ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXII, 8 SECTIO A 1978

Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

Mieczysław POLAK

On Properties of Some Classes of Discrete Distributions

O własnościach pewnych klas rozkładów dyskretnych

О свойствах некоторых классов дискретных распределений

1. Introduction. The Binomial-Poisson distribution is the distribution of the sum $S_N = X_0 + X_1 + X_2 + \ldots + X_N$, where $X_0 = 0$, *a, e,* and $X_1, X_2, \ldots$ are independent random variables having the same Poisson distribution with parameter λ, and N is a binomial variate with parameter n, p, distributed independently of $X_1, X_2, \ldots$. It is well know that the distribution of S_N is given by

$$(1) \qquad P[S_N = x] = \frac{\lambda^x}{x!} \sum_{k=0}^{n} \binom{n}{k} k^x [pe^{-\lambda}]^k (1-p)^{n-k}$$

where $0 < p < 1, \lambda > 0$.

This distribution was introduced by Khatri and Patel [2] as a special case of the distribution of 'Type B'. Johnson and Katz [1] investigated the first four moments of this distribution. The above distribution has useful applications in life insurance lapsation phenomena.

2. Certain types of distributions. The distribution (1) can be considered as a member of the class of discrete distributions which are distributions of a random sum of independent random variables. This follows from Lemma 1 given further on.

Let $[X_j, j \in T]$, $T = N \cup [0]$, $N = [1, 2, \ldots]$, be a sequence of i. i. d. random variables which have a power series distribution (PSD), i. e.

$$(2) \qquad P[X_1 = k] = \frac{a(k)}{f(\theta)} \theta^k, \; k = 0, 1, 2, \ldots,$$

where $f(\theta) = \sum_{k=0}^{\infty} a(k)\theta^k$, $a(k) \geq 0$, and there exists a natural number k such that $a(k) > 0$.

We consider also a random variable N having a PSD, and independent of $\{X_j, j \in T\}$. Let us denote

$$P[N=j] = \frac{b(j)}{g(z)} z^j, \; j \in T, \tag{3}$$

where, of course, $g(z) = \sum_{j \in T} b(j) z^j$, $b(j) \geq 0$, and there exists a natural number j such that $b(j) > 0$.

Lemma 1. *If $\{X_j, j \in T\}$ are i. i. d. variates with distribution (2) such that $X_0 = 0$ a. s., and N is a random variable with the probability function (3) and independent of $\{X_j, j \in T\}$, then the distribution of the sum $S_N = X_1 + X_2 + \ldots + X_N$ has the form*

$$P[S_N = x] = \theta^x \sum_{j \in T} \frac{a^{(j)}(x) b(j)}{f_j(\theta) g(z)} z^j, \; x = 0, 1, 2, \ldots, \tag{4}$$

where $a^{(j)}(x)$ is the coefficient of θ^x in the j-th convolution of $f(\theta)$.

Putting in (4) $T = \{0, 1, 2, \ldots, n\}$, $f(\theta) = e^\theta$, $\theta = \lambda$ and $g(z) = (1+z)^n$, $z = p/1-p$, we obtain (1).

Recently interest has arisen in the so-called inflated power series distribution (IPSD) see e. g. Singh [3] and [4]. It is easy to show that if N has an inflated binomal distribution, i. e.

$$P[N=j] = \begin{cases} \beta + \alpha\binom{n}{j} p^j (1-p)^{n-l}, & j = l, \\ \alpha\binom{n}{j} p^j (1-p)^{n-j}, & j = 0, \ldots, l-1, l-2, \ldots, n, \end{cases}$$

where $\alpha + \beta = 1$, $0 < \alpha \leq 1$, then the distribution of S_N is given by

$$P[S_N = x] = \alpha \frac{\lambda^x}{x!} \sum_{j=0}^{n} \binom{n}{j} j^x [pe^{-\lambda}]^j (1-p)^{n-j} + \tag{5}$$

$$+ \beta \frac{(\lambda l)^x}{x!} e^{-\lambda l}, \; x = 0, 1, 2, \ldots .$$

The distribution (5) belongs to the class of discrete distribution defined in the following.

Lemma 2. *If $\{X_j, j \in T\}$ are i. i. d. variates with distribution (2) such that $X_0 = 0$ a. s. and N is a random variable, independent of $\{X_j, j \in T\}$, with probability function*

$$P[N=j]=\begin{cases}\beta+\alpha\dfrac{b(j)}{g(z)}z^j, & j=l,\\[2mm] \alpha\dfrac{b(j)}{g(z)}z^j, & j\in T,\text{ and } j\neq l\end{cases}$$

where $\alpha+\beta=1, 0<\alpha\leqslant 1$, then the distribution of the sum S_N has the form

$$P[S_N=x]=\alpha\theta^x\sum_{j\in T}\frac{a^{(j)}(x)b(j)}{f_j(\theta)g(z)}z^j+\beta\theta^x\frac{a^{(l)}(x)}{f_l(\theta)}. \tag{6}$$

Obviously, putting $\alpha=1$ in (6) we get (4). Similarly putting in (6) $T=[0,1,2,\dots,n]$, $f(\theta)=e^\theta$, $\theta=\lambda$, $g(z)=(1+z)^n$, $z=p/1-p$, $0<p<1$, we obtain (5).

3. Recurrence relations for the ordinary and the central moments. Varde [5] gave recurrence relations for the ordinary and the central moments of the distribution (1) in terms of derivative with respect to p and λ. Here we derive recurrence relations of a different kind, which are in some cases helpful, because they are combinations of moments of the Poisson distribution.

a) Recurrence relations for the ordinary moments. Let m_r denote the ordinary moment of order r of the distribution (1) and α_k the ordinary moment of order k of the Poisson distribution with parameter λ.

Theorem 1. *If S_N has the distribution defined in (1), then for an arbitrary r equation*

$$m_{r+1}=p[\lambda n m_r+\sum_{j=0}^{r-1}\binom{r}{j}[\lambda n m_j\sum_{k=0}^{r-j}\binom{r-j}{k}\alpha_k-m_{j+1}\alpha_{r-j}]] \tag{7}$$

defines a relation between the first r + 1 moments.

Proof. Let $\phi_{S_N}(t)$ denote the characteristic function of S_N. It is easy to show that

$$\phi_{S_N}(t)=[pe^{\lambda(e^{it}-1)}+q]^n .$$

Using the relation between the ordinary moments and the characteristic function, we have

$$\sum_{j=0}^{\infty}\frac{\theta^j}{j!}m_j=[pe^{\lambda(e^\theta-1)}+q]^n$$

where $\theta=it$.

Differentiating the above relation with respect to θ, we obtain

$$[pe^{\lambda(e^{\theta}-1)}+q]\sum_{j=0}^{\infty}\frac{\theta^j}{j!}m_{j+1}=\lambda npe^{\lambda(e^{\theta}-1)+\theta}\phi_{S_N}(\theta)$$

Expanding the expressions containing θ as power series and using the relation

(8) $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{i+j}}{i!\,j!}=\sum_{i=0}^{\infty}\frac{\theta^i}{i!}\sum_{j=0}^{i}\binom{i}{j},$$

we get

$$p\sum_{i=0}^{\infty}\frac{\theta^i}{i!}\sum_{j=0}^{i}\binom{i}{j}m_{j+1}\alpha_{i-j}+q\sum_{j=0}^{\infty}\frac{\theta^j}{j!}m_{j+1}=$$

$$=\lambda np\sum_{i=0}^{\infty}\frac{\theta^i}{i!}\sum_{j=0}^{i}\binom{i}{j}m_j\sum_{k=0}^{i-j}\binom{i-j}{k}\alpha_k$$

Equating coefficients of θ^r complets the proof.

Putting $r=0$ in (7) gives $m_1=\lambda np$. Similarly, for $r=1$, we have $m_2=\lambda np[(n-1)\lambda p+$ $+\lambda+1]$.

Theorem 2. *Equation*

(9) $$\frac{\partial m_r}{\partial p}=\sum_{j=0}^{r-1}\binom{r}{j}\alpha^{r-j}[nm_j-p\frac{\partial m_j}{\partial p}]$$

defines a relation between the first r ordinary moments and their derivatives.

Proof. Using the relation between the ordinary moments and the characteristic function, we have

$$\sum_{j=0}^{\infty}\frac{\theta^j}{j!}m_j=[p(e^{\lambda(e^{\theta}-1)}-1)+1]^n$$

Differentiating with respect to p, and using the fact that $\partial m_0/\partial p=0$, we obtain

$$[pe^{\lambda(e^{\theta}-1)}+q]\sum_{j=0}^{\infty}\frac{\theta^{j+1}}{(j+1)!}\frac{\partial m}{\partial p}j+1=n[e^{\lambda(e^{\theta}-1)}-1]\sum_{j=0}^{\infty}\frac{\theta^j}{j!}m_j.$$

Expanding $e^{\lambda e^{\theta}}$ in power series, and using (8)we have

$$p \sum_{i=1}^{\infty} \frac{\theta^i}{i!} \sum_{j=1}^{i} \binom{i}{j} \frac{\partial m_j}{\partial p} \alpha_{i-j} + q \sum_{j=1}^{\infty} \frac{\theta^j}{j!} \frac{\partial m_j}{\partial p} =$$

$$= n \sum_{i=0}^{\infty} \frac{\theta^i}{i!} \sum_{j=0}^{i} \binom{i}{j} m_j \alpha_{i-j} - n \sum_{j=0}^{\infty} \frac{\theta^j}{j!} m_j .$$

Equating coefficients of θ^r completes the proof.

Putting $r = 1$ in (9) we have $\partial m_1 / \partial p = \lambda n$. Similarly, for $r = 2$, we have

$$\frac{\partial m_2}{\partial p} = \lambda^2 n[2p(n-1)+1] + \lambda n .$$

b) Recurrence relations for the central moments. Let μ_r denote the central moment of order r of the random variable S_N. The following two theorems give the relations between the central moments of S_N.

Theorem 3. *Equation*

(10) $$\mu_{r+1} = \lambda n p[\sum_{j=0}^{r} \binom{r}{j} \mu_j [\sum_{k=0}^{r-j} \binom{r-j}{k} \alpha_k - p\alpha_{r-j}] - q\mu_r] +$$

$$+ p \sum_{j=0}^{r-j} \binom{r}{j} \mu_{j+1} \alpha_{r-j}$$

defines relations between the first r + 1 central moments.

Proof. We introduce the random variable

$$Y_N = S_N - m_1 .$$

The characteristic function of Y_N has the form

$$\phi Y_N{}^{(\theta)} = e^{-\theta \lambda n p} [pe^{\lambda(e^{\theta}-1)} + q]^n$$

where $\theta = it$.

As in the proceeding theorems, differentiating the equation

$$\sum_{j=0}^{\infty} \frac{\theta^j}{j!} \mu_j = e^{-\theta \lambda n p} [pe^{\lambda(e^{\theta}-1)} + q]^n$$

with respect to θ, we obtain

$$[pe^{\lambda(e^{\theta}-1)}+q]\sum_{j=0}^{\infty}\frac{\theta^j}{j!}\mu_{j+1}=\lambda npe^{\lambda(e^{\theta}-1)+\theta}\phi Y_N^{(\theta)}+$$

$$+\lambda np[pe^{\lambda(e^{\theta}-1)}+q]\phi_{Y_N}{}^{(\theta)}.$$

Expanding the expression containing θ as power series, we get

$$p\sum_{i=0}^{\infty}\frac{\theta^i}{i!}\sum_{j=0}^{i}\binom{i}{j}\mu_{j+1}\alpha_{i-j}+q\sum_{j=0}^{\infty}\frac{\theta^j}{j!}\mu_{j+1}=$$

$$=\lambda np[\sum_{i=0}^{\infty}\frac{\theta^i}{i!}\sum_{j=0}^{i}\binom{i}{j}\mu_j\sum_{k=0}^{i-j}\binom{i-j}{k}\alpha_k-q\sum_{j=0}^{\infty}\frac{\theta^j}{j!}\mu_{j+1}+$$

$$+p\sum_{i=0}^{\infty}\frac{\theta^i}{i!}\sum_{j=0}^{i}\binom{i}{j}\mu_j\alpha_{i-j}].$$

Equating coefficient of θ^r completes the proof.

In the special case $r=0$ and $r=1$, we have $\mu_1=0$ and $\mu_2=\lambda np(\lambda q+1)$.

Theorem 4. *Equation*

$$\frac{\partial\mu_r}{\partial p}=\sum_{j=0}^{r-1}\mu_j[n\binom{r}{j}\alpha_{r-j}-\lambda rnp\binom{r}{j}\alpha_{r-j-1}+$$
$$+p\binom{r}{j}\frac{\partial\mu_j}{\partial p}\alpha_{r-j}]-\lambda nqr\mu_{r-1} \tag{11}$$

defines a relation between the first r central moments and their derivatives.

Proof. The proof is similar to that of Theorem 2.

In the case $r=1$, we have $\partial\mu_1/\partial p=0$, and in the case $r=2$, $\partial\mu_2/\partial p=\lambda^2nq+\lambda(n+1)$.

If we asumme that in the sequence $[X_j, j\in T]$, $X_0\neq 0$, then (1) takes the form

$$P[S_N=x]=\frac{\lambda^x}{x!}e^{-\lambda}\sum_{k=0}^{n}\binom{n}{k}(k+1)^x[pe^{-\lambda}]^k(1-p)^{n-k}$$

Then the recurrence relations for the ordinary and central moments are given, respectively, as follows:

$$m_{r+1}=\lambda\sum_{j=0}^{r}\binom{r}{j}m_j[p(n+1)\sum_{i=0}^{r-j}\binom{r-j}{i}\alpha_i+q]-p\sum_{j=0}^{r-1}\binom{r}{j}m_{j+1}\alpha_{r-j} \tag{7'}$$

(9′)
$$\frac{\partial m_r}{\partial p} = \sum_{j=0}^{r-1} \binom{r}{j} \alpha_{r-j} [n m_j - p \frac{\partial m_j}{\partial p}],$$

(10′)
$$\mu_{r+1} = \lambda \sum_{j=0}^{r} \binom{r}{j} [q \mu_j - p(1+np) \mu_j \alpha_{r-j} +$$
$$+ p(1+n) \mu_j \sum_{i=0}^{r-j} \binom{r-j}{i} \alpha_i] - p \sum_{j=0}^{r-1} \binom{r}{j} \mu_{j+1} \alpha_{r-j} +$$
$$+ \lambda q (1+np) \mu_r ,$$

(11′)
$$\frac{\partial \mu_r}{\partial p} = \sum_{j=0}^{r-1} \mu_j [n \binom{r}{j} \alpha_{r-j} - \lambda r n p \binom{r}{j} \alpha_{r-j-1} - p \binom{r}{j} \frac{\partial \mu_j}{\partial p} \alpha_{r-j}] + \lambda n q r \mu_{r-1} .$$

c) Recurrence relations for the moments of the distribution (5).
We now gove equation for the ordinary and the central moments of the distribution (5). As in a), m_r denotes the ordinary moment of the order r.

Theorem 5. *If S_N has the distribution defined in* (5), *then for arbitrary r the equation*

(12)
$$m_{r+1} = \lambda [\frac{\partial m_r}{\partial \lambda} + p n m_r + (1-p) \frac{\partial m_r}{\partial p} + (l - np) \beta k_l]$$

where $k_l = \sum_{x=0}^{\infty} x^r \frac{(\lambda l)^x}{x!} e^{-\lambda l}$,

defines a recurrence relation for the ordinary moments of S_N.

Proof. Using (5), we have

$$m_r = \alpha \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \sum_{x=0}^{\infty} x^r \frac{(\lambda j)^x}{x!} e^{-\lambda j} + \beta \sum_{x=0}^{\infty} x^r \frac{(\lambda l)^x}{x!} e^{-\lambda l} .$$

Differentiating the above equation with respect to λ and p respectively, we have

(13)
$$\frac{\partial m_r}{\partial \lambda} = \lambda^{-1} m_{r+1} - \alpha t_r - l \beta k_l ,$$

and

(14)
$$\frac{\partial m_r}{\partial p} = (1-p)^{-1} [p^{-1} \alpha t_r - n m_r + n \beta k_l],$$

where $t_r = \sum_{j=0}^{n} jp^j(1-p)^{n-j}k_j$.

The theorem then follows from (13) and (14).

We now denote the central moments of order r by μ_r.

Theorem 6. *The following equation defines the recurrence relation for the central moment of* (5).

$$(15)\qquad \mu_{r+1} = \lambda\,[\frac{\partial\mu_r}{\partial\lambda} + \beta(np+l)\,\mu_{r+1} + p[\alpha rn[(1+\lambda)(1-p) + \beta rl]\,\mu_{r-1} + (1-p)\frac{\partial\mu_r}{\partial p}] + \beta(l-np)M_l],$$

where

$$M_l = \sum_{x=0}^{\infty}(x-m_1)^r\frac{(\lambda l)^x}{x!}e^{-\lambda l}.$$

Proof, We have

$$\mu_r = \alpha\sum_{j=0}^{n}\binom{n}{j}p^j(1-p)^{n-j}\sum_{x=0}^{\infty}(x-m_1)^r\frac{(\lambda j)^x}{x!}e^{-\lambda j} + (x-m_1)^r\frac{(\lambda l)^r}{x!}e^{-\lambda l}.$$

Putting $m_1 = \lambda(\alpha np + \beta l)$ and differentiating with respect to λ and p, we obtain

$$(16)\qquad \frac{\partial\mu_r}{\partial\lambda} = \lambda^{-1}[\mu_{r+1} + \lambda(\alpha np+\beta l)\,\mu_r] - r(\alpha np+\beta l)\,\mu_{r-1} + \alpha W_r - \beta l M_l.$$

and

$$(17)\qquad \frac{\partial\mu_r}{\partial p} = (1-p)^{-1}[p^{-1}\alpha W_r - n\mu_r] - \lambda\alpha nr\mu_{r-1} + n(1-p)^{-1}\beta M_l.$$

where

$$W_r = \sum_{j=0}^{n} j\binom{n}{j}p^j(1-p)^{n-j}M_j.$$

Comparsion of equations (16) and (17) completes the proof.

Putting $\alpha = 1$ in (12) and (15), we obtain the recurrence relations for the oridinary moments and the central moments given by Varde [3].

d) A limiting case. Letting $n \to \infty$ and $p \to 0$ in (12) and (15) in such a way that $np = a$ (a – is a constant), we have

$$m_r = \lambda \left[\!\left[\frac{\partial m_r}{\partial \lambda} + a[m_r + \frac{\partial m_r}{\partial a}] + (l-a)\beta k_l \right]\!\right]$$

and

$$\mu_{r+1} = \lambda \left[\!\left[\frac{\partial \mu_r}{\partial \lambda} + \beta(a+l)\,\mu_{r+1} + a[\alpha r(1+\lambda+rl\beta)]\mu_{r-1} + \frac{\partial \mu_r}{\partial a} + \beta(l-a)M_l \right]\!\right].$$

REFERENCES

[1] Johson and Kotz., *Diserete Distributions*, 1970, 186–187.

[2] Khatri, C. G. and Patel, I. R., *Three classes of univariate diserete distributions*, Biometries, 17 (1961), 567–576.

[3] Singh, M. P., *A note on generalized inflated binomial distribution*, Sankhya, Ser. A, 28, No 1 (1966), 93.

[4] Singh, M. P., *Inflated binomial distribution*, J. Sci. Res. Banaras Hindu Univ., XVI, (1965/1966), 87–90.

[5] Varde, V., *A note on moment recurrence relations and limiting forms of Binomial Poisson distribution*, Metrika, 23 (1974), 204–210.

STRESZCZENIE

W pracy rozważano rozkład dwumianowo-poissonowski jako szczególny przypadek szerszej klasy rozkładów złożonych. Podano także wzory rekurencyjne na momenty zwykłe i centralne tego rozkładu.

РЕЗЮМЕ

В настоящей работе изучается биномиально-пуассоновское распределение как частный случай класса сложных распределений. Приводится также рекуррентные формулы на момент.