ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXII, 3 SECTIO A 1978

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On a Certain Integrodifferential System with Delay

O pewnym układzie równań różniczkowo-całkowych z opóźnieniem

Об некоторой системе интегрально-дифференциальных уравнений с запаздыванием

1. Notations. In this paper we shall use the following notations: R is the real line; R^+ and R^- are the positive, respektively negative, half–axis of R; M_{nxn} is the set of square matrices of order n with the Euclidian norm;

$$L^1([0;+\infty)) = [x/x : R^+ \longrightarrow R^n\ ;\ \|x\|_{L^1} = \int_0^{+\infty} \|x(t)\|\, dt < +\infty]$$

2. Preliminaries. In some papers C. Corduneau [1–3], R. K. Miller [8–10], S. I. Grossman [11] and N. Luca [6–7] have established various variation of constants formula for some classes of integrodifferential systems.

For various classes of integrodifferential systems have been studied by the help of theese formula the existence, uniqueness and behaviour of the solutions of some problems with initial conditions. In the following sections we shall establish a variation of constants formula for the case of some integrodifferential systems with delay whose kernel isn't a translation.

Using then this result we shall study some problems concerning a class of nonlinear systems with delay.

3. Let us consider the integrodifferential equation:

$$(\mathrm{E}_0) \qquad \dot{x}(t) = A\cdot x(t) + B\cdot x(t-\tau) + \int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot x(\zeta)\, d\zeta \quad \text{with } t \geqslant \tau \geqslant 0$$

where A, B are real matrices from M_{nxn}, $K(t) \in M_{nxn}$ is a given matrix function defined on R^+ and $K(t) = 0 \in M_{nxn}$ for $t \in R^-$, where 0 is the null matrix of M_{nxn}, $\tau \in R^+$ is given number and $h : R^+ \to R$ is a given function.

The main problem is to establish some conditions under which the equation (E_0) with the initial conditions:

$$(3.1) \qquad x(t) = g(t),\ t < \tau \ \text{ and } \ x(\tau+) = x^0$$

has solutions and what is the behaviour of theese solutions with respect to initial data.

With this purpose let us consider the nonhomogeneous equations

$$(E) \qquad \dot{x}(t) = A\cdot x(t) + B\cdot x(t-\tau) + \int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot x(\zeta)\,d\zeta + f(t), \quad t \geqslant \tau \geqslant 0$$

and the initial conditions (3.1)

where A, B, h, K, g, τ, are previously defined and $f: R^+ \to R^n$ is a given function. In connection with this problem the main result is contained in the theorem 3.1.

Theorem 3.1. *Assume that*

(i) *There is a unique continuous matrix function of two variables $Y(t, u)$ which satisfies the condition:*

$$(3.2) \qquad \frac{\partial Y}{\partial t}(t,u) = A\cdot Y(t,u) + B\cdot Y(t-\tau, u) + \int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot Y(\zeta, u)\,d\zeta$$

$$(3.3) \qquad \frac{\partial Y}{\partial t}(t,u) = -Y(t,u)\cdot A - Y(t, u+\tau)\cdot B - h(u)\cdot \int_u^t Y(t,\zeta)\cdot K(\zeta - u)\,d\zeta$$

for $t \geqslant \tau$ and for, $u \geqslant \tau$.

$$(3.4) \qquad Y(a,a) = I,\ a \geqslant \tau$$

$$(3.5) \qquad Y(t,u) = 0,\ u > t$$

(ii) *$h: R^+ \to R$ and $f: R^+ \to R^n$ are given continuous functions*

(iii) *$g: (-\infty; \tau) \to R^n$ is a given continuous function;*

(iv) $\| K(t) \| \in L^1([0; +\infty))$,

Then there is a unique solution of the equation (E) which satisfies conditions (3.1) *and this solution is given by*

$$x(t) = Y(t,\tau)\cdot x^0 + \int_0^\tau Y(t, u+\tau)\cdot B\cdot g(u)\,du + \int_\tau^t Y(t,u)\cdot f(u)\,du, \quad t \geqslant \tau$$

$$(3.6) \qquad x(t) = g(t), \quad t < \tau$$

Formula (3.6) *is reffered to as 'the variation of constants formula' for equation* (E).

Proof. We assume that the equation (E) with initial values (3.1) has solutions and we shall prove that any solution has the form (3.6). Then we shall prove that the function $x(t)$ defined by the formula (3.6) verifirs the equation (E) and tne initial conditions (3.1).

Integrating between τ and t in the both members of the identity:

$$\frac{\partial}{\partial u}\,[Y(t,u)\cdot x(u)\,] = \frac{\partial Y}{\partial u}(t,u)\cdot x(u) + Y(t,u)\cdot \dot{x}(u)$$

we obtain

$$Y(t,u)\cdot x(u)\,\Bigg|_{u=\tau}^{u=t} = \int_{\tau}^{t}\left[\frac{\partial}{\partial u}Y(t,u)\cdot x(u) + Y(t,u)\cdot \dot{x}(u)\right] du\,,$$

that is

$$Y(t,t)\cdot x(t) - Y(t,\tau)\cdot x(\tau+) = \int_{\tau}^{t}\,[\,\frac{\partial Y}{\partial u}(t,u)\cdot x(u) + Y(t,u)\cdot \dot{x}(u)]\,du,$$

whence, according to (3.4), (3.1) and (E) one can obtain the representation:

$$x(t) = Y(t,\tau)\cdot x^0 + \int_{\tau}^{t}\Bigg[\frac{\partial Y}{\partial u}(t,u)\cdot x(u) + Y(t,u)\cdot A\cdot x(u) +$$

$$+ Y(t,u)\cdot B\cdot x(u-\tau) + Y(t,u)\cdot \int_{\tau}^{u} h(\zeta)\cdot K(u-\zeta)\cdot x(\zeta)d\zeta +$$

$$+ Y(t,u)\cdot f(u)\Bigg]\,du$$

or

$$x(t) = Y(t,\tau)\cdot x^0 + \int_{\tau}^{t}\left[\frac{\partial Y}{\partial u}(t,u)\cdot x(u)\right] du + \int_{\tau}^{t} Y(t,u)\cdot A\cdot x(u)\,du +$$

(3.7)
$$+ \int_{\tau}^{t} Y(t,u)\cdot B\cdot x(u-\tau)\,du + \int_{\tau}^{t} Y(t,u)\cdot(\int_{\tau}^{u} h(\zeta)\cdot K(u-\zeta)\cdot x(\zeta)\,d\zeta)\,du +$$

$$+ \int_{\tau}^{t} Y(t,u)\cdot f(u)\,du\,.$$

Taking into account (3.5) we can write:

$$\int_\tau^t Y(t,u)\cdot B\cdot x(u-\tau)\,du=\int_0^{t-\tau} Y(t,u+\tau)\cdot B\cdot x(u)\,du=\int_0^\tau Y(t,u+\tau)\cdot B\cdot$$

(3.8)

$$\cdot x(u)\,du+\int_\tau^{t-\tau} Y(t,u+\tau)\cdot B\cdot x(u)\,du=\int_0^\tau Y(t,u+\tau)\cdot B\cdot g(u)\,du+\int_\tau^t Y(t,u+\tau)\cdot$$

$$\cdot B\cdot x(u)\,du.$$

According to (3.8) and (3.1) from (3.7) we obtain:

$$x(t)=Y(t,\tau)\cdot x^0+\int_\tau^t\Big[\frac{\partial Y}{\partial u}(t,u)+Y(t,u)\cdot A+Y(t,u+\tau)\cdot B\Big]\cdot x(u)\,du+$$

$$+\int_0^\tau Y(t,u+\tau)\cdot B\cdot g(u)\,du+\int_\tau^t Y(t,u)\cdot\Big(\int_\tau^u h(\zeta)\cdot K(u-\zeta)\cdot x(\zeta)\,d\zeta\Big)\,du+ \tag{3.9}$$

$$+\int_\tau^t Y(t,u)\cdot f(u)\,du.$$

But

$$\int_\tau^t Y(t,u)\cdot\Big(\int_\tau^u h(\zeta)\cdot K(u-\zeta)\cdot x(\zeta)\,d\zeta\Big)\,du=\int_\tau^t h(\zeta)\cdot\Big(\int_\zeta^t Y(t,u)\cdot K(u-\zeta)\,du\Big)\cdot$$

$$\cdot x(\zeta)\,d\zeta=\int_\tau^t h(u)\cdot\Big(\int_u^t Y(t,\zeta)\cdot K(\zeta-u)\,d\zeta\Big)\cdot x(u)\,du$$

and therefore (3.9) can be written:

$$x(t)=Y(t,\tau)\cdot x^0+\int_\tau^t\Big[\frac{\partial Y}{\partial u}(t,u)+Y(t,u)\cdot A+Y(t,u+\tau)\cdot B+h(u)\cdot$$

(3.10)

$$\cdot\int_u^t Y(t,\zeta)\cdot K(\zeta-u)\,d\zeta\Big]\cdot x(u)\,du+\int_0^\tau Y(t,u+\tau)\cdot B\cdot g(u)\,du+\int_\tau^t Y(t,u)\cdot f(u)\,du.$$

Finally, according to (3.3) the formula (3.10) becomes:

$$x(t)=Y(t,\tau)\cdot x^0+\int_0^\tau Y(t,u+\tau)\cdot B\cdot g(u)\,du+\int_\tau^t Y(t,u)\cdot f(u)\,du,$$

which is just the first formula (3.6).

Therefore any solution of the equation (E) with the initial conditions (3.1) has the form (3.6). This fact proves the uniqueness of solution of equation (E) with initial conditions (3.1).

Let us prove the existence of a solution for this problem

For this we have sufficiently to prove that $x(t)$ definet by the formula (3.6) is a solution of the equation (E) and satisfies the initial conditions (3.1).

Deriving the both members of this equality we obtain:

(3.11)

$$\dot{x}(t) = \frac{\partial Y}{\partial t}(t,\tau)\cdot x^0 + Y(t,t)\cdot f(t) + \int_\tau^t \frac{\partial Y}{\partial t}(t,u)\cdot f(u)\,du + \int_0^\tau \frac{\partial Y}{\partial t}(t,u+\tau)\cdot B\cdot g(u)\,du.$$

Then on the base of condition (3.5) we can write:

$$\int_\tau^t \left(\int_\tau^\zeta h(\zeta)\cdot K(t-\zeta)\cdot Y(\zeta,u)\cdot f(u)\,du\right) d\zeta = \int_\tau^t\left(\int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot Y(\zeta,u)\cdot f(u)\,d\zeta\right) du =$$

$$(3.12)\qquad = \int_\tau^t \left(\int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot Y(\zeta,u)\, f(u)\,d\zeta\right) du,$$

$$\int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot\left(\int_0^\tau Y(\zeta)\,u+\tau)\cdot B\cdot g(u)\,du\right) du =$$

(3.13)

$$= \int_0^\tau \left(\int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot Y(\zeta, u+\tau)\,d\zeta\right)\cdot B\cdot g(u)\,du.$$

Now according to (3.2), (3.5), (3.12) and (3.13), the relation (3.11) becomes:

$$(3.14)\quad \dot{x}(t) = \frac{\partial Y}{\partial t}(t,\tau)\cdot x^0 + f(t) + \int_\tau^t \frac{\partial Y}{\partial t}(t,u)\cdot f(u)\,du + \int_0^\tau \frac{\partial Y}{\partial t}(t,u+\tau)\cdot B\cdot g(u)\,du =$$

$$= A\cdot Y(t,\tau)\cdot x^0 + \int_\tau^t Y(t,u)\cdot f(u)\,du + \int_0^\tau A\cdot Y(t,u+\tau)\cdot B\cdot g(u)\,du + B\cdot Y(t-\tau,\tau)\cdot x^0 +$$

$$+ \int_\tau^{t-\tau} B\cdot Y(t-\tau,u)\cdot f(u)\,du + \int_0^\tau B\cdot Y(t-\tau,u+\tau)\cdot B\cdot g(u)\,du + \int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot Y(\zeta,\tau)\cdot$$

$$\cdot x^0\,d\zeta + \int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot\left(\int_\tau^\zeta Y(t,u)\cdot f(u)\,du\right) d\zeta + \int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot\left(\int_0^\tau Y(\zeta,u+\tau)\cdot\right.$$

$$\left.\cdot B\cdot g(u)\,du\right) d\zeta + f(t).$$

On the base of formula (3.6) we can write:

$$(3.15)\quad A\cdot x(t) + B\cdot x(t-\tau) + \int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot x(\zeta)\,d\zeta = A\cdot Y(t,\tau)\cdot x^0 + \int_0^\tau A\cdot Y(t,u+\tau)\cdot$$

$$\cdot B\cdot g(u)\,du + \int_\tau^t A\cdot Y(t,u)\cdot f(u)\,du + B\cdot Y(t-\tau,\tau)\cdot x^0 + \int_0^\tau B\cdot Y(t-\tau,u+\tau)\cdot B\cdot g(u)\,du +$$

$$+ \int_\tau^{t-\tau} B\cdot Y(t-\tau,u)\cdot f(u)\,du + \int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot Y(\zeta,\tau)\cdot x^0\,d\zeta + \int_\tau^t h(\zeta)\cdot K(t-\zeta)\cdot$$

$$\cdot\left(\int_0^{\tau} Y(\zeta, u+\zeta)\cdot B\cdot g(u)\,du\right)d\zeta + \int_{\tau}^{t} h(\zeta)\cdot K(t-\zeta)\cdot\left(\int_{\tau}^{\zeta} Y(\zeta, u)\cdot f(u)\,du\right)d\zeta .$$

Comparing (3.14) with (3.15) we obtain:

$$\dot{x}(t) = A\cdot x(t) + B\cdot x(t-\tau) + \int_{\tau}^{t} h(\zeta)\cdot K(t-\zeta)\cdot x(\zeta)\,d\zeta + f(t), \quad t > \tau$$

i. e. (3.6) is a solution of the equation (E).

Let us prove that $x(t)$ given by the formula (3.6) verifies the initial conditions (3.1). From the first relation (3.6) we obtain:

$$x(\tau+) = x^0 + \lim_{\substack{t\to\tau \\ t>\tau}} \int_0^{\tau} Y(t, u+\tau)\cdot B\cdot g(u)\,du .$$

Taking into account the hyppotheses (i) – (iii), (3.5) and Lebeque's criterion of domination we have:

$$\lim_{\substack{t\to\tau \\ t>\tau}} \int_0^{\tau} Y(t, u+\tau)\cdot B\cdot g(u)\,du = \int_0^{\tau} Y(\tau, u+\tau)\cdot B\cdot g(u)\,du = 0 .$$

Therefore $x(\tau+) = x^0$, which, together with the second relation (3.6) constitutes the relations (3.1).

Let us consider the equation:

$$(E_1) \qquad \dot{x}(t) = A\cdot x(t) + B\cdot x(t-\tau) + \int_{\tau}^{t} h(\zeta)\cdot K(t-\zeta)\cdot x(\zeta)\,d\zeta + f(t, x(t)), \quad t \geq \tau \geq 0$$

with the initial conditions (3.1).

With respect to this problem one can easily obtain the following result:

Theorem 3.2. *If besides the hyppotheses of the theorem* 3.1 *the conditions* (i1) *and* (i2) *are satisfied and if* $L\cdot C<1$, *then the equation* (E_1) *with initial conditions* (3.1) *has an unique solution belonging to* $L^1([0; +\infty))$, *and this solution satisfies the nonlinear Volterra integral equation:*

$$(3.16) \qquad x(t) = Y(t,\tau)\cdot x^0 + \int_0^{\tau} Y(t, u+\tau)\cdot B\cdot g(u)\,du + \int_{\tau}^{t} Y(t,u)\cdot f(u, x(u))\,du$$

(i1) $\int_0^{+\infty} \| Y(t,u) \|\, dt \leq C, \ \forall\, u \geq 0$ *where C is a positive constant.*

(i2) $f: R^+ \times R^n \to R^n$ *is Lipschitzian and its Lipschitz constant is L.*

Proof. One can easily observe that the operator $A : L^1([0;+\infty)) \to L^1([0;+\infty))$ defindet by the formula:

$$(Ax)(t) = Y(t,\tau)\cdot x^0 + \int_0^\tau Y(t, u+\tau)\cdot B\cdot g(u)\,du + \int_\tau^t Y(t,u)\cdot f(u, x(u))\,du$$

is a contraction.

Indeed from:

$$\| Ax - Ay \|_{L^1} = \int_0^{+\infty} \| \int_\tau^t Y(t,u)\cdot [f(u,x(u)) - f(u,y(u))]\,du \|\,dt \leqslant$$

$$\leqslant \int_0^{+\infty} L\cdot \| x(u) - y(u) \| \cdot \left(\int_0^{+\infty} \| Y(t,u) \|\,dt\right) du \leqslant L\cdot C\cdot \| x - y \|_{L^1}$$

and because $L \cdot C < 1$ we obtain that A is a contraction and so it has an unique fix point in $L^1([0;+\infty))$.

By the help of the formula (3.6) one can obtain some results with respect to the behaviour of the solutions of the homogeneous equation (E_0).

Obviously, the equation (E_0) with initial condition (3.1) $g(t) = 0$ for $t < \tau$, $x^0 = 0$, has only the solution $x = 0$ which is called the trival solution of the homogeneous problem.

Now we want to study various stability types of the trival solution of this problem. For this, we shall femind the necessary definitions:

Definition 3.1. Let $x(t, \tau, g, x^0)$ be the solution of the equation (E_0) with initial condition (3.1).

a) The trival solution of the homogeneous problem is called stable with respect to initial values (τ, x^0, g) if it is defined on R and for every $\tau \geqslant 0$ and for any $\epsilon > 0$ there exists a number $\delta(\epsilon, \tau) > 0$ such that for all $g \in L^\infty([0;+\infty))$ with $|g|_\tau \leqslant \delta$ and for all x^0 with $\| x^0 \| \leqslant \delta$, $x(t, \tau, g, x^0)$ is defined for $t \geqslant \tau$ and $\| x(t, \tau, g, x^0) \| \leqslant \epsilon$ for $t \geqslant \tau$, where

$$|g|_\tau = \sup_{t \leqslant \tau} \| g(t) \|.$$

b) The trival solution of the homogeneous problem is called uniformly stable with respect to initial values (τ, x^0, g) if it is stable and δ can be chosen indepdndent of τ.

c) The trival solution of the homogeneuous problem is called asymptotically stable with respect to initial values (τ, x^0, g), if it is stable and if for any given (τ, x^0, g) one has:

$$\lim_{t \to +\infty} \| x(t, \tau, g, x^0) \| = 0,$$

d) The trival solution of the homogeneuous problem is called uniformly asymptotically stable with respect to (τ, x^0, g), if it is uniformly stable and if there exists a number $A > 0$, such that for any given $\epsilon > 0$ there exists $T(\epsilon)$ such that:

$$\|x(t + T(\epsilon), \tau, g, x^0)\| \leqslant \epsilon,$$

uniformly for all $t \geqslant \tau$, all $\tau \geqslant 0$ and all $x^0 \in R^n$, $g \in L^\infty([0; +\infty))$ such that:

$$\| x^0 \| \text{ and } | g |_\tau \leqslant A.$$

The main results with respect to the stability of the trival solution of the homogeneous problem are included in the following theorems:

Theorem 3.3. *In the hyppotheses of the theorem 3.1 the necesary and sufficient conditions such that the trival solution of the homogeneous problem is uniformly stable are:*

$\int_\tau^{2\tau} \| Y(t, u) \| du$ *and* $Y(t, \tau)$ *are uniformly bounded with respect to* τ.

Proof. (a) From (3.6) we obtain that the solution $x(t, \tau, g, x^0)$ has the form:

$$x(t, \tau, g, x^0) = Y(t, \tau)\cdot x^0 + \int_0^\tau Y(t, u + \tau)\cdot B\cdot g(u)\, du \tag{3.17}$$

from where

$$\| x(t, \tau, g, x^0) \| \leqslant \| Y(t, \tau) \| \cdot \| x^0 \| + \| B \| \cdot | g |_\tau \cdot \int_0^\tau \| Y(t, u + \tau) \| du =$$

$$= \| Y(t, \tau) \| \cdot \| x^0 \| + \| B \| \cdot | g |_\tau \cdot \int_\tau^{2\tau} \| Y(t, u) \| du$$

Because $\| Y(t, \tau) \| \leqslant C_2$ for all $t \geqslant \tau \geqslant 0$ and $\int_\tau^{2\tau} \| Y(t, u)\| du \leqslant C_1$ for all $t \geqslant \tau \geqslant 0$ we obtain:

$$\| x(t, \tau, g, x^0) \| \leqslant C_2 \cdot \| x^0 \| + \|B \| \cdot C_1 \cdot | g |_\tau$$

Therefore, taking $\delta(\epsilon) = \min \left[\frac{\epsilon}{2C_2}, \frac{\epsilon}{2\cdot\| B \|\cdot C_1} \right]$ which is obviously independent of τ, then if $\| x^0 \| \leqslant \delta$ and $| g |_\tau \leqslant \delta$ one can obtain $\| x(t, \tau, g, x^0) \| \leqslant \epsilon$.

(b) We assume that the trival solution os uniformly stable. If $\epsilon_0 > 0$ is fixed, then we have:

$$(3.18)\qquad \| x(t,\tau,g,x^0) \| = \| Y(t,\tau)\cdot x^0 + \int_0^T Y(t,u+\tau)\cdot B\cdot g(u)\,du \| \leqslant \epsilon_0\,,\quad t \geqslant \tau$$

for every $\tau \geqslant 0$ and for all (g, x^0) with $|g|_\tau \leqslant \delta_0, \|x^0\| \leqslant \delta_0$ $(\delta_0 = \delta_0(\epsilon_0)$ is a constant).

Taking $g \equiv 0$ and $x^0 \neq 0 \in R^n$ in (3.18), we obtain:

$$(3.19)\qquad \| Y(t,\tau)\cdot x^0 \| \leqslant \epsilon_0 \text{ for } t \geqslant \tau \text{ and for every } \tau \geqslant 0 \text{ and for all } x^0 \text{ with } \| x^0 \| < \delta_0,$$

from where, according to the principle of uniform boundness (see [12]) one can obtain:

$$(3.20)\qquad \| Y(t,\tau) \| \leqslant C_2 \text{ for } t \geqslant \tau \geqslant 0.$$

Now we take $x^0 = 0 \in R^n$ and $g \in L^\infty([0;+\infty))$ in (3.18), and we obtain:

$$\| \int_0^T Y(t,u+\tau)\cdot B\cdot g(u)\,du \| \leqslant \epsilon_0$$

from where, acording to the theorem of representation of the linear functionales, we have:

$$\int_0^T \| Y(t,u+\tau)\cdot B \|\,du \leqslant \epsilon_0$$

from where

$$\int_\tau^{2T} \| Y(t,u)\cdot B \|\,du \leqslant \epsilon_0$$

which obviously involves:

$$\int_\tau^{2T} \| Y(t,u) \|\,du \leqslant C_1 \quad \text{for every } \tau \geqslant 0\,.$$

Theorem 3.4. *In the hyppotheses of the theorem 3.1 the necessary and sufficient conditions such that the trival solution of the homogeneous problem is uniformly asymptotically stable are:*

$\int_\tau^{2T} \| Y(t,u) \|\,du$ *and* $\| Y(t,\tau) \| \to 0$ *when* $t \to +\infty$ *uniformly with respect to* τ, *i.e. for any* $\epsilon \geqslant 0$ *there exists* $T(\epsilon) > 0$ *such that for every* $\tau \geqslant 0$ *one has*

$$\int_\tau^{2T} \| Y(t,u) \|\,du < \epsilon \quad \textit{and} \quad \| Y(t,\tau) \| < \epsilon \quad \textit{for } t \geqslant T(\epsilon)\,.$$

Proof. Obviously, in the hyppotheses of the theorem 3.4 the trival solution of the homogeneuous problem is uniformly stable.

(a) From (3.17) we have:

$$\| x(t, \tau, g, x^0) \| \leqslant \| Y(t, \tau) \| \cdot \| x^0 \| + \| B \| \cdot | g |_\tau \cdot \int_\tau^{2\tau} \| Y(t, u) \| \, du .$$

Because $$\int_\tau^{2\tau} \| Y(t, u) \| \, du < \frac{\epsilon}{2A} \quad \text{if } t > T_0(\epsilon), \, t \geqslant \tau, \, \forall \, \tau \geqslant 0 ,$$

$$\| Y(t, \tau) \| < \frac{\epsilon}{2 \cdot \| B \| \cdot A} \quad \text{if } t > T_0(\epsilon) \, ; \, t \geqslant 0, \, \forall \, \tau \geqslant 0 .$$

taking $| g |_\tau < A$ and $\| x^0 \| < A$, we have:

$$\| x(t, \tau, g, x^0) \| \leqslant \epsilon, \forall t \geqslant T_0(\epsilon).$$

(b) We assume that the trival solution is uniformly asymptotically stable and this assumption implies that if $\epsilon > 0$ is fixed then for $\| x^0 \| < A$ and $| g |_\tau < A$ there exists $T(\epsilon)$ such that

$$\| x(t, \tau, g, x^0) \| < \epsilon \text{ for } t \geqslant T(\epsilon), \, t \geqslant \tau \geqslant 0.$$

Taking $g \equiv 0$ and $x^0 \neq 0 \in R^n$ in (3.18) we obtain:

$$\| Y(t, \tau) \cdot x^0 \| \leqslant \epsilon, \, t \geqslant \tau \geqslant 0, t \geqslant T(\epsilon).$$

Therefore $\| Y(t, \tau) \| \to 0$ for $t \to +\infty$ uniformly with respect to τ.

If in (3.18) we take $x^0 = 0 \in R^n$ and $g \in L^\infty([0; +\infty))$ one can obtain that $\forall \epsilon > 0$, there exists $T(\epsilon)$ such that

$$\| \int_0^\tau Y(t, u + \tau) \cdot B \cdot g(u) \, du \| \leqslant \epsilon .$$

Therefore one can easily obtain:

$$\int_\tau^{2\tau} \| Y(t, u) \cdot B \| \, du \leqslant \epsilon ,$$

$$\text{i.e. } \int_\tau^{2\tau} \| Y(t, u) \| \, du \to 0 \quad \text{for } t \to +\infty$$

(uniformly with respect to τ), with the proof of the theorem is finished.

4. Let us consider the equation

(E2) $\dot{x}(t) = A\cdot x(t) + B\cdot x(t-\tau) + \int_{\tau}^{t} h(\zeta)\cdot K(t-\zeta)\cdot x(\zeta)\,d\zeta + f(t, x(t))$, $T \geq t \geq \tau \geq 0$

with initial conditions (3.1).

With respect to this problem one can easily obtain the following result:

Theorem 4.1. *If besides the hyppotheses of the theorem 3.1. the conditions:*

(j1) $$\| Y(t,u) \| \leq C.\ \forall\, t, u \in [0;T], C \geq 1;$$

(j2) *$f(t,x)$ is a boundet real function defined on $D = [0, T] \times R^n$ which satisfies the Lipschitz's condition:*

$$\| f(t,x) - f(t,y) \| \leq G(t) \cdot \| x - y \|$$

where $G(t)$ is an integrable function on $[0;T]$

$$f : [0;T] \times R^n \to R^n,$$

are satisfied then the equation (E2) *with initial condition* (3.1) *has a unique solution belonging to $C([0;T])$.*

Proof. For proof will be necessary the following results.

Definition (see [5]). Let F be a transformation of B into itself, where B is a Banach space with the norm $\| \, \|_0$. The transformation F will be called 'a strong contraction' if for every number $\epsilon > 0$ there exists a norm $\| \, \|_\epsilon \sim \| \, \|_0$ such that for every $x, y \in B$,

$$\| Fx - Fy \|_\epsilon \leq \epsilon \cdot \| x - y \|_\epsilon .$$

Theorem 4.2 (see [5]). *A 'strong contration' has a unique fix–point. We remark then that the equation* (E2) *with initial data* (3.1) *are equivalent with nonlinear Volterra integral equation:*

$$x(t) = Y(t,\tau)\cdot x^0 + \int_0^{\tau} Y(t, u+\tau)\cdot B\cdot g(u)\,du + \int_{\tau}^{t} Y(t,u)\cdot f(u, x(u))\,du .$$

Now we consider the operator $A : C([0;T]) \to C([0;T])$ defined by the formula:

$$(Ax)(t) = Y(t,\tau)\cdot x^0 + \int_0^{\tau} Y(t, u+\tau)\cdot B\cdot g(u)\,du + \int_{\tau}^{t} Y(t,u)\cdot f(u, x(u))\,du .$$

$C([0;T])$ is a Banach space with the norm: $\| x \|_0 = \sup_{[0;T]} \| x(t) \|$.

Define:

$$L(t) \stackrel{\text{def}}{=\!=\!=} G(t)\cdot C, \quad t \in [0;T].$$

On $C([0;T])$ we take the norms (see [5])

$$\| x \|_p = \sup_{(0;T)} e^{-p\int_0^t L(\zeta)\,d\zeta} \cdot \| x(t) \| \ , \ x \in C([0;T]) \ , \ p > 0$$

and we obserye that $\| \cdot \|_p \sim \| \cdot \|_0$

and we observe that $\| \cdot \|_p \sim \| \cdot \|_0$.

Now we shall prove that A is a 'strong contraction'. This statement results from the following inequalities:

$$\| \mathring{A}_x - \mathring{A}_y \|_p = \sup_{(0,T)} e^{-p\cdot\int_0^t L(\zeta)\,d\zeta} \cdot \| \int_{\tau}^{t} Y(t,u)\cdot[f(u,x(u)) - f(u,g(u))]\, du \ \| \leqslant$$

$$\leqslant \sup_{(0,T)} e^{-p\cdot\int_0^t L(\zeta)\,d\zeta} \cdot \int_0^t \| Y(t,u) \| \cdot G(u) \cdot \| x(u) - \dot{y}(u) \| \, du \leqslant$$

$$\leqslant \sup_{(0,T)} e^{-p\cdot\int_0^t L(\zeta)\,d\zeta} \cdot \int_0^t G(u)\cdot C \cdot \| x(u) - y(u) \| \, du \ =$$

$$= \sup_{(0,T)} e^{-p\int_0^t L(\zeta)\,d\zeta} \cdot \int_0^t L(u)\cdot e^{p\cdot\int_0^u L(\zeta)\,d\zeta} \cdot e^{-p\cdot\int_0^u L(\zeta)\,d\zeta} \cdot \| x(u) -$$

$$- y(u) \| \, du \leqslant \| x - y \|_p \cdot \sup_{(0,T)} e^{-p\cdot\int_0^t L(\zeta)\,d\zeta} \cdot \int_0^t L(u)\cdot e^{p\cdot\int_0^u L(\zeta)\,d\zeta} \, du =$$

$$= \| x - y \|_p \cdot \sup_{(0,T)} e^{-p\cdot\int_0^t L(\zeta)\,d\zeta} \cdot \left(\frac{1}{p} \cdot e^{p\cdot\int_0^u L(\zeta)\,d\zeta} \Big|_0^t \right) =$$

$$= \| x - y \|_p \cdot \sup_{(0,T)} \left(\frac{1}{p} - \frac{1}{p} \cdot e^{-p\cdot\int_0^t L(\zeta)\,d\zeta} \right) \leqslant \frac{1}{p} \cdot \| x - y \|_p .$$

By virtue theorem 4.2. we conclude that the operator has a unique fix-point in $C([0;T])$ which is just the solution of equation (E2) with initial conditions (3.1).

Let us consider the equation (E2) with initial conditions (3.1)

Other result with respect to this problem is contained in following theorem:

Theorem 4.3. *If besides the hyppotheses of the theorem* 3.1. *the conditions:*

(k 1) $f: [0; +\infty) \times R^n \to R^n$ *is a given continuous and bounded function,*

(k 2) $\| Y(t, u) \| \leq C_0 . \forall t, u \in [0; +\infty)$,

(k 3) *there exist the numerical sequences* $[\alpha_n]$, $[L_n]$ – *and a constant* $A > 0$ *such that,*

$$\| f(t, x) - f(t, y) \| \leq L_n \cdot \| x - y \|^{\alpha_n} ,$$

$$\lim_{n \to +\infty} \alpha_n = 1 , \quad \lim_{n \to +\infty} L_n = +\infty$$

$$L_i \leq \frac{A}{C_0} \cdot S_i , \, i \in N \quad where$$

$$S_n = 1 + \alpha_1 \alpha_2 \ldots \alpha_n + \alpha_2 \alpha_3 \ldots \alpha_n + \ldots + \alpha_{n-1} \alpha_n + \alpha_n =$$

$$= ([[(1 + \alpha_1) \cdot \alpha_2 + 1] \cdot \alpha_3 + 1] \ldots) \alpha_n + 1$$

are satisfied, then the equation (E 2) *with initial data* (3.1) *has no more than one solution defined on* $R_+ = [0; +\infty)$.

Proof. First we remark that the equation (E 1) with initial data (3.1) is equivalent with the nonlinear Volterra integral equation (3.16)

$$(3.16) \qquad x(t) = Y(t, \tau) \cdot x^0 + \int_0^\tau Y(t, u + \tau) \cdot B \cdot g(u) \, du + \int_\tau^t Y(t, u) \cdot f(u, x(u)) \, du .$$

Then we observe that condition $\lim_{n \to +\infty} \alpha_n = 1$ implies $\lim_{n \to +\infty} S_n = +\infty$

Let x, y be two solutions of our problem.
We shall prove that $x(t) = y(t)$ for every $t \in [0; +\infty)$.
We consider $T > 0$ and we shall prove that $x(t) = y(t)$ for every $t \in [0; T]$.
Because f is bounded we have:

$$\| x(t) - y(t) \| \leq \int_0^t \| Y(t, u) \| \cdot \| f(u, x(u)) - f(u, y(u)) \| \, du \leq 2 M C_0 \, t .$$

Then, from the cонsition (k 3) we have:

$$\| x(t) - y(t) \| \leq \int_0^t \| Y(t, u) \| \cdot \| f(u, x(u)) - f(u, y(u)) \| \, du \leq$$

$$\leq C_0 \cdot \int_0^t L_1 \cdot \| x(u) - y(u) \|^{\alpha_1} \quad du \leq C_0 \cdot \int_0^t L_1 (2 M C_0 u)^{\alpha_1} \; du .$$

or

$$\| x(t) - y(t) \| \leq \frac{L_1 \cdot C_0 (2 M C_0)^{\alpha_1} \cdot t^{1+\alpha_1}}{1 + \alpha_1} \leq A \cdot (2 M C_0)^{\alpha_1} \cdot t^{1+\alpha_1} =$$

$$= (At)^{1+\alpha_1} \cdot (\frac{2 M C_0}{A})^{\alpha_1} .$$

Taking into account (k 2) and (k 3) for $n = 2$, we obtain:

$$\| x(t) - y(t) \| \leq \int_0^t C_0 \cdot \| f(u, x(u)) - f(u, y(u)) \| \; du \leq$$

$$\leq C_0 \cdot \int_0^t L_2 \cdot \| x(u) - y(u) \|^{\alpha_2} du \leq C_0 \cdot \int_0^t L_2 (\frac{L_1 C_0 \cdot (2 M C_0)^{\alpha_1} \cdot u^{1+\alpha_1}}{1 + \alpha_1})^{\alpha_2} du \leq$$

$$\leq \frac{C_0 \cdot L_2 \cdot L_1^{\alpha_2} \cdot C_0^{\alpha_2} \cdot (2 M C_0)^{\alpha_1 \alpha_2}}{(1 + \alpha_1)^{\alpha_2}} \cdot \int_0^t u^{(1+\alpha_1) \cdot \alpha_2} du =$$

$$= \frac{C_0 \cdot L_2 \cdot L_1^{\alpha_2} \cdot C_0^{\alpha_2} \cdot (2 M C_0)^{\alpha_1 \alpha_2}}{(1 + \alpha_1)^{\alpha_2} \cdot [(1 + \alpha_1) \cdot \alpha_2 + 1]} \cdot t^{(1+\alpha_1)\alpha_2 + 1} \leq A^{\alpha_2} \cdot A \cdot t^{S_2} \cdot (2 M C_0)^{\alpha_1 \alpha_2} =$$

$$= (At)^{S_2} \cdot (\frac{2 M C_0}{A})^{\alpha_1 \alpha_2} .$$

In the same way we get:

$$\| x(t) - y(t) \| \leq (At)^{S_n} \cdot (\frac{2 M C_0}{A})^{\alpha_1 \alpha_2 \dots \alpha_n} .$$

Because $(2MC_0/A)^{\alpha_1 \alpha_2 \dots \alpha_n}$ is bounded and $\lim_{n \to +\infty} (At)^{S_n} = 0$ for $t < 1/A$ making in

$(*)$ $n \to +\infty$ we obtain $x(t) = y(t)$ on $(0; 1/A)$.
If $(0; T) \not\subset [0; 1/A)$ then we shall repeat the previous reasoning on the interval $[1/A, 2/A)$ and so on.

Finally we obtain that $x(t) = y(t)$ on $[0; T]$. Since T was chosen arbitrarilly there resuls that $x(t) = y(t)$ on $[0; +\infty)$. q. e. d.

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STRESZCZENIE

W pracy podano wzór na uzmiennianie stałej w przypadku układu równań różniczkowo-całkowych (E_0) z opóźnionym argumentem. Przy pomocy tego wzoru podano warunki konieczne i wystarczające na różne rodzaje stabilności rozwiązań układu (E_0).

РЕЗЮМЕ

В работе представлено формулу неподвижной изменчивости в случае схемы интегрально-дифференциальных уравнений (E_0) с замедленным аргументом. При помощи этой формулы представлено необходимые и достаточные условия для разного рода стабильности решения схемы (E_0).