covector field on M.

The map $\nabla_{(-)}(\pi_{\vee}\theta)$ denotes the linear mapping from the module off vector fields on M onto itself: i. e. for arbitrary vector fields u, $\nabla_u(\pi_{\vee}\theta)$ is the vector field on M.

Let ∇ be the linear connection on M and let π be a non-singular tensor field of the type (0, 2) on M.

Definition. The connection ∇^* on M which is given by the formula

$$\nabla^*_v u := \nabla_v u + \pi_{\vee}(\nabla_v(\pi^{\wedge} u)), \tag{10}$$

is said to be a π-conjugate connection with respect to the given connection ∇. In the local chart (U, x) the formula (10) takes the form

$$G^i_{ks} v^k u^s = (\pi^{pi}\nabla_k \pi_{ps} + \overset{i}{\underset{ks}{\Gamma}}) v^k u^s, \tag{11}$$

where (G^i_{ks}) are the local coefficients of the connection ∇^*.

We have the following.

Lemma. *The following identity is valid*

$$(\pi_{\vee}(\nabla_v(\pi^{\wedge} u)))^C = \pi^C_{\vee}(\nabla^C_{v^C}(\pi^{C\wedge} u^C)). \tag{13}$$

Proof. If the use the local frame $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ then we may write

$$\pi_{\vee}(\nabla_v \pi^{\wedge} u)|_m = \sum_{h=1}^{n} (\pi^{hs}\nabla_k \pi_{si} v^k u^i)\mathbf{x}_h|_m .$$

This follows by formula (3) in the left-hand member of (13):

$$(\pi_{\vee}(\nabla_v(\pi^{\wedge} u)))^C_{|Z} = \sum_{h=1}^{n} (\pi^{sh}\nabla_k \pi_{si} v^k u^i)\mathbf{x}_h + \sum_{h=1}^{n} \partial_z(\pi^{sh}\nabla_k \pi_{si} v^k u^i)\mathbf{x}_{n+h} \tag{14}$$

for each $Z \in p^{-1}(U)$ where $m = p(Z)$.

We have for the right-hand member of formula (13) the equality

$$\pi^C_{\vee}(\nabla^C_{v^C}(\pi^{C\wedge} u^C))|_Z = \sum_{H=1}^{2n} (\tilde{\pi}^{HS} V^K \tilde{\pi}_{SJ|K} - V^K \tilde{\Gamma}^T_{KS}\tilde{\pi}_{TJ} - V^K \tilde{\Gamma}^T_{KJ}\tilde{\pi}_{ST}) U^J \mathbf{X}_H \tag{15}$$

where $(V^K)_{K=1,\ldots,2n}$ are the local components of the vector v^C (see formula (3)), $(\tilde{\pi}_{HS})$ and $(\tilde{\pi}^{IJ})$ are local components of the tensor π^C and $(\pi^{-1})^C$ respectively. These are given by (5), (6) and the coefficients $(\tilde{\Gamma}^H_{IJ})$ are given by (8). We decompose the right-hand member of the formula (15) into the form

$$\sum_{h=1}^{n} (\tilde{\pi}^{hS}\tilde{\nabla}_K \tilde{\pi}_{SJ} U^J V^K)|_Z \mathbf{X}_h + \sum_{\bar{h}=n+1}^{2n} (\tilde{\pi}^{\bar{h}S}\tilde{\nabla}_K \tilde{\pi}_{SJ} U^J V^K)\mathbf{X}_{\bar{h}}$$

From direct calculations we have the following identities

(16) $$(\tilde{\pi}^{hS}\tilde{\nabla}_K\tilde{\pi}_{SJ}U^J V^K)|_Z = (\pi^{hs}\nabla_k\pi_{sj}u^j v^k)|_Z$$

(17) $$(\tilde{\pi}^{\bar{h}S}\tilde{\nabla}_K\tilde{\pi}_{SJ}U^J V^K)|_Z = \partial_Z(\pi^{hs}\nabla_k\pi_{sj}u^j v^k)|_Z$$

Proof of the formula (17):

(18) $$\tilde{\pi}^{\bar{h}S}\tilde{\nabla}_K\tilde{\pi}_{SJ}U^J V^K = \tilde{\pi}^{\bar{h}s}(\tilde{\nabla}_K\tilde{\pi}_{sJ}V^K U^J) + \tilde{\pi}^{\bar{h}\bar{s}}(\tilde{\nabla}_K\tilde{\pi}_{\bar{s}J}V^K U^J) =$$

$$= \pi^{hs}(\tilde{\nabla}_k\tilde{\pi}_{sj}V^k U^j + \tilde{\nabla}_{\bar{k}}\tilde{\pi}_{sj}V^{\bar{k}}U^j + \tilde{\nabla}_k\tilde{\pi}_{s\bar{j}}V^k U^{\bar{j}} + \tilde{\nabla}_{\bar{k}}\tilde{\pi}_{s\bar{j}}V^{\bar{k}}U^{\bar{j}}) +$$

$$+ Z^r\pi^{hs}_{|r}(\tilde{\nabla}_k\tilde{\pi}_{\bar{s}j}V^k U^j + \tilde{\nabla}_{\bar{k}}\tilde{\pi}_{\bar{s}j}V^{\bar{k}}U^j + \tilde{\nabla}_k\tilde{\pi}_{\bar{s}\bar{j}}V^k U^{\bar{j}} + \tilde{\nabla}_{\bar{k}}\tilde{\pi}_{\bar{s}\bar{j}}V^{\bar{k}}U^{\bar{j}})$$

We have the obvious identities

(18.1) $$(\tilde{\nabla}_k\tilde{\pi}_{si})|_Z = \sum_{p=1}^{n} Z^p(\nabla_k\pi_{si})|_p$$

(18.2) $$(\tilde{\nabla}_{\bar{k}}\tilde{\pi}_{si})|_Z = \nabla_k\pi_{si}$$

(18.3) $$(\tilde{\nabla}_k\tilde{\pi}_{s\bar{i}})|_Z = \nabla_k\pi_{si}$$

(18.4) $$(\tilde{\nabla}_k\tilde{\pi}_{\bar{s}i})|_Z = \nabla_k\pi_{si}$$

and

(18.5)–(18.8) $$(\tilde{\nabla}_{\bar{k}}\tilde{\pi}_{s\bar{i}})|_Z = (\tilde{\nabla}_{\bar{k}}\tilde{\pi}_{\bar{s}i})|_Z = (\tilde{\nabla}_k\tilde{\pi}_{\bar{s}\bar{i}})|_Z = (\tilde{\nabla}_{\bar{k}}\tilde{\pi}_{\bar{s}\bar{i}})|_Z = 0$$

If we apply the above identities to the right-hand side of formula (18) then we obtain:

(19) $$\pi^{hs}(v^k u^i Z^p(\nabla_k\pi_{si})|_p + Z^p v^k_{|p}u^i\nabla_k\pi_{si} + Z^p v^k u^i_{|p}\nabla_k\pi_{si}) +$$

$$+ Z^p\pi^{hs}_{|p}(v^k u^i\nabla_k\pi_{si})$$

for $h = 1, \dots, n$. The above expressions (19) are differentials of real functions $(\pi^{hs}\nabla_k\pi_{si}\cdot u^i v^k)_{h=1,\dots,n}$ with respect to the vector $Z\in p^{-1}(U)$. All functions $(\pi^{hs}\nabla_k\pi_{si}u^i v^k)_{h=1,\dots,n}$ are defined in a certain open subset of R^n. The expression (19) may be written in the form

(20) $$\partial_Z(\pi^{hs}\nabla_k\pi_{si}u^i v^k)_{h=1,\dots,n}$$

which completes proof of the formula (17). In order to get a proof of (16) it is sufficient to writte the left-hand member of these indentities in the explicit form.

Theorem. *If the linear connections* ∇^{*} *and* ∇ *are* π*-conjugate in the sense of the formula* (10) *then the connections* ∇^{*C} *and* ∇^{C} *are* π^{C}*-conjugate on TM.*

Proof. The condition for π^{C}-conjugation has the following local form

$$\nabla^{*C}_{v^C}u^C = \nabla^{C}_{v^C}u^C + \pi^{C}_{\vee}(\nabla^{C}_{v^C}(\pi^C \wedge u^C)), \tag{22}$$

for arbitrary vector fields u, v on M. The complete lift of the formula (10) yields us

$$\nabla^{*C}_{v^C}u^C = \nabla^{C}_{v^C}u^C + (\pi_{\vee}(\nabla_v(\pi \wedge u)))^C \tag{23}$$

We have

$$\nabla^{*C}_{v^C}u^C = \nabla^{C}_{v^C}u^C + \pi^{C}_{\vee}(\nabla^{C}_{v^C}(\pi^C \wedge U^C)) \tag{24}$$

from the lemma (12) and the above formula. That completes the proof of theorem (21).

3. Corollaries concerning curves in ***TM*** **which are related to curves in** ***M*****.** Let (M, ∇, π) be a structure like one considerated in 2. We assume that $\nabla\pi \neq 0$ every on M. Let's take consider the linear connection ∇^* defined ba the formula (10) on the manifold M with a so defined structure. Let $\gamma : R \supset I \to M$ be a parametrization of the curve k on M. The 1-jet of the map γ, $j^1_{t\,|\,s}\gamma(t) = T_{\gamma(s)}$, is the tangent vector of the curve k at the point $\gamma(s)$. We say that curve k on M is a π-geodesic ([1], [3]) if

$$\nabla_T T + \pi_{\vee}(\nabla_T(\pi \wedge T)) = \lambda T \tag{25}$$

λ being some real function.

We are going to consider a structure (TM, ∇^C, π^C) where ∇^C and π^C are complete lifts of ∇ and π respectively.

Lemma. *If* $\nabla\pi \neq 0$ *then* $\nabla^C\pi^C \neq 0$. *Moreover* $\nabla^C\pi^C = 0$ *iff* $\nabla\pi = 0$.

Proof. In a local chart $(p^{-1}(U), X)$ formulas (18.1 – 18.8) imply

$$(\nabla^C_k \pi^C_{si})|_Z = \sum_{p=1}^{n} Z^p (\nabla_k \pi_{si})|_p$$

$$(\nabla^C_{\bar{k}} \pi^C_{\bar{s}\bar{i}})|_Z = \nabla_k \pi_{si}$$

$$(\nabla^C_k \pi^C_{s\bar{i}})|_Z = \nabla_k \pi_{si}$$

$$(\nabla^C_{\bar{k}} \pi^C_{\bar{s}i})|_Z = \nabla_k \pi_{si}$$

$$(\nabla^C_{\bar{k}} \pi^C_{s\bar{i}})|_Z = 0$$

$$(\nabla^C_{\bar{k}} \pi^C_{\bar{s}\bar{i}})|_Z = 0$$

$$(\nabla^C_k \pi^C_{\bar{s}\bar{i}})|_Z = 0$$

$$(\nabla^C_{\bar{k}} \pi^C_{\bar{s}i})|_Z = 0$$

We obtain from above identities the implication

$$(\nabla\pi \neq 0) \Rightarrow (\nabla^C \pi^C \neq 0)$$

If $\nabla^C \pi^C = 0$ then all above identies are equal to zero. We have the equivalence

$$(\nabla_k \pi_{si} \equiv 0) \equiv (\nabla^C_K \pi^C_{SJ} \equiv 0)$$

from $(\nabla^C_k \pi^C_{si})|_Z = \nabla_k \pi_{si} = 0$ which completes the proof of lemma (26).

Let the tensor field π on (M, ∇) satisfy the condition: $\nabla\pi \neq 0$ every where on M. Then the connection ∇^* defined by the formula (10) is different from the connection ∇.

Let TM be a manifold with connection ∇^C and symetric non-singular tensor field π^C of the type (0,2). We define the new connection ∇^{C*} by putting

$$(27) \qquad \nabla^{C*}_{v^C} u^C = \nabla^C_{v^C} u^C + \pi^C_{\vee}(\nabla^C_{v^C}(\pi^{C\wedge} u^C))$$

for arbitrary vector fields v, u on M. The above defined connection ∇^{C*} is called the π^C--conjugate with respect to the given connection ∇^C. We have

Theorem. *The complete lift of the connection ∇^* given by the formula* (10) *ia identical to the π^C-conjugate connection ∇^{C*} of the connection ∇^C which is given by the formula* (27), *i. e.* $(\nabla^{*C}) \equiv \nabla^{C*}$.

Proof. We have from the formulas (22) and (27)

$$\nabla^{*C}_{v^C} u^C = \nabla^{C*}_{v^C} u^C$$

which gives us at once the statement of theorem (28).

Let $\gamma : I \to M/t \to \gamma(t)$ be the description of the curve k on M. The mapping $\gamma^* : I \to TM/s \to j^1_{t \mid s}\gamma(t)$ paremetrizes the curve k^* on TM. This is a natural lift of the curve k. The natural lift k^* of the geodesic on (M, ∇) is the geodesic on TM with respect to the connection ∇^C.

Example. Let k be π-geodesic on (M, ∇, π), i. e. k is a geodesic on M with respect to the connection ∇^*. The natural lift k^* is the π^C-geodesic on (TM, ∇^C) and it is a geodesic on TM with respect to the connection ∇^{*C}.

Let's consider the arbitrary $\tilde{k}$ with the local parametrization $\tilde{\gamma} : I \to TM$ and $j^1\tilde{\gamma} = \tilde{T}$ which satisfies the equality

$$\nabla^C_{\tilde{T}} \tilde{T} + \pi^C_{\vee}(\nabla_{\tilde{T}}(\pi^{C\wedge}\tilde{T})) = \tilde{\lambda}\tilde{T} \tag{29}$$

where $\tilde{\lambda}$ is a certain smooth real function on TM.

We have from the theorem (21).

Corollary. *The curve $\tilde{k}$ is a geodesic on TM with respect to the connection ∇^{C^*} defined by the formula* (27).

Proof. Making use of the identity (22) to the left hand member of formula (29) we have

$$\nabla^{*C}_{\tilde{T}} \tilde{T} = \tilde{\lambda}\tilde{T}$$

and by virtue of the theorem (28) we get

$$\nabla^{C^*}_{\tilde{T}} \tilde{T} = \tilde{\lambda}\tilde{T}$$

The last equation describes a geodesic on TM with respect to the connection ∇^{C^*} q.e.d.

Let M and TM be manifolds with connections ∇ and ∇^C respectively. Let k be a curve on TM which is parametrized by $\tilde{\gamma} = w \cdot \gamma : I \to TM$, where γ is a parametrization of k, k being a certain curve on M and w is some vector field on M. The curve $\tilde{k}$ is geodesic on TM with respect to the given connection if, and only if

1) γ parametrizes some geodesic k on (M, ∇)

2) $w \mid \mathrm{im}\gamma$ is a Jacobi vector field along the curve k (see [4] Prop. 9.1.)

There exists a unique connection on M which has a zero torsion and it has the same geodesics as the given connection ∇. Then we may take a torsion less connections ∇ for studying geodesics.

Let ∇ be a connection with zero torsion on M and γ be a parametrization of some geodesic k. A smooth vector field w on M is called a Jacobi field along k, if there holds along k

(31) $$\nabla_T \nabla_T w = R_T w^T$$

where $T = j^1 \gamma$.

For the arbitrary vector fields u, v, w, on M

(32) $$R_{uv} w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w .$$

Now we introduce some formulas for to determinate geodesic on TM. Let $\gamma : I \to M$ be a parametrization of a geodesic k on M and β be a fixed real number from $I \subset R$, $(e_1, \dots, e_n)$ be a given basis in the vector space $T_{\gamma(\beta)}M$. The vector T is the tangent vector along the curve k. Let $(u_1, \dots, u_n)$ denote the parallel transport of $(e_1, \dots, e_n)$ along the geodesic k, then for each $t \in I$ the vectors $(u_1(t), \dots, u_n(t)) \subset T_{\gamma(t)}M$ form a basis of $T_{\gamma(t)}M$. We assume that $T = u_n$. The Jacobi field w can be written uniquely in the form $w = \sum_{i=1}^{n} w^i u_i$ where the $(w^1, \dots, w^n)$ are real-valued functions defined on I. Because (u_i) are vector fields which are transplate of (e_i) along the geodesic k thus we have

(33) $$\nabla_T u_i = 0 \quad \text{for } i = 1, \dots, n.$$

We calculate covariant derivatives

(34) $$\nabla_T w = \sum_{i=1}^{n} (w^i)' u_i$$

and

(35) $$\nabla_T \nabla_T w = \sum_{i=1}^{n} (w^i)'' u_i$$

From the formulas (31), (35) and from equation

$$R_{u_j u_k} u_i = R_{ijk}{}^{l} u_l$$

we have

(36) $$\sum_{l=1}^{n} (w^l)'' u_l = \sum_{k,l=1}^{n} (R_{nnk}{}^{l} w^k) u_l$$

We interprete formulas (36) as a system of linear equations of second order with coefficients $(R_{nnk}{}^{l})$ $kl = 1, \dots, n$ and with the following initial conditions

(37) $$w(\beta) = \mathbf{t} \quad \text{and} \quad (\nabla_T w)|_\beta = \mathbf{k}$$

where **t**, **k** are certain fixed vectors in $T_{\gamma(\beta)}M$. We deduce the existence and uniqueness of solution $(w^1, \dots, w^n)$ of system (36from the theory of differential equations.

If we consider the solution w of (36) with initial conditions (37), and the curve k which is parametrized by $\gamma : I \to M$ then the composition

$$w \cdot \gamma : I \to TM$$

gives us parametrization of a certain geodesic on (TM, ∇^C). If $(R_{nnk}{}^l)kl = 1, \dots, n$ are components of the curvante tensor of the connection ∇ then the composition $w \cdot \gamma$ parametrizes a certain geodesic on (TM, ∇^C). If $(R_{nnk}{}^l)kl = 1, \dots, n$ are components of a curvante tensor of a connection ∇^* then the composition $w \cdot \gamma$ parametrizes a certain geodesic on (TM, ∇^{*C}). The latest statement is an example of π^C-geodesic on (TM, ∇^C, π^C).

4. The horizontal lifts of π-conjugate connections. Besides cosiderating complete lifts of geometric objects (1,2) we take into consideration also the horizontal lift geometric objects (see [4], chapter II).

The linear connection ∇^H on TM which is defined by the formulas

$$\nabla^H_{v^V} u^V = 0, \quad \nabla^H_{v^V} u^H = 0,$$
(38)
$$\nabla^H_{v^H} u^V = (\nabla_v u)^V, \quad \nabla^H_{v^H} u^H = (\nabla_v u)^H$$

for any vector fields u, v on M, is called the horizontal lift of the linear connection ∇.

Remark. The vector field v^H defined as in 1, satisfies the equality

$$v^H(Z) = v^C(Z) - (\nabla_Z v)^V \quad \text{for each } Z \in TM \text{ ([4] p. 87)}$$

It follows from the above that the connection ∇^H is well defined on the module of vector fields on TM.

In the following part of our paper we will consider certain relations between the horizontal lifts of the π-conjugate connections. Our considerations will be performed in local coordinates of a chart $(p^{-1}(U), X)$.

The local coefficients of the connection ∇^H are

$$\bar{\Gamma}^k_{ji} = \Gamma^k_{ji}, \quad \bar{\Gamma}^k_{\bar{j}i} = 0, \quad \bar{\Gamma}^k_{j\bar{i}} = 0, \quad \bar{\Gamma}^k_{\bar{j}\bar{i}} = 0 = \bar{\Gamma}^{\bar{k}}_{\bar{j}\bar{i}},$$
(39)
$$\bar{\Gamma}^{\bar{k}}_{ji} = Z^r \Gamma^k_{ji} - Z^r R_{rji}{}^k = 0, \quad \bar{\Gamma}^{\bar{k}}_{\bar{j}i} = \Gamma^k_{ji}, \quad \bar{\Gamma}^{\bar{k}}_{j\bar{i}} = \Gamma^k_{ji}.$$

where (Γ^k_{ji}) are the local coefficients of the connection ∇, $(R_{rji}{}^h)$ are the coordinates of the curvante tensor of the connection ∇ and (Z^r) are the coordinates of the vector $Z \in p^{-1}(U)$ with respect to the frame $(\mathbf{X}_1, \dots, \mathbf{X}_n)$.

Let A be a tensor field of the type (0,2) on M. A tensor field A^H on TM (of the type (0,2)) defined by formulas

$$\begin{aligned} A^H(v^C, u^C)|_Z &= A(\nabla_Z v, u) + A(v, \nabla_Z u) \\ A^H(v^C, u^V)|_Z &= A(\nabla_Z v, u) \\ A^H(v^V, u^C)|_Z &= A(v, \nabla_Z u) \\ A^H(v^V, u^V)|_Z &= 0 \end{aligned} \tag{40}$$

is called the horizontal lift of A. A vectors v, u are arbitrary vector fields on M.

Let B be a tensor field of the type (2,0) on M. A tensor field B^H on TM, defined by the formulas

$$\begin{aligned} B^H(\omega^C, \sigma^C)|_Z &= B(\nabla_Z \omega, \sigma) + B(\omega, \nabla_Z \sigma) \\ B^H(\omega^C, \sigma^V)|_Z &= B(\nabla_Z \omega, \sigma) \\ B^H(\omega^V, \sigma^C)|_Z &= B(\omega, \nabla_Z \sigma) \\ B^H(\omega^V, \sigma^V)|_Z &= 0 \end{aligned} \tag{41}$$

is called a horizontal lift of the tensor field B. The tensor B^H is of the type (2,0). A covectors ω, σ are arbitrary covectors fields on M.

Let π be a symmetric non-singular tensor field of the type (0,2) on M, and let (π_{ij}) be a matrix of a local components of π. The tensor field π^H which is a horizontal lifting of π and is defined by means of the formula (40) has the following matrix of local coordinates:

$$(\bar{\pi}_{IJ})(Z) := \begin{pmatrix} Z^r \pi_{ij|r} - \nabla_Z \pi_{ij}, & \pi_{ij} \\ \pi_{ij}, & 0 \end{pmatrix} \tag{42}$$

The tensor field $(\pi^{-1})^H$ defined by the formula (41) has the following local coordinates

$$(\bar{\pi}^{IJ})(Z) := \begin{pmatrix} 0, & \pi^{ij} \\ \pi^{ij}, & Z^r \pi^{ij}_{|r} - \nabla_Z \pi^{ij} \end{pmatrix} \tag{43}$$

Let M be the smooth differential manifold with the connection ∇. There is given symmetric non-singular tensor field π of the type (0,2) on M and the linear connection ∇^* which is defined the formula (10).

Theorem. *The horizontal lifts of π-conjugate connections ∇^* and ∇ on M ia a pair of π^C-conjugate connections ∇^{*H} and ∇^H on TM.*

Proof. Let (G^K_{JI}) be local coefficients of ∇^{*H} and (Γ^K_{JI}) be local coefficients of connection ∇^H defined by (39). The coordinates $(\tilde{\pi}_{JI})$ and $(\tilde{\pi}^{JI})$ of the tensor fields π^C and $(\pi^{-1})^C$ respectively are defined in (5) and (6).

(45) $$\bar{G}^H_{JI} = \bar{\Gamma}^H_{JI} + \sum_{S=1}^{2n} \tilde{\pi}^{SH}(\bar{\nabla}_J \tilde{\pi}_{SI})$$

After some simple calculations we get

(45.1) $$\bar{G}^h_{ji} = \Gamma^h_{ji} + \pi^{sh}(\nabla_j \pi_{si})$$

(45.2) $$\bar{G}^h_{\bar{j}i} = 0$$

(45.3) $$\bar{G}^h_{j\bar{i}} = 0$$

(45.4) $$\bar{G}^h_{\bar{j}\bar{i}} = 0$$

(45.5) $$\bar{G}^{\bar{h}}_{ji} = \bar{\Gamma}^{\bar{h}}_{ji} + \tilde{\pi}^{s\bar{h}}(\bar{\nabla}_j \tilde{\pi}_{si}) + \tilde{\pi}^{\bar{s}\bar{h}}(\bar{\nabla}_j \tilde{\pi}_{\bar{s}i})$$

(45.6) $$\bar{G}^{\bar{h}}_{\bar{j}i} = \Gamma^h_{ji} + \pi^{sh}(\nabla_j \pi_{si})$$

(45.7) $$\bar{G}^{\bar{h}}_{j\bar{i}} = \Gamma^h_{ji} + \pi^{sh}(\nabla_j \pi_{si})$$

(45.8) $$\bar{G}^{\bar{h}}_{\bar{j}\bar{i}} = 0$$

In the local coordinates we have formula

(46) $$G^h_{ji} = \Gamma^h_{ji} + \pi^{sh}(\nabla_j \pi_{si})$$

Making use of the formulas (39) to the right-hand side of the formula (46) we get proofs of the equalities (45.1 – 45.4) and (45.6 – 45.8). Now it remains to prove (45.5). The formulas (39) and (46) imply

(47) $$\bar{\bar{G}}^{h}_{ji}(Z) := \sum_{p=1}^{n} [Z^p(\Gamma^h_{ji} + \pi^{hs}(\nabla_j \pi_{si}))|_p - Z^p B_{pji}{}^h]$$

where B denotes the curvanture tensor of ∇^*. From the formulas (47) and (45.5) we have

(48) $$\bar{\bar{G}}^{h}_{ji}(Z) = (\bar{\bar{\Gamma}}^{h}_{ji} + (\bar{\nabla}_j \tilde{\pi}_{si})\tilde{\pi}^{s\bar{h}} + (\bar{\nabla}_j \tilde{\pi}_{\bar{s}i})\tilde{\pi}^{\bar{s}\bar{h}})|_Z$$

where ∇_j is the operctor of the covariant differentiation with respect to the connection ∇, and $\bar{\nabla}_J$ is the operator of the covariant differentiation with respect to the given connection ∇^H. The right-hand side of the formula (48) is:

(49) $$\bar{\Gamma}^{\bar{h}}_{ji} + (\bar{\nabla}_j \tilde{\pi}_{si})\tilde{\pi}^{sh} - (\bar{\nabla}_j \tilde{\pi}_{\bar{s}i})\tilde{\pi}^{\bar{s}\bar{h}} = (Z^k \pi_{si|k})_{|j}\pi^{sh} -$$
$$- \Gamma^r_{js} Z^k \pi_{ri|k}\pi^{sh} - (Z^k \Gamma^r_{js|k} - Z^k R_{kjs}{}^r)\pi_{ri}\pi^{sh} -$$
$$- \Gamma^r_{ji} Z^k \pi_{sr|k}\pi^{sh} + \nabla_j \pi_{si}(Z^k \pi^{sh}_{|k}).$$

From the formulas

(A) $$\pi^{sh}_{|k} = \nabla_k \pi^{sh} - \Gamma^h_{kt}\pi^{st} - \Gamma^s_{kt}\pi^{th}$$

(B) $$R_{kji}{}^h = \Gamma^h_{ji|k} - \Gamma^h_{ki|j} + \Gamma^h_{kt}\Gamma^t_{ji} - \Gamma^h_{jt}\Gamma^t_{ki}$$

(C) $$\pi_{ri|k} = \nabla_k \pi_{ri} + \Gamma^t_{kr}\pi_{ti} + \Gamma^t_{ki}\pi_{rt}$$

and from the partial derivatives of the functions

$$g_{si}(x^1, \dots, x^n, Z^1, \dots, Z^n) := \sum_{k=1}^{n} Z^k \pi_{si}(x^1, \dots, x^n)_{|k}$$

with respect to the variables $(x^1, \dots, x^n)$ for s, $i = 1, \dots, n$ the right-hand side of the formula (49) is of the form:

$$[(Z^k \nabla_k \pi_{si})_{|j} + Z^k \Gamma^r_{ks|j}\pi_{ri} + Z^k \Gamma^r_{ks}\pi_{ri|j} + Z^k \Gamma^r_{ki|j}\pi_{sr} + Z^k \Gamma^r_{ki}\pi_{sr|j}]\pi^{sh} -$$
$$- \Gamma^r_{js} Z^k (\nabla_k \pi_{ri} + \Gamma^t_{kr}\pi_{ti} + \Gamma^t_{ki}\pi_{rt})\pi^{sh} - Z^k \Gamma^r_{js|k}\pi_{ri}\pi^{sh} + (Z^k \Gamma^r_{js|k} - Z^k \Gamma^r_{ks|j} +$$
$$+ \Gamma^r_{kt}\Gamma^t_{js} - \Gamma^r_{jt}\Gamma^t_{ks})\pi_{ri}\pi^{sh} - \Gamma^r_{ji}(Z^k \nabla_k \pi_{sr} + Z^k \Gamma^t_{ks}\pi_{tr} + Z^k \Gamma^t_{kr}\pi_{st})\pi^{sh} +$$
$$+ Z^k \nabla_j \pi_{si}(\nabla_k \pi^{sh} - \Gamma^h_{kt}\pi^{st} - \Gamma^s_{kt}\pi^{th}).$$

After some abbreviation,

$$(50)\qquad Z^k (\nabla_k \pi_{si})_{|j}\, \pi^{sh} + Z^k \Gamma^h_{ji|k} - Z^k R_{kji}{}^h + Z^k \Gamma^r_{ki} \nabla_j \pi_{sr}\, \pi^{sh} -$$

$$- Z^k \Gamma^r_{js} \nabla_k \pi_{ri}\, \pi^{sh} - Z^k \Gamma^r_{ji} \nabla_k \pi_{sr}\, \pi^{sh} + Z^k \nabla_j \pi_{si} \nabla_k \pi^{sh} - Z^k \Gamma^h_{kt} \nabla_j \pi_{si}\, \pi^{st}$$

The left-hand side of the formula (48) takes the form
(51)

$$Z^k G^h_{ki|j} - Z^k G^h_{kt} G^t_{ji} - Z^k G^h_{jt} G^t_{ki} = Z^k \Gamma^h_{ki|j} + Z^k (\nabla_k \pi_{si})_{|j}\, \pi^{sh} + Z^k \nabla_k \pi_{si}\, \pi^{sh}_{|j} -$$

$$- Z^k \Gamma^h_{kt} \Gamma^t_{ji} - Z^k \Gamma^h_{kt} \nabla_j \pi_{ri}\, \pi^{rt} - Z^k \nabla_k \pi_{st}\, \pi^{sh} \Gamma^t_{ji} - Z^k \nabla_k \pi_{st}\, \pi^{sh} \nabla_j \pi_{ri}\, \pi^{rt} +$$

$$+ Z^k \Gamma^h_{jt} \Gamma^t_{ki} + Z^k \Gamma^h_{jt} \nabla_k \pi_{ri}\, \pi^{rt} + Z^k \nabla_j \pi_{qt}\, \pi^{qh} \Gamma^t_{ki} + Z^k \nabla_j \pi_{qt}\, \pi^{qh} \nabla_k \pi_{ri}\, \pi^{rt}$$

Let π be non-singular tensor field on M. By a covariant derivating of the identity $\pi_{rl}\, \pi^{rh} = \delta^h_1$ we get

$$\pi^{sl}(\nabla_j \pi_{rl})\, \pi^{rh} = - \nabla_j \pi^{sh}\,.$$

Consequently

$$(\mathrm{D})\qquad \nabla_k \pi_{si} \nabla_j \pi^{sh} = -(\pi^{st} \nabla_k \pi_{si})(\pi^{rh} \nabla_j \pi_{rt})$$

with respect to any local chart (U, x) on M.

From the identity (A), (B) and (D) in (51) we have

$$(52)\qquad \bar{G}^{\bar{h}}_{ji}(Z) := Z^k \Gamma^h_{ji} - Z^k R_{kji}{}^h + Z^k (\nabla_k \pi_{si})_{|j}\, \pi^{sh} -$$

$$- Z^k \nabla_k \pi_{si} \Gamma^s_{jt}\, \pi^{th} - Z^k \Gamma^h_{kt} \nabla_j \pi_{si}\, \pi^{st} - Z^k \Gamma^t_{ji} \nabla_k \pi_{st}\, \pi^{sh} -$$

$$- Z^k \nabla_k \pi_{st}\, \pi^{sh} \nabla_j \pi_{ri}\, \pi^{rt} + Z^k \Gamma^t_{ki} \nabla_j \pi_{st}\, \pi^{sh}.$$

By means of the idensity (D) it is easy to see: the equality of the right-hand member of (50) and (52) holds. We obtain the equality

$$Z^k G^h_{ji|k} - Z^k B_{kji}{}^h = (\bar{\Gamma}^{\bar{h}}_{ji} + (\bar{\nabla}_j \tilde{\pi}_{si})\, \tilde{\pi}^{s\bar{h}} + (\bar{\nabla}_j \tilde{\pi}_{si})\, \tilde{\pi}^{\bar{s}\bar{h}})_{|Z}$$

what finishes the proof of the theorem (44).

Let $\nabla \pi \neq 0$ on M and let π^H be the horizontal lift of the tensor field π.

Remark. Connections ∇^{*H} and ∇^H on TM are not necessarily the π^H-conjugate connections on TM.

Proof. Suppose that connections ∇^{*H} and ∇^{H} are π^{H}-conjugate. That would imply

(53′) $$\bar{G}^{K}_{JI} = \bar{\Gamma}^{K}_{JI} + \sum_{S=1}^{2n} \bar{\pi}^{SH}(\bar{\nabla}_J \bar{\pi}_{SJ})$$

in an arbitrary local chart $(p^{-1}(U), X)$ on TM. The last term of the formula (53′) may be written in the form of the sum of the 8 terms as in (45.1–45.8). We take one term of this sum, e.g.

(54) $$\bar{G}^{\bar{h}}_{\bar{j}i}(Z) = (\bar{\Gamma}^{\bar{h}}_{\bar{j}i} + \bar{\pi}^{s\bar{h}}(\bar{\nabla}_j \bar{\pi}_{si}) + \bar{\pi}^{\bar{s}\bar{h}}(\bar{\nabla}_{\bar{j}} \bar{\pi}_{\bar{s}i}))_{|Z}.$$

A simple calculation yields

$$(\bar{\nabla}_{\bar{j}} \bar{\pi}_{si})_{|Z} = (Z^k \pi_{si|k} - \nabla_Z \pi_{si})_{|\bar{j}} - \bar{\Gamma}^{\bar{r}}_{\bar{j}s} \bar{\pi}_{ri} - \bar{\Gamma}^{\bar{r}}_{\bar{j}s} \bar{\pi}_{\bar{r}i} -$$

$$- \bar{\Gamma}^{r}_{\bar{j}i} \bar{\pi}_{sr} - \bar{\Gamma}^{\bar{r}}_{\bar{j}i} \bar{\pi}_{s\bar{r}} = \pi_{si|j} - \nabla_j \pi_{si} - \Gamma^{r}_{js} \pi_{ri} - \Gamma^{r}_{ji} \pi_{sr} = 0.$$

Thus we have

(55) $$\bar{\nabla}_{\bar{j}} \bar{\pi}_{si} = 0$$

and analoguously

(56) $$\bar{\nabla}_{\bar{j}} \bar{\pi}_{\bar{s}i} = 0.$$

We obtain from formulas (54–56)

(57) $$\bar{G}^{\bar{h}}_{\bar{j}i} = \bar{\Gamma}^{\bar{h}}_{\bar{j}i} = \Gamma^{h}_{ji}.$$

On the other hand the application of formulas (39) to the connection given by formula (46) yields

(58) $$\bar{G}^{\bar{h}}_{\bar{j}i} = \Gamma^{h}_{ji} + \pi^{hs}(\nabla_j \pi_{si}).$$

From the formulas (57) and (58) we obtain an inequality

(59) $$\Gamma^{h}_{ji} + \pi^{hs}(\nabla_j \pi_{si}) \neq \Gamma^{h}_{ji}$$

because of $\nabla_j \pi_{si} \neq 0$. This inequality gives a contradiction, what completes the proof of the remark (53).

Let (TM, ∇^H, π^H) be the horizontal lift of the structure (M, ∇, π) i.e. the horizontal lift with respect to the given linear connection on M. In view of theorem (44) and of remark (53) we conclude that ∇^{*H} and ∇^H are π^C-conjugate always, while ∇^{*H} and ∇^H are π^H-conjugate iff $\nabla\pi = 0$ on M.

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STRESZCZENIE

K. Yano i S. Isihara wprowadzili zupełne i horyzontalne podnoszenie obiektów geometrycznych z rozmaitości różniczkowej M na rozmaitość TM [4].

W tej pracy przedstawiono pewne własności par koneksji liniowych na TM danych poprzez zupełne oraz horyzontalne podniesienie par koneksji π-sprzężonych z rozmaitości M. Za pomocą metod danych w [4] zostały podniesione i zbadane pewne koneksje oraz π-geodezyjne na rozmaitości TM.

Wyniki tej pracy zawarte są w twierdzeniach (21) i (44). Wszystkie rozważania prowadzone są w kategorii C^∞.

РЕЗЮМЕ

К. Яно и С. Ишихара ввели совершенные и горизонтальные поднятия геометрических объектов из дифференциального разнообразия M на многообразие TM [4].

В этой работе представляется некоторые свойства пар линейной связности на TM данных через совершенные и горизонтальные поднятия пар П — спряженных связности с M. При помощи методов представленных в [4] переносится некоторые результаты касающиеся П связности и П геодезийных из M на TM.

Результаты этой работы содержатся в утверждениях [21] и [44] и все рассуждения ведутся в категории C^∞.