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A Relative Growth of Modulus of Derivatives for Majorized Fuctions

O względnym wzroście modułów pochodnych dla funkcji zmajoryzowanych

–ruski–

1. Introduction. Let f, F be two holomorphic functions in the disc $K_R = \{z : |z| < R\}$. We say that a function f is subordinate to F in K_R and write $f \lesssim F$ in K_R, if there exists a holomorphic function ω such that $|\omega(z)| \leq |z|$ and $f(z) = F(\omega(z))$ for $z \in K_R$. We say that f is majorized by F in K_R and write $f \ll F$ in K_R, if there exists a holomorphic function ϕ such that $|\phi(z)| \leq 1$ and $f(z) = \phi(z)F(z)$ for $z \in K_R$.

M. Biernacki [1] investigated the relation between subortination $f \lesssim F$ in K_1 and majorization of derivatives $f' \ll F'$ in some smaller disc K_{r_0} if the functions f, F are univalent in K_1. This problem was also investigated by G. M. Goluzin (cf. [3] p. 330) and Shah Tao-shing [6]. Z. Lewandowski [5] investigated an analogous problem in the case when $f \ll F$ in K_1 and the functions F are univalent or starlike in K_1. He proved that

$$f \ll F \text{ in } K_1 \Rightarrow f' \ll F' \text{ in } K_{3-2\sqrt{2}}.$$

Z. Bogucki and J. Zderkiewicz [2] sovelt this problem for convex functions. They proved that

$$f \ll F \text{ in } K_1 \Rightarrow f' \ll F' \text{ in } K_{1/3}.$$

These problems may be generalized in the following way. Let A, B be two fixed classes of holomorphic functions in K_1. Find the smallest function $T(r) = T(r; A, B)$, $r \in \langle 0, 1)$ such that for every pair of functions $f \in A$, $F \in B$ the implication

$$f \ll F \text{ in } K_1 \Rightarrow |f'(z)| \leq T(r; A, B)\, |F'(z)| \text{ for } |z| = r < 1$$

holds.

Now if we want to find the radius of majorization of derivatives then it is enough to solve the inequality

$$T(r; A, B) \leqslant 1.$$

In the same way we may generalize and majorization of its derivatives:

Find the smallest function $G(r) = G(r; A, B)$, $r \in \langle 0, 1)$ such that for every pair of functions $f \in A$, $F \in B$ the implication

$$f \lesssim F \text{ in } K_1 \Rightarrow |f'(z)| \leqslant G(r; A, B)\, |F'(z)| \text{ for } z = r < 1.$$

In this paper we are going to determine the functions $T(r; A, B)$ for some specjal classes of holomorphic functions.

2. Main results. Let S denote a class of function $F(z)$-holomorphic and univalent in K_1 and normalized by the conditions $F(0) = 0, F'(0) = 1$

Let us put

$$S^* = [\, F \in S \;;\; \operatorname{Re} \frac{zF'(z)}{F(z)} > 0 \text{ for } z \in K_1 \,],$$

$$S^c = [\, F \in S \;;\; \operatorname{Re}\left(1 - \frac{zF''(z)}{F'(z)}\right) > 0 \text{ for } z \in K_1 \,].$$

$$H(K_1) = [\, f \;;\; f\text{–holomorphic in } K_1 \,].$$

For given number $n = 0, 1, 2, \ldots$ let us put

$$N_n = [\, f ; f(z) = a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots \in H(K_1)]$$

$$\Omega_n = [\, \omega ; \omega(z) = \alpha_n z^n + \alpha_{n+1} z^{n+1} + \ldots \in H(K_1), |\omega(z)| \leqslant 1] \text{ for } z \in K_1$$

Theorem 1. *Let $f \in N_0$, $F \in S^c$. If $f \ll F$ in K_1 and $|z| = r < 1$ then*

$$|f'(z)| \leqslant T(r; N_0, S^c)\, |F'(z)| \tag{1}$$

where

$$T(r; N_0, S^c) = \begin{cases} 1 & \text{for } r \in \langle 0, 1/3 \rangle \\ \dfrac{5r^2 - 2r + 1}{4r(1-r)} & \text{for } r \in (1/3, 1) \end{cases} \tag{2}$$

The result is sharp. For $z_0 = r_0 e^{i\theta_0}$, $r_0 > 1/3$ *a pair of functions*

$$f(z) = e^{i\theta_1} \frac{r_0 z^{-i\theta_0} + 1 - 2r_0}{r_0 + (1-2r_0) z e^{-i\theta_0}} \cdot \frac{z}{1 + e^{-i\theta_0} z} \tag{3}$$

$$F(z) = \frac{z}{1 + z e^{-i\theta_0}}, \tag{4}$$

where θ_1 *is an arbitrary real number, is the extremal pair. For* $r_0 \in \langle 0, 1/3 \rangle$ *every pair of functions* $f(z) = e^{i\theta_1} F(z), F(z)$, where $F \in S^c$, *is the extremal pair.*

Remark 1. From the theorem 1 we can obtain immediately some generalization of the result of paper [2]. Namely, we can omit the condition of $f'(0) \geqslant 0$.

Proof of theorem 1. If $f \ll F$ in K_1 then there exists a function $\phi \in \Omega_0$ such that

$$f(z) = \phi(z) F(z) \quad \text{for } z \in K_1 \tag{5}$$

It is known (cf. [3] p. 286) that if $\phi \in \Omega_0$, then

$$|\phi'(z)| \leqslant \frac{1 - |\phi(z)|^2}{1 - |z|^2} \qquad \text{for} \quad z \in K_1 \tag{6}$$

If $F \in S^c$ and $|z| = r < 1$ then (cf. [4] p 13)

$$r(1-r) \leqslant \left| \frac{F(z)}{F'(z)} \right| \leqslant r(1+r) \ . \tag{7}$$

Differentiating the equality (5) and dividing it by $F'(z)$ we obtain

$$\frac{f'(z)}{F'(z)} = \phi'(z) \qquad \frac{F(z)}{F'(z)} + \phi(z) \tag{8}$$

Now from (8) by (6) and (7) we have for $|z| = r < 1$

$$\left| \frac{f'(z)}{F'(z)} \right| \leqslant \frac{1 - |\phi(z)|^2}{1 - r^2} r(1+r) + |\phi(z)| = -\frac{r}{1-r} |\phi(z)|^2 + |\phi(z)| + \frac{r}{1-r} \tag{9}$$

If $|z| = r$ is fixed then the right hand side of this inequality is a square function of variable $u = |\phi(z)|$

$$P(u) = -\frac{r}{1-r}u^2 + u + \frac{r}{1-r}$$

By the condition that $\phi \in \Omega_0$ it is implied that u may take values only in the interval $\langle 0,1\rangle$. The function $P(u)$ has its maximum at the point $u = 1$ when $r \in \langle 0, 1/3\rangle$ and at the point $u = (1-r)/2r$ when $r \in (1/3, 1)$.
Consequently

$$(10) \qquad \max_{u \in \langle 0,1\rangle} P(u) = \begin{cases} 1 & \text{for } r \in \langle 0, 1/3\rangle \\ \dfrac{5r^2 - 2r + 1}{4r(1-r)} & \text{for } r \in (1/3, 1) \end{cases}$$

Thus from (9) and (10) we have (1).

By simple calculation we can check that the function (3) and (4) satisfy the condition

$$\left| \frac{f'(z_0)}{F'(z_0)} \right| = \frac{5r_0^2 - 2r_0 + 1}{4r_0(1-r_0)}$$

and the proof is completed.

Theorem 2. *Let $f \in N_0$ and $F \in S$. If $f \prec F$ in K_1 and $|z| = r < 1$ then*

$$(11) \qquad |f'(z)| \leqslant T(r; N_0, S)\,|F'(z)|,$$

where

$$(12) \qquad T(r; N_0, S) = \begin{cases} 1 & \text{for } r \in \langle 0, 2-\sqrt{3}\,\rangle \\ \dfrac{4r^2 + (1-r)^4}{4r(1-r)^2} & \text{for } r \in (2-\sqrt{3}\,;\ 1) \end{cases}$$

The result is sharp. For $r \in \langle 0, 2-\sqrt{3}\rangle$ every pair of functions, $f(z) = e^{i\theta_1}F(z)$, $F(z)$ where $F \in S$ is extremal. For $r \in (2-\sqrt{3}, 1)$, $z_0 = r_0 e^{i\theta_0}$ a pair of functions,

$$(13) \qquad f(z) = e^{i\theta_1}\frac{r_0(1+2r_0-r_0^2)ze^{-i\theta_0} + (1-2r_0-r_0^2)}{r_0(1+2r_0-r_0^2) + (1-2r_0-r_0^2)ze^{-i\theta_0}} \cdot \frac{z}{(1+ze^{-i\theta_0})^2}$$

(14) $$F(z) = \frac{z}{(1 + ze^{-i\theta_0})^2}$$

where θ_1 is an arbitrary real number, is extremal.

Remark 2. The extremal function (14) belongs to the class $S^* \subset S$ and therefore the result of this theorem can not be improved by stronger condition that $F \in S^*$.

Remark 3. From the theorem 2 we can obtain immediately a generalization of Lewandowski's. Namely we can omit the condition $f'(0) > 0$ in the theorems $1'$ and $1''$ of paper [5].

Proof of theorem 2. By analogy to the way applied in the theorem 1 we have

(15) $$\frac{f'(z)}{F'(z)} = \phi'(z)\,\frac{F(z)}{F'(z)} + \phi(z)$$

where $\phi \in \Omega_0$ and $F \in S$. If $F \in S$ then by (18) p. 113 of [3] we have for $|z| = r < 1$

(16) $$r\,\frac{1-r}{1+r} < \left|\frac{F(z)}{F'(z)}\right| < r\,\frac{1+r}{1-r}.$$

Now from (15) by (6) and (16) we have for $|z| = r < 1$

(17) $$\left|\frac{f'(z)}{F'(z)}\right| < \frac{1 - |\phi(z)|^2}{1-r^2}\;\frac{r(1+r)}{1-r} + |\phi(z)|$$

If $|z| = r$ is fixed, then the right side of this inequality is a square function of a variable $u = |\phi(z)|$,

$$P(u) = -\frac{r}{(1-r)^2}\,u^2 + u + \frac{r}{(1-r)^2}\,.$$

The variable u may take values only in the interval $\langle 0, 1\rangle$. Now the function $P(u)$ has its maximum at the point $u = 1$ when $r \in \langle 0, 2-\sqrt{3}\,\rangle$ and at the point $u = (1-r)^2/2r$ when $r \in (2-\sqrt{3}, 1)$.

Therefore

$$(18)\qquad \max_{u\in\langle 0,1\rangle} P(u) = \begin{cases} 1 & \text{for } r\in\langle 0, 2-\sqrt{3}\,) \\ \dfrac{4r^2+(1-r)^4}{4r(1-r)^2} & \text{for } r\in(2-\sqrt{3}, 1). \end{cases}$$

Thus by (17) and (18) we obtain (11).

For every $z_0 = r_0 e^{i\theta_0}$, $r_0 \in (2-\sqrt{3}, 1)$ the functions (13) and (14) satisfy the equality

$$\left|\frac{f'(z_0)}{F'(z_0)}\right| = \frac{4r_0^2+(1-r_0)^4}{4r_0(1-r_0)^2}$$

and therefore the result is sharp.

In this part we determine the functions $T(r;N_n,S^c)$, $T(r;N_n,S)$ and $T(r;N_n,S)$ for $n\geqslant 1$.

Theorem 3. *Let $n\geqslant 1$, $f\in N_n$ and $F\in S^c$. If $f \ll F$ in K_1 and $|z| = r < 1$ then*

$$(19)\qquad |f'(z)| \leqslant T(r;N_n,S^c)\,|F'(z)|.$$

A function T is given by the formula

$$(20)\qquad T(r;N_n,S^c) = \begin{cases} (nr+n+1)r^n & \text{for } r\in\langle 0,\sigma_n\rangle \\ \dfrac{(nr+n+1)^2(1-r)^2+4r^2}{4(1-r)}\,r^{n-1} & \text{for } r\in(\sigma_n, 1) \end{cases}$$

where

$$(21)\qquad \sigma_n = \frac{2(n+1)}{3+\sqrt{4n^2+4n+9}}$$

The result is sharp. For $z_0 = r_0 e^{i\theta_0}$, $r_0\in\langle 0,\delta_n\rangle$ a pair of functions

$$(22)\qquad f(z) = \frac{z^{n+1}e^{i\theta_1}}{1+ze^{-i\theta_0}}, \quad F(z) = \frac{z}{1+ze^{-i\theta_0}}$$

where θ_1 an arbitrary real number is extremal. For $r_0\in(\delta_n, 1)$ the extremal pair is the following pair of functions

(23) $$f(z) = e^{i\theta_0} \frac{ze^{-i\theta_0} + a}{1 + aze^{-i\theta_0}} \frac{'' z^{n+1}}{1 + ze^{-i\theta_0}}, \quad F(z) = \frac{z}{1 + ze^{-i\theta_0}}$$

where, θ_1 -arbitrary real number and

(24) $$a = \frac{(n+2)r_0 - n - 1}{r_0(n - 1 - nr_0)} \in (-1, 1)$$

Proof. The fact that $f \in N_n$, $F \in S^c$ and $f \ll F$ in K_1 implies that there exists a function $\phi \in \Omega_n$ which satisfies the condition (5) and (8). If $\phi \in \Omega_n$ then

(25) $$|\phi(z)| \leqslant |z|^n \quad \text{for } z \in K_1$$

and the function

$$\Psi(z) = \phi(z)/z^n \in \Omega_0$$

Applying to the function $\Psi(z)$ we obtain

$$\left| \frac{\phi'(z)}{z^n} - \frac{n\,\phi(z)}{z^{n+1}} \right| \leqslant \frac{1 - \left| \frac{\phi(z)}{z^n} \right|^2}{1 - |z|^2}$$

Thus we have

(26) $$|\phi'(z)| \leqslant \frac{n\,|\phi(z)|}{|z|} + \frac{|z|^{2n} - |\phi(z)|^2}{|z|^n (1 - |z|^2)}, \quad z \in K_1$$

Now from (8) using (7) and (26) we obtain

(27) $$\left| \frac{f'(z)}{F'(z)} \right| \leqslant \frac{n\,|\phi(z)|}{r} + \frac{r^{2n} - |\phi(z)|^2}{r^n (1 - r^2)} r(1+r) + |\phi(z)| =$$

$$= -\frac{1}{r^{n-1}(1-r)} |\phi(z)|^2 + (nr + n + 1)|\phi(z)| + \frac{r^{n+1}}{1-r}$$

for $|z| = r < 1$

The right hand side of inequality (27) is a square function of variable $u = |\phi(z)|$. Let us denote it by $P(u)$. From (25) we have that u ranges the interval $\langle 0, r^n\rangle$. If $|z| = r$ is fixed then the function $P(u)$ takes its maximum at the point $u = r^n$ or at the point $u = \frac{1}{2}(nr + n + 1)(1 - r)r^{n-1}$ and therefore

$$\sup_{\phi \in \Omega_n} P(|\phi(z)|) = \max_{u \in \langle 0, r^n \rangle} P(u) = T(r; N_n, S^c)$$

where $T(r;N_n,S^c)$ is given by (20). This proves the theorem. A simple calculation shows that for the pairs of functions (22) and (23) we have equality in (19) for $z = z_0$.

Theorem 4. *Let $n \geqslant 1, f \in N_n, F \in S$. If $f \ll F$ in K_1 and $|z| = r < 1$ then*

$$|f'(z)| \leqslant T(r;N_n,S)\,|F'(z)|. \tag{28}$$

A function T is given by the formula

$$T(r;N_n,S) = \begin{cases} \dfrac{(n-1)r + n + 1}{n-1}\, r^n & \text{for } r \in \langle 0, \rho_n \rangle \\[2ex] \dfrac{3\,[n + 1 - 2r - (n-1)r^2]^3 + 2r^2}{4(1-r)^2}\, r^{n-1} & \text{for } r \in (\rho_n, 1) \end{cases} \tag{29}$$

where

$$\rho_n = \frac{n+1}{\sqrt{n^2+3}+2} \tag{30}$$

The result is sharp. For $z_0 = r_0 e^{i\theta_0}$, $r_0 \in \langle 0, \rho_n \rangle$ the pair of functions

$$f(z) = \frac{e^{i\theta_1} z^{n+1}}{(1 + ze^{-i\theta_0})^2}, \qquad F(z) = \frac{z}{(1 + ze^{-i\theta_0})^2}, \tag{31}$$

where θ_1 -arbitrary real number, is extremal. For $r_0 \in (\rho_n, 1)$, the extremal functions are the following

$$f(z) = \frac{e^{i\theta_1}(ze^{-i\theta_0} + b)\, z^{n+1}}{(1 + bze^{-i\theta_0})(1 + ze^{-i\theta_0})^2}, \qquad F(z) = \frac{z}{(1 + ze^{-i\theta_0})^2} \tag{32}$$

where θ_1 -arbitrary real number and

$$b = \frac{(n+1)(1-r_0^2) - 2r_0}{r_0(n-1)(r_0^2-1) + 2r_0} \in (-1, 1) \tag{33}$$

Proof. The assumptions imply that there exists a function $\phi \in \Omega_n$ which satisfies the identity (15). Then from (15) using (16) and (26) we have for $|z| = r < 1$

$$\left|\frac{f'(z)}{F'(z)}\right| < \frac{n\,|\phi(z)|}{r} + r\frac{r+1}{1-r}\;\frac{r^{2n} - |\phi(z)|^2}{r^n(1-r^2)} + |\phi(z)| =$$

$$= -\frac{|\phi(z)|^2}{r^{n-1}(1-r)^2} + \frac{n+1+(n-1)r}{1-r}\,|\phi(z)| + \frac{r^{n+1}}{(1-r)^2} = P(|\phi(z)|) \tag{34}$$

If $|z| = r$ is fixed then the right hand side of (34) takes its maximum at the point $|\phi(z)| = r^n$ or $|\phi(z)| = ½n + 1 + (n-1)r(1-r)r^{n-1}$. Therefore

$$\sup_{\phi\in\Omega_n} P(|\phi(z)|) = \max_{u\in\langle 0, r^n\rangle} P(u) = T(r; N_n, S) \tag{35}$$

where $T(r; N_n, S)$ is given by (29). Now by (35) and (34) we obtain (28). The functions (31) and (32) where b is given by (33) give the equality in (28) for $z_0 = r_0 e^{i\theta_0}$;

Remark 4. The function $F(z)$ given by (31) or (32) is starlike. Therefore the result of theorem 4 can not be improved if we replace S by S^*.
Now we can obtain some results concerning the relation between majorization of function $f \ll F$ and their derivatives $f' \ll F'$

Corollary 1. *Let $n = 1, 2, \ldots, f \in N_n, F \in S^c$.*

Then

$$f \ll F \text{ in } K_1 \Rightarrow f' \ll F' \text{ in } K_{r_n}$$

where r_n is the unique positive root of the equation

$$nr^{n+1} + (n+1)r^n - 1 = 0 \tag{36}$$

In particular

$$r_1 = \sqrt{2} - 1, r_2 = ½$$

The sequence $\{r_n\}$ *is increasing to* 1 *when n tends to* ∞ *and*

$$(37)\qquad (2n+1)^{-1/n} \leqslant r_n \leqslant (2n+1)^{-1/n+1} .$$

Proof. To find r_n it suffices to solve the inequality $T(r;\, N_n,\, S^c) \leqslant 1$. First we can prove that $r_n \leqslant \delta_n$, and then that (36) is implied immediately by (20). The inequalities (37) are obtained from the inequalities

$$nr^{n+1} + (n+1)r^{n+1} - 1 \leqslant nr^{n+1} + (n+1)r^n - 1 \leqslant nr^n + (n+1)r^n - 1$$

for $r \in \langle 0, 1\rangle$ and from the fact that these three functions are increasing in $\langle 0, 1\rangle$.

Corollary 2. *Let* $n = 1, 2, \ldots, f \in N_n, F \in S$ (or S^*).

Then

$$f \prec F \text{ in } K_1 \Rightarrow f' \prec F' \text{ in } K_{R_n}$$

where R_n *is the unique root of the equation*

$$(38)\qquad (n-1)r^{n+1} + (n+1)r^n + r - 1 = 0.$$

In particular

$$R_1 = {}^1/_3,\ R_2 = \sqrt{2} - 1 .$$

Proof. To find R_n it suffices to solve the inequality $T(r;\, N_n,\, S) \leqslant 1$. It is easy to show that $R_n \leqslant \delta_n$ and then (38) is immediately by (29).

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STRESZCZENIE

Mówimy, że funkcja f jest zmajoryzowana przez F w kole $K_1 = [z : |z| < 1]$ i piszemy $f \ll F$ w K_1, jeżeli dla każdego $z \in K_1$ zachodzi nierówność $|f(z)| \leqslant |F(z)|$.

W pracy tej rozważany jest następujący problem: niech A, B będą ustalonymi klasami funkcji holomorficznych w kole K_1. Wyznaczyć możliwie najmniejszą funkcję $T(r) = T(r; A, B)$ taką, żeby dla każdej pary funkcji $f, F (f \in A, F \in B)$ prawdziwa była następująca implikacja

$$f \ll F \text{ w } K_1 \Rightarrow |f'(z)| \leqslant T(r)\,|F'(z)| \text{ dla } |z| = r < 1$$

Problem ten rozwiązaliśmy całkowicie, gdy A jest klasą wszystkich funkcji holomorficznych w K_1 i mających rozwinięcie $f(z) = a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots, n = 0,1,2,\dots$ natomiast B jest klasą funkcji gwiaździstych lub klasą funkcji wypukłych w kole K_1.

РЕЗЮМЕ

Говорим, что функция f является змаëризованной через F в кругу $K_1 = \{z : |z| < 1\}$ и пишем $f \ll F$ в K_1, если для каждого $z \in K_1$ исполнено неравенство $|f(z)| \leqslant |F(z)|$.

В данной работе выступает следующая проблема: Пусть A, B будут определенными классами голоморфных функции в кругу K_1. Обозначить возможно самую малую функцию $T(r) = T(r; A, B)$ такую, чтобы для каждой пары функций $f, F(f \in A, F \in B$ была правдивой следующая импликация: $f \ll F$ в $K_1 \Rightarrow |f'(z)| \leqslant T(r)\,|F'(z)|$ для $|z| = r < 1$

Эту проблему разрешено вполне, если A является классом всех голоморфных функций в K_1 и имеющих разложение $f(z) = a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots$, $n = 0, 1, 2, \dots$, а B является классом звездных функций или классом выпуклых функций в круге K_1.