ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA

LUBLIN–POLONIA

VOL. XXXII, 5 SECTIO A 1978

Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

Janusz KAPTUR

On Certain Boundary Value Problems for Partial Differential Equations

O pewnych problemach brzegowych dla równań różniczkowych cząstkowych

Об некоторых краевых задачах для дифференциальных уравнений с частными производными

1. In this paper we consider some types of boundary value problems for certain partial differential equations, using a method developed by T. Lezański (see [2], [3]).

For the sake of clearity we shall briefly describe this method. Let $(H, (\ldots))$ be a real Hilbert space and let M be its dense linear subsed on which is defined another scalar product $(\ldots)_*$. If $(H_1, (\ldots)_*)$ is a unitary completion of $(M, (\ldots)_*)$ and $\Psi : M \times M \to R$ is a real valued functional satisfying the following conditions

(1.1) for every $u \in M$ the functional $\Psi(u, .)$ is linear and bounded in the norm $\|\cdot\|_*$,

(1.2) there exists a positive constant b such that

$$\bigwedge_{u,v,h \in M} |\Psi(u+v, h) - \Psi(u, h)| \leq b \cdot \|v\|_* \cdot \|h\|_* ,$$

(1.3) there exists a positive constant a such that

$$\bigwedge_{u,h \in M} \Psi(u+h, h) - \Psi(u, h) \geq a \cdot \|h\|_*^2 ,$$

then a functional $\tilde{\Psi} : H_1 \times H_1 \to R$ defined for $u, h \in H_1$ by placing

(1.4) $$\tilde{\Psi}(u, h) = \lim_{n \to \infty} \Psi(u_n, h_n) .$$

where $u_n \in M$ $(n = 1, 2, \ldots)$ and $h_n \in M$ $(n = 1, 2, \ldots)$ are sequences convergent in the norm $\|\cdot\|_*$ to u and h respectively, enjoys the same conditions (1.1), (1.2), (1.3) (with obvious changes), and the equation

$$\bar{\Psi}(u, h) = 0 \text{ for every } h \in H_1 \tag{1.5}$$

has a unique solution $\bar{u}$ in H_1. If the space M, scalar product $(\ldots)_*$ and the functional Ψ are properly chosen, the solution $\bar{u}$ may sometimes be a solution of an appropriate boundary value problem.

In his papers [2], [3] T. Lezański solved some types of boundary value problems with the aid of the above method. The characteristic feature of his papers [2], [3] is the relation $C \cdot \|u\| \leqslant \|u\|_*$ $(u \in M)$, with a positive constant C; in the present paper this relation usually does not hold, but despite of that the method may be successfully used and even more general problems may be treated.

In the following two passages we shall investigate boundary value problems for certain partial differential equations of order $2 \cdot N$ (where N is a positive integer). In the last passage we shall indicate a case when it may be effectively compute a sequence of elements u_j $(j = 1, 2, \ldots)$ in M which converges in the norm $\|\cdot\|_*$ to a solution of considered boundary value problems.

2. Let R^n denote the space of sequences $\xi = (\xi_1, \ldots, \xi_n)$, $\xi_i \in R$ with a scalar product

$$\xi \cdot \eta = \sum_{i=1}^{n} \xi_i \cdot \eta_i \tag{2.1}$$

and let Ω be a simply-connected, bounded region in R^n, with a boundary $S = \partial\Omega$ which is a regular surface of the class C^1. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, where α_i are non-negative integers, we shall denote

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \ldots \partial \xi_n^{\alpha_n}} \quad \text{for } |\alpha| > 0$$

and

$$D^\alpha = \text{identity operator for } |\alpha| = 0.$$

Let N be a fixed natural number. We shall denote by m the cardinality of the set of all multi-inedices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leqslant N$

(2.2) $$m = \text{card}\,[\alpha = (\alpha_1, \ldots, \alpha_n) : |\alpha| \leqslant N]$$

it is seen that $m = \binom{n+N}{n}$. Let

(2.3) $$\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)}$$

where $\alpha^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_n^{(i)})$ with $\alpha_j^{(i)}$ non-negative integers, be a fixed enumeration of all multi-indeces α with $|\alpha| \leqslant N$.

Definition 2.4. $(H, (\ldots))$ denotes the space $L^2(\Omega)$ with its scalar product

$$(u, v) = \int_\Omega u(\xi)\cdot v(\xi)\, d\xi \quad \text{for } u, v \in H;\ \text{besides } \|u\| = \sqrt{(u, u)}$$

Now, we define a linear substed M of H.

Definition 2.5. A real valued function u belongs to M if $u \in C^{(2N)}(\bar{\Omega})$ and $D^\alpha u|S = 0$ for every multi-index α with $|\alpha| < N$.

It is readily seen that the set M is dense in H in the norm $\|\cdot\|$. We are going to determine on M a new scalar product $(\ldots)_*$. Let $[p_1, p_2, \ldots, p_m]$ be a sequence of non-negative functions p_i which satisfy the following conditions

(2.6) $$p_i \in C^{(N)}(\Omega) \text{ for } i = 1, 2, \ldots, m,$$

(2.7) there exists a number $k \in [0, 1, \ldots, N]$ such that $\text{mes}\,([\xi \in \Omega : p_j(\xi) = 0]) = 0$ for all $j \in [\![i \in [1, 2, \ldots, m] : |\alpha^{(i)}| = k]\!]$,

$\alpha^{(i)}$ $(i = 1, 2, \ldots, m)$ being the sequence of multi-indices fixed above (see (2.3))

Definition 2.8. Let
$$(u, v)_* = \sum_{i=0}^{N} (u, v)_i\,,$$
where
$$(u, v)_i = \int_\Omega \sum_{[j : |\alpha^{(j)}| = i]} p_j(\xi)\cdot D^{\alpha^{(j)}} u(\xi)\cdot D^{\alpha^{(j)}} v(\xi)\, d\xi$$
$(i = 0, 1, \ldots, N)$, for every $u, v \in M$.

We shall prove that the linear set M and the form $(\ldots)_*$ constitute a unitary space.

Lemma 2.9. *The form $(\ldots)_*$ is a scalar product on the linear set M.*

Proof. It is evident that each of the form $(\ldots)_j$ $(j = 0, 1, \ldots, N)$ is bilinear and positive and so is $(\ldots)_*$ as their sum. We shall demonstrate that if $(u, u)_k = 0$ for an element $u \in M$, then $u = 0$. Ineed, let $u \in M$ and let $(u, u)_k = 0$ i.e.

$$\int_\Omega \sum_{\{j: |\alpha^{(j)}| = k\}} p_j(\xi) \, |D^{\alpha^{(j)}} u(\xi)|^2 \, d\xi = 0.$$

From this we obtain by (2.7) and by the condition $u \in C^{(2N)}(\bar{\Omega})$, $D^\alpha u(\xi) = 0 (\xi \in \Omega)$ for all multi-indices α with $|\alpha| = k$. If $k = 0$ this means that $u = 0$; if $k > 0$, then by the condition $D^\beta u|_S = 0$ for all multi-indices β with $|\beta| = k - 1$, we get $D^\beta u(\xi) = 0$ for every $\xi \in \Omega$ and for all multi-indices β with $|\beta| = k - 1$, because all partial derivatives of $D^\beta u$ are equal to zero. Continuing this proces, if needed, we obtain after k steps $u = 0$, so the form $(\ldots)_k$ is really a scalar product. Now we may prove that $(\ldots)_*$ is also a scalar product. If $(u, u)_* = 0$ for an element $u \in M$, then by the condition

$$0 \leqslant (u, u)_k \leqslant \sum_{i=0}^{N} (u, u)_i = (u, u)_* = 0,$$

we get $(u, u)_k = 0$, which, by the first part of the proof, implies that $u = 0$. This ends the proof of Lemma 2.9.

At present we shall define on the set $M \times M$ a real valued functional Ψ. Let $f_j(t_1, t_2, \ldots, \ldots, t_m, \xi)$ $(j = 1, 2, \ldots, m)$ be real valued functions, $t_j \in \mathcal{R}$, $\xi \in \Omega$. We assume these functions satisfy the following conditions

(2.10) $$f_j \in C^{(|\alpha^{(j)}|)}(\mathcal{R}^m \times \Omega) \qquad (j = 1, 2, \ldots, m),$$

(2.11) for every function $\nu \in C(\Omega)$ it holds

$$\int_{\operatorname{supp} f_j(0, \ldots, 0, \cdot)} |f_j(0, \ldots, 0, \xi)| \cdot |\nu(\xi)| \, d\xi \leqslant \int_{\operatorname{supp} p_j} |f_j(0, \ldots, 0, \xi)| \cdot |\nu(\xi)| \, d\xi$$
$$(j = 1, 2, \ldots, m),$$

(2.12) $$\vartheta_j = \int_{\operatorname{supp} p_j} \frac{|f_j(0, \ldots, 0, \xi)|^2}{p_j(\xi)} \, d\xi < +\infty \quad (j = 1, 2, \ldots, m).$$

To formulate next conditions let us put

$$f_{ij}(t_1, \ldots, t_m, \xi) = \frac{\partial}{\partial t_j}(t_1, \ldots, t_m, \xi) \qquad (j, i = 1, 2, \ldots, m).$$

We also assume that the functions f_{ij} comply with the next two conditions

(2.13) there exists a positive constant a such that

$$\sum_{i,j=1}^{m} f_{ij}\cdot pj\cdot s_i\cdot s_j \geqslant a \sum_{j=1}^{m} pj\cdot s_j^2,$$

(2.14) there exists a positive constant b such that

$$\left| \sum_{i,j=1}^{m} f_{ij} pj\cdot s_i\cdot r_j \right| \leqslant b^2 \left(\sum_{j=1}^{m} pj\cdot s_j^2\right)\cdot\left(\sum_{j=1}^{m} pj\cdot r_j^2\right).$$

Let $q \in H = L^2(\Omega)$ be a real valued function satisfying the following condition

$$\Theta = \int_\Omega \frac{|g(\xi)|}{pj_0(\xi)} d\xi < +\infty, \tag{2.15}$$

where $j_0 \in \{1, 2, \ldots, m\}$ is such that $|\alpha^{(j_0)}| = 0$ (see (2.3)).

Definition 2.16. Let

$$\Psi(u, h) = \Psi_0(u, h) + (q, h),$$

where

$$\Psi_0(u, h) = \int_\Omega \sum_{j=1}^{m} f_j(p_1(\xi)\cdot D^{\alpha^{(1)}} u(\xi), \ldots, p_m(\xi), \xi)\cdot D^{\alpha^{(j)}} h(\xi)\, d\xi,$$

for every $u, h \in M$.

It is evident that the functional $\Psi(u, h)$ is linear with respect to h for each $u \in M$. It also holds.

Lemma 2.17. *The functional Ψ satisfies the following conditions:*

1) *for every $u \in M$ there exists a positive constant C_u such that*

$$\bigwedge_{h \in M} \quad |\Psi(u, h)| \leqslant C_u\cdot \|h\|_*,$$

2) $$\bigwedge_{u, v, h \in M} \quad |\Psi(u + v, h) - \Psi(u, h)| \leqslant b\cdot \|v\|_*\cdot \|h\|_*.$$

3) $$\bigwedge_{u, h \in M} \quad \Psi(u + h, h) - \Psi(u, h) \geqslant a\cdot \|h\|_*^2.$$

Proof. First we shall prove 2). Let $u, v, h \in M$. Then by (2.14), we have

$$|\Psi(u+v, h) - \Psi(u, h)| = |\Psi_0(u, h) - \Psi_0(u+v, h)| = |\int_0^1 \frac{\partial}{\partial t}\Psi_0(u + t\cdot v, h)\,dt| =$$

$$= |\int_0^1 \int_\Omega \sum_{j,i=1}^m f_{ij}(p_1(\xi)\cdot D^{\alpha^{(1)}}(u(\xi) + t\cdot v(\xi)), \dots, p_m(\xi)\cdot D^{\alpha^{(m)}}(u(\xi) + t\cdot v(\xi)), \xi)\cdot$$

$$\cdot pj(\xi) D^{\alpha^{(j)}}\cdot v(\xi)\cdot D^{\alpha^{(i)}} h(\xi)\,dt| \leqslant b\cdot \int_0^1\int_\Omega (\sum_{j=1}^m pj(\xi)\cdot |D^{\alpha^{(j)}} v(\xi)|^2)^{1/2}\cdot$$

$$\cdot (\sum_{j=1}^m pj(\xi)\cdot |D^{\alpha^{(j)}} h(\xi)|^2)^{1/2}\,d\xi\,dt \leqslant b\cdot \int_0^1 (\int_\Omega \sum_{j=1}^m pj(\xi)\cdot |D^{\alpha^{(j)}} v(\xi)|^2\,d\xi)^{1/2}\cdot$$

$$\cdot (\int_\Omega \sum_{j=1}^m pj(\xi)\cdot |D^{\alpha^{(j)}} h(\xi)|^2\,d\xi)^{1/2}\,dt = \int_0^1 b\cdot \|v\|_*\cdot \|h\|_*\,dt = b\cdot \|v\|_*\cdot \|h\|_* .$$

which proves 2). Now, using 2), we shall demonstrate 1). Since for $u, h \in M$ it holds

$$|\Psi(u, h)| \leqslant |\Psi(u, h) - \Psi(0, h)| + |\Psi(0, h)| \leqslant$$

$$\leqslant b\cdot \|u\|_*\cdot \|h\|_* + |\Psi_0(0, h)| + |(q, h)|$$

it suffices to estimate $|\Psi_0(0, h)|$ and $|(q, h)|$ for $h \in M$. By (2.11) and (2.12) we obtain the following inequalities

$$|\Psi_0(u, h)| \leqslant \int_\Omega \sum_{j=1}^m |f_j(0, \dots, 0, \xi)|\cdot |D^{\alpha^{(j)}} h(\xi)|\,d\xi \leqslant \sum_{j=1}^m \int_{\text{supp } pj} |f_j(0, \dots, 0, \xi)|\cdot$$

$$\cdot |D^{\alpha^{(j)}} h(\xi)|\,d\xi \leqslant \sum_{j=1}^m \int_{\text{supp } pj} \frac{|f_j(0, \dots, 0, \xi)|}{\sqrt{pj(\xi)}}\cdot \sqrt{pj(\xi)}\cdot |D^{\alpha^{(j)}} h(\xi)|\,d\xi \leqslant$$

$$\leqslant \sum_{j=1}^m (\int_{\text{supp } pj} \frac{|f_j(0, \dots, 0, \xi)|^2}{pj(\xi)}\,d\xi)^{1/2}\cdot (\int_{\text{supp } pj} pj(\xi)\cdot |D^{\alpha^{(j)}} h(\xi)|^2\,d\xi)^2 \leqslant$$

$$\leqslant (\sum_{j=1}^m \vartheta_j)^{1/2}\cdot (\sum_{j=1}^m \int_\Omega pj(\xi)\cdot |D^{\alpha^{(j)}} h(\xi)|^2\,d\xi)^{1/2} = (\sum_{j=1}^m \vartheta_j)^{1/2}\cdot \|h\|_* .$$

On the other hand by (2.15), we get

$$|(g, h)| \leqslant \int_\Omega \frac{|g(\xi)|}{\sqrt{pj_0(\xi)}}\cdot \sqrt{pj_0(\xi)}\cdot |h(\xi)|\,d\xi \leqslant$$

$$(\int_\Omega \frac{|g(\xi)|}{pj_0(\xi)}\,d\xi)^{1/2}\cdot (\int_\Omega pj_0(\xi)|h(\xi)|^2\,d\xi)^{1/2} = \sqrt{\theta}\,\|h\|_0 \leqslant \sqrt{\theta}\,(\sum_{j=0}^N \|h\|_j^2) \leqslant$$

$$\leqslant \sqrt{\theta}\cdot \sqrt{N+1}\,(\sum_{j=0}^N \|h\|_j^2)^{1/2} = \sqrt{\theta(N+1)}\cdot \|h\|_* .$$

Hence finally, for every $u, h \in M$

$$|\Psi(u,h)| \leq [\, b\cdot\|u\|_* + \sqrt{\sum_{j=1}^{m} \vartheta_j} + \sqrt{\theta\cdot(N+1)}\,]\cdot\|h\|_* ,$$

which ends the proof of 1). To prove 3), we shall take advantage of (2.13). Let $u, h \in M$, then

$$\Psi(u+h,h)-\Psi(u,h)=\Psi_0(u+h,h)-\Psi_0(u,h)=\int_0^1 \frac{\partial}{\partial t}\,\Psi_0(u+t\cdot h,h)\,dt=$$

$$=\int_0^1\int_\Omega \sum_{ij=1}^{m} f_{ij}(p_1(\xi)\cdot D^{\alpha^{(1)}}(u+th)(\xi),\dots,p_m(\xi)\cdot D^{\alpha^{(m)}}(u+t\cdot h)(\xi),\xi)\cdot pj(\xi)\cdot$$

$$\cdot D^{\alpha^{(j)}}h(\xi)\cdot D^{\alpha^{(i)}}h(\xi)\,d\xi\,dt \geq a\int_0^1\int_\Omega\sum_{j=1}^{m} pj(\xi)\cdot|D^{\alpha^{(j)}}h(\xi)|^2\,d\xi\,dt = a\int_0^1\|h\|_*^2\,dt=$$

$$=a\cdot\|h\|_*^2$$

which proves 3) and completes the proof of Lemma 2.17.

At present we sha l find a different formula for the functional Ψ. After integrating $\Psi_0(u,h)$ by parts (see Definition 2.16), we get

$$\Psi_0(u,h)=\int_\Omega\sum_{j=1}^{m}(-1)^{|\alpha^{(j)}|}D^{\alpha^{(j)}}\cdot f_j(p_1(\xi)\cdot D^{\alpha^{(1)}}u(\xi),\dots,p_m(\xi)\cdot D^{\alpha^{(m)}}u(\xi),\xi)\cdot h(\xi)d\xi,$$

for $D^a h|_S = 0$ for multi-indices α with $|\alpha| < N$. So, if we define an operator $U{:}M \to H$ by

$$(U(u))(\xi)=\sum_{j=1}^{m}(-1)^{|\alpha^{(j)}|}\cdot D^{\alpha^{(j)}}f_j(p_1(\xi)\cdot D^{\alpha^{(1)}}u(\xi),\dots,p_m(\xi)\cdot D^{\alpha^{(m)}}u(\xi),\xi)+ \tag{2.18}$$
$$+g(\xi),$$

where $\xi \in \Omega$, then we may write for $u, h \in M$

$$\Psi(u,h)=(U(u),h). \tag{2.19}$$

Let $(H_1, (.,.)_*)$ denote a unitary completion of $(M, (.,.)_*)$ and let $\tilde{\Psi}$ be the extension of Ψ defined in passage 1.

Now, we shall prove.

Theorem 2.20. *Let $u \in H_1$ be such that $\tilde{\Psi}(u, h) = 0$ for every $h \in H_1$. If $u \in M$, then the function is a solution of the boundary value problem*

(i) $\sum_{j=1}^{m} (-1)^{|\alpha^{(j)}|} \cdot D^{\alpha^{(j)}} f_j(p_1(\xi) \cdot D^{\alpha^{(1)}} u(\xi), \dots, p_m(\xi) \cdot D^{\alpha^{(m)}} u(\xi), \xi) + g(\xi) = 0$

for every $\xi \in \Omega$,

(ii) $D^{\alpha} u(\xi) = 0$ *for every* $\xi \in S$ *and multi-indices* α *with* $|\alpha| < N$. *Besides, the problem* (i), (ii) *has at most one solution in the class* $C^{(2N)}(\bar{\Omega})$.

Proof. By the assuption $\tilde{\Psi}(u, h) = 0$ for every $h \in H_1$ and by the condition $u \in M$, we obtain $\Psi(u, h) = 0$ for every $h \in M$ i. e., thanks to (2.19) it holds $(U(u), h) = 0$ for every $h \in M$. By the condition $U(u) \in H$ and from the density of M in H in the norm $\|\cdot\|$ it follows that $U(u) = 0$; so the element u is a solution of the equation (i). The element u also satisfies the condition (ii) because $u \in M$. To prove the last part of Theorem 2.20 let us observe that if $u \in C^{(2N)}(\bar{\Omega})$ is a solution of the boundary value problem (i), (ii), t en $u \in M$ and $(U(u), h) = 0$ for every $h \in M$. Hence by (2.19) and by the definition of the functional $\tilde{\Psi}$ we obtain $\tilde{\Psi}(u, h) = 0$ for every $h \in H_1$. Since the equation $\tilde{\Psi}(u, h) = 0$ for every $h \in H_1$ has a unique solution, the same proerty has the boundary value problem (i), (ii) in the class $C^{(2N)}(\bar{\Omega})$. This completes the proof of Theorem 2.20.

Remark 2.21. If an element $u \in H_1$, being a solution of the equation $\tilde{\Psi}(u, h) = 0$ for every $h \in H_1$, were called a generalized solution of the boundary value problem (i), (ii) then the following statement would be true 'the boundary value problem (i), (ii) has always a unique generalized solution'. It follows f om the above proof that so defined generalized solution u would be a classical one, if $u \in M$; conversely, any classical solution o the problem would be a generalized one.

3. In this passage the symbols R^n, $\xi \cdot \eta$, Ω, S, H, M, $(.,.)$, D^{α}, $|\alpha|$, numbers N and m, sequence of multi-indices ($j = 1, 2, \dots, m$) retain their meaning (see (2.1), (2.2), (2.3), Definition 2.4, Definition 2.5), but this time a scslar product in M will be defined differently.

Let $[p_1, p_2, \dots, p_m]$ be a sequence of real valued functions p_i which satisfy the following conditions

(3.1) $$p_i \in C^{(N)}(\Omega) \quad (i = 1, 2, \dots, m),$$

(3.2) there exists a number $k \in [0, 1, \dots, N]$ such that for all numbers $j \in [\![i \in [1, 2, \dots, m] : |\alpha^{(i)}| = k]\!]$

$$\text{mes}([\xi \in \Omega : p_j(\xi) = 0]) = 0.$$

Let us notice that now we do not assume the functions $p_1, \dots, p_m$ to be non-negative.

Definition 3.3. Let

$$(u, v)_* = \sum_{i=0}^{N} (u, v)_i ,$$

where

$$(u, v)_i = \int_\Omega \sum_{\{j: |\alpha^{(j)}| = i\}} p_j^2(\xi) \cdot D^{\alpha^{(j)}} u(\xi) \cdot D^{\alpha^{(j)}} v(\xi)\, d\xi \quad (i = 0, 1, \ldots, N),$$

for every $u, v \in M$.

It is seen that Lemma 2.9 and its proof retain their validity in the case of the scalar product $(.,.)_*$ defined in Definition 3.3

Now, we are go ng to define on $M \times M$ a real valued functional Ψ. Let $f_j(t_1, t_2, \ldots \ldots, t_m, \xi)$ $(j = 1, 2, \ldots, m)$, $t_i \in \mathcal{R}$, $\xi \in \bar{\Omega}$, be real valued functions such that

(3.4) $$f_j \in C^{(|\alpha^{(j)}|)} (\mathcal{R}^m \times \bar{\Omega}) \quad \text{for } j = 1, 2, \ldots, m.$$

Let the symbols f_{ij} $(i, j = 1, 2, \ldots, m)$ have the same meaning as in passage 2. We shall also assume that the functions f_{ij} satisfy the following two conditions

(3.5) there exists a positive constant a such t at

$$\sum_{i,j=1}^{m} f_{ij} s_i \cdot s_j \geq a \cdot \left(\sum_{j=1}^{m} s_j^2\right),$$

(3.6) there exists a positive constant b such that

$$\left| \sum_{i,j=1}^{m} f_{ij} \cdot s_i \cdot r_j \right|^2 \leq b^2 \left(\sum_{j=1}^{m} s_j^2\right)\left(\sum_{j=1}^{m} r_j^2\right).$$

Let $q \in H = L^2(\Omega)$ be a real valued function fulfilling

(3.7) $$\theta = \int_\Omega \frac{|g(\xi)|^2}{|p_{j_0}(\xi)|^2}\, d\xi < +\infty$$

where $j_0 \in [1, 2, \ldots, m]$ is such that $|\alpha^{(j_0)}| = 0$ (see (2.3)).

Definition 3.8. Let

$$\Psi(u, h) = \Psi_0(u, h) + (q, h),$$

where

$$\Psi_0(u, h) = \int_\Omega \sum_{j=1}^{m} p_j(\xi) \cdot f_j(p_1(\xi) \cdot D^{\alpha^{(1)}} u(\xi), \dots, p_m(\xi) \cdot D^{\alpha^{(m)}} u(\xi), \xi) \cdot D^{\alpha^{(j)}} h(\xi)\, d\xi$$

for every $u, h \in M$.

Now, we shall prove the following.

Lemma 3.9. *The functional Ψ satisfies the following conditions*

1) *for every $u \in M$ there exists a positive constant C_u such that*

$$\bigwedge_{h \in M} \quad |\Psi(u, h)| \leqslant C_u \cdot \| h \|_* ,$$

2)

$$\bigwedge_{u, v, h \in M} \quad |\Psi(u+v, h) - \Psi(u, h)| \leqslant b \cdot \| v \|_* \cdot \| h \|_* ,$$

3)

$$\bigwedge_{u, h \in M} \quad \Psi(u+h, h) - \Psi(u, h) \geqslant a \cdot \| h \|_*^2 .$$

Proof. First we shall prove 2). Let $u, v, h \in M$. By (3.6), we have the following estimates:

$$|\Psi(u+v, h) - \Psi(u, h)| = |\Psi_0(u, h) - \Psi_0(u+v, h)| = |\int_0^1 \text{——} \Psi_0(u + t\cdot v, h)\, dt| \leqslant$$

$$\leqslant \int_0^1 \int_\Omega | \sum_{i,j=1}^{m} p_i(\xi)\cdot f_{ij}(p_1(\xi)\cdot D^{\alpha^{(1)}} (u+tv)(\xi), \dots, p_m(\xi)\cdot D^{\alpha^{(j)}} v(\xi)\cdot D^{\alpha^{(i)}} h(\xi) |\, d\xi\, dt \leqslant$$

$$\leqslant \int_0^1 \int_\Omega b \cdot (\sum_{j=1}^{m} p_j^2(\xi)\, |D^{\alpha^{(j)}} v(\xi)|^2)^{1/2} \cdot (\sum_{j=1}^{m} p_j^2(\xi) \cdot |D^{\alpha^{(j)}} h(\xi)|^2)^{1/2}\, d\xi\, dt \leqslant$$

$$\leqslant b \cdot \int_0^1 (\int_\Omega \sum_{j=1}^{m} p_j^2(\xi)\, |D^{\alpha^{(j)}} v(\xi)|^2\, d\xi)^{1/2} \cdot (\int_\Omega \sum_{j=1}^{m} p_j^2(\xi) \cdot |D^{\alpha^{(j)}} h(\xi)|^2\, d\xi)^{1/2}\, dt =$$

$$= b \cdot \| v \|_* \cdot \| h \|_* .$$

Now, using 2), we shall show 1). Since for $u, h \in M$

$$|\Psi(u,h)| \leqslant |\Psi(u,h) - \Psi(0,h)| + |\Psi(0,h)| \leqslant b \cdot \|u\|_* \cdot \|h\|_* + |\Psi_0(0,h)| + \\ + |(q,h)|,$$

it is enough to estimate $|\Psi_0(0,h)|$ and $|(q, h)|$. By virtue of the continuity of f_j, there exists a positive constant K such that $|f_j(0, \dots, 0, \xi)| \leqslant K$ for $\xi \in \bar{\Omega}$ and $j = 1,2, \dots, m$. Hence by the Schwarz inequality:

$$|\Psi_0(0,h)| \leqslant \int_\Omega |\sum_{j=1}^m p_j(\xi) \cdot f(0, \dots, 0, \xi) \cdot D^{\alpha^{(j)}} h(\xi)|\, d\xi \leqslant K \int_\Omega \sum_{j=1}^m |p_j(\xi)| \cdot |D^{\alpha^{(j)}} h(\xi)|\, d\xi \leqslant$$

$$\leqslant K \cdot \sqrt{m} \int_\Omega \sqrt{\sum_{j=1}^m |p_j(\xi) \cdot D^{\alpha^{(j)}} h(\xi)|^2}\, d\xi \leqslant K \cdot \sqrt{m} \left(\int_\Omega d\xi\right)^{1/2} \left(\int_\Omega \sum_{j=1}^m p_j^2(\xi) \cdot\right.$$

$$\left. \cdot |D^{\alpha^{(j)}} h(\xi)|^2\, d\xi\right)^{1/2} = K \cdot \sqrt{m} \cdot \sqrt{mes\Omega} \cdot \|h\|_*.$$

On the other hand by (3.7) and by Definition 3.3, we have for $h \in M$:

$$|(q,h)| \leqslant \int_\Omega |q(\xi)| \cdot |h(\xi)|\, d\xi \leqslant \int_\Omega \frac{|q(\xi)|}{|pj_0(\xi)|} \cdot |pj_0(\xi)| \cdot |h(\xi)|\, d\xi \leqslant$$

$$\leqslant \left(\int_\Omega \frac{|q(\xi)|^2}{|pj_0(\xi)|^2} d\xi\right)^{1/2} \cdot \left(\int_\Omega p_j^2(\xi) \cdot |h(\xi)|^2\, d\xi\right)^{1/2} = \sqrt{\theta} \cdot \|h\|_0 \leqslant \sqrt{\theta} \left(\sum_{j=0}^N \|h\|_j\right) \leqslant$$

$$\leqslant \sqrt{\theta} \cdot \sqrt{N+1} \left(\sum_{j=0}^N \|h\|_j^2\right)^{1/2} = \sqrt{\theta \cdot (N-1)} \cdot \|h\|_*,$$

which gives in the end for $h \in M$:

$$|\Psi(u,h)| \leqslant \left(b \cdot \|u\|_* + K \cdot \sqrt{m \cdot mes\Omega} + \sqrt{\theta \cdot (N+1)}\right) \cdot \|h\|_*,$$

thus 1) is proved. To prove 3) let us take $u, h \in M$; by virtue of (3.5) we get

$$\Psi(u+h,h) - \Psi(u,h) = \Psi_0(u+h,h) - \Psi_0(u,h) = \int_0^1 \frac{\partial}{\partial t} \Psi_0(u + t \cdot h, h)\, dt =$$

$$= \int_0^1 \int_\Omega \sum_{i,j=1}^m p_i(\xi) \cdot f_{ij}(p_1(\xi) \cdot D^{\alpha^{(1)}}(u + t \cdot h)(\xi), \dots, p_m(\xi) \cdot D^{\alpha^{(m)}}(u + t \cdot h)(\xi), \xi)\, pj(\xi) \cdot$$

$$\cdot D^{\alpha^{(j)}} h(\xi)\, d\xi\, dt \geqslant a \cdot \int_0^1 \int_\Omega \sum_{j=1}^m p_j^2(\xi) \cdot |D^{\alpha^{(j)}} h(\xi)|\, d\xi\, dt = \int_0^1 a \cdot \|h\|_*^2\, dt = a \cdot \|h\|_*^2.$$

This ends the proof of Lemma 3.9.

Les us notice that integrating $\Psi_0(u, h)$ by parts (see Definition 3.8), we obtain for every $u, h \in M$

$$\Psi_0(u, h) = \int_\Omega \sum_{j=1}^{m} (-1)^{|\alpha^{(j)}|} \cdot D^{\alpha^{(j)}} [p_j(\xi) \cdot f_j(p_1(\xi) \cdot D^{\alpha^{(1)}} u(\xi), \dots, p_m(\xi) \cdot D^{\alpha^{(m)}} u(\xi), \xi)] \cdot$$

$$\cdot h(\xi)\, d\xi,$$

since $D^\alpha h|_S = 0$ for multi-indices α with $|\alpha| < N$. Let us define an operation $U: M \to H$ by placing

$$(3.10) \qquad (U(u))(\xi) = \sum_{j=1}^{m} (-1)^{|\alpha^{(j)}|} \cdot D^{\alpha^{(j)}} [p_j(\xi) \cdot f_j(p_1(\xi) \cdot D^{\alpha^{(1)}} u(\xi), \dots, p_m(\xi) \cdot$$

$$\cdot D^{\alpha^{(m)}} u(\xi), \xi)] + q(\xi),$$

where $\xi \in \Omega$. Using this operation we may express the functional Ψ by the formula

$$(3.11) \qquad \Psi(u, h) = (U(u), h) \quad \text{for } u, h \in M.$$

Let $(H_1, (.,.)_*)$ denote a unitary completion of $(M, (.,.)_*)$ and let $\tilde{\Psi}$ be the extension of Ψ defined in passage 1.

Theorem 3.12. *Let $u \in H_1$ and let $\tilde{\Psi}(u, h) = 0$ for all $h \in H_1$. If the element u belongs to M, then it is a solution of the boundary value problem*

$$\text{(i)} \qquad \sum_{j=1}^{m} (-1)^{|\alpha^{(j)}|} \cdot D^{\alpha^{(j)}} [p_j(\xi) \cdot f_j(p_1(\xi) \cdot D^{\alpha^{(1)}} u(\xi), \dots, p_m(\xi) \cdot D^{\alpha^{(m)}} u(\xi), \xi) +$$

$$+ q(\xi) = 0 \quad \text{for } \xi \in \Omega,$$

(ii) *$D^\alpha u(\xi) = 0$ for $\xi \in S$ and multi-indices α with $|\alpha| < N$. The problem* (i), (ii) *has at most one solution in the class $C^{(2N)}(\bar{\Omega})$.*

The proof of Theorem 3.12 is quite similar to the one of Theorem 2.20 so we omit it.

4. All symbols used in passage 2 retain their meaning in the present passage. In this passage we are going to give a sufficient condition for the existence of an orthonormal and linearly dense sequence $e_j \in M$ $(j = 1,2, \dots)$ in the space $(M, (.,.)_*)$. Such systems are important in applications, because using them we ma construct a sequence of elements $u_j \in M$ $(j = 1,2, \dots)$ which approximate in the norm $\|\cdot\|_*$ the solution $\bar{u} \in H_1$ of the equation $\tilde{\Psi}(u, h) = 0$ for all $h \in H_1$, i.e. theore holds $\lim_{j \to \infty} \|u_j - u\|_* = 0$. Namely, if e_j $(j = 1,2, \dots)$ is an orthonormal and linearly dense in $(M, (.,.)_*)$, then the element $u_j \in \text{lin}(e_1, e_2, \dots, e_j)$ is defined as a solution of the equation

(4.1) $\Psi(u, h) = 0$ for every $h \in \mathrm{lin}(e_1, e_2, \ldots, e_j)$ $(j = 1, 2, \ldots)$.

As it is known the equation (4.1) has always a unique solution $u_j \in \mathrm{lin}(e_1, e_2, \ldots, e_j)$ (for a detailed treatment of a numerical solving equations of the type (4.1) see T. Leżański [2]). Now, we are passing on to a lengthy considerations.

Definition 4.2. Let

$$(u, v)_{\square} = \sum_{k=0}^{N} (u, v)_{0,k},$$

where

$$(u, v)_{0,k} = \int_{\Omega} \sum_{|\alpha|=k} D^{\alpha} u(\xi) \cdot D^{\alpha} v(\xi)\, d\xi \qquad (k = 0, 1, \ldots, N),$$

for every $u, v \in M$.

If all the functions $p_1, p_2, \ldots, p_m$ (see (2.6)) are bounded, then there exists a positive constant K_1 such that

(4.3) $$\|u\|_* \leq K_1 \cdot \|u\|_{\square} \quad \text{for } u \in M.$$

On the other hand it follows from the well known Friedrichs inequality that there exists a positive constant C_p such that for every $u \in M$

(4.4) $$\|u\|_{0,p} \leq C_p \cdot \|u\|_{0,p+1} \quad (p = 0, 1, \ldots, N-1).$$

therefore it also holds

(4.5) $$\|u\|_{\square} \leq K_2 \cdot \|u\|_{0,N} \quad \text{for every } u \in M,$$

where K_2 is an appropriate constant.

Let $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial \xi_j^2}$. It is readily seen that the operator $\Delta_p = \underbrace{\Delta \cdot \Delta \cdot \ldots \cdot \Delta}_{p\text{-times}}$, regarded as acting on $C^{(2p)}(\Omega)$ (with p natural), may be represented in the form

(4.6) $$\Delta^p = \sum_{|\alpha|=p} K_{\alpha}^{(p)} D^{2\cdot\alpha},$$

where $K_{\alpha}^{(p)}$ are positive integers.

Let $u, v \in M$. Integrating $((-1)^N \cdot \Delta^N u, v)$ N times by parts, we obtain

(4.7) $$((-1)^N \cdot \Delta^N u, v) = \sum_{|\alpha|=N} K_{\alpha}^{(N)} \int_{\Omega} D^{\alpha} u(\xi) \cdot D^{\alpha} v(\xi)\, d\xi.$$

Definition 4.8. Let $u, v \in M$ and let

$$(u, v)_{**} := ((-1)^N \cdot \Delta^N u, v).$$

If $K_3 = \sup [K_\alpha^{(N)} : |\alpha| = N]$, then

(4.9) $$\|u\|_{0,N} \leqslant \|u\|_{**} \leqslant K_3 \cdot \|u\|_{0,N} \quad \text{for every } u \in M.$$

Hence finally by (4.3), (4.5) and (4.9), we get

(4.10) $$\|u\|_{*} \leqslant K \cdot \|u\|_{**} \quad \text{for } u \in M,$$

where $K = K_1 \cdot K_2$.

By the last considerations, we obtain.

Lemma 4.11. *If the functions $p_1, p_2, \ldots, p_m$ are bounded and elements $e_j \in M$ ($j = 1, 2, \ldots$) constitute a linearly dense set in $(M, (.,.)_{**})$, then the sequence e_j ($j = 1, 2, \ldots$) is a linearly dense set in $(M, (.,.)_{*})$.*

Linearly dense systems in $(M, (.,.)_{**})$ has been constructed by L. Kantorovitch (cf. e. g. [1], 295–306 or [4], 368–369) under certain assumptions concerning the region Ω. As it follows from the above Lemma 4.11, the sam systems are also linearly dense in $(M, (.,.)_{*})$. Hence after having been orthonor alized with respect to the scalar product $(.,.)_{*}$ these systems may be used to construct a sequence of elements $u_j \in M$ ($j = 1, 2, \ldots$) such that $\lim_{j \to \infty} \|u_j - \bar{u}\| = 0$, where $\bar{u} \in H_1$ is a solution of the problem $\tilde{\Psi}(u, h) = 0$ for every $h \in H_1$; thus if the boundary value problem (i), (ii), (see Theorem 2.20) has a solution in the class $C^{(2N)}(\bar{\Omega})$, the sequence u_j ($j = 1, 2, \ldots$) converges to the solution in the norm $\|\cdot\|_{*}$ (this is obviously true if the region Ω satisfies the conditions needed in the above mentioned L. Kantorovitch's construction).

Remark 4.12. Let us notice that what we have told about linearly dense systems in the space $(M, (.,.)_{*})$ considered in passage 2 applies as well to the space $(M, (.,.)_{*})$ considered in passage 3.

REFERENCES

[1] Канторович, А. В., Крылов, В. И., Приближенные методы высшего анализа, Москва 1962.

[2] Leżański, T., *Sur les solutions generalisees des equations quasi-lineraires*, Studia Math., 29 (1968), 133–142.

[3] ———, *Sur les equations du type* $\Psi(x,h) = 0$, Studia Math., 59 (1976), 155–175.

[4] Михлин, С. Т., Вариационные методы в математической физике, Москва 1970.

STRESZCZENIE

W pracy niniejszej badane są jednorodne problemy brzegowe dla dwóch typów równań różniczkowych cząstkowych rzędu parzystego (problem (i), (ii) z Twierdzenia 2.20 oraz problem (i), (ii) z Twierdzenia 3.12). Wykazano, że jeśli rozważane problemy mają rozwiązanie to jest ono jedyne. Ponadto wskazano przypadek, gdy problemy te mogą być rozwiątane w sposób przybliżony; mianowicie możliwe jest efektywne wyliczenie elementów ciągu zbieżnego (w normie $\|\cdot\|_*$) do rozwiązań powyższych problemów (o ile te rozwiązania istnieją).

РЕЗЮМЕ

В работе рассмотрены однородные краевые проблемы для двух типов дифференциальных уравнений с частными производными чётного порядка (проблема (i), (ii) из Теоремы 2.20 и проблема (i), (ii) из Теоремы 3.12). Доказано, что рассматриваемые проблемы имеют только одно решение. Кроме того, показано случай, когда эти проблемы могут быть приближенно решены; именно возможно эффективно вычислить члены последовательности, сходящей (в норме $\|\cdot\|_x$) к решению рассматриваемых проблем (если это решение существует).