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Quasisubordination and quasimajorization

Quasipodporządkowanie a quasimajoryzacja

Квазиподчинение а квазимажорация

1. Introduction

Let B denote the class of functions regular and bounded by 1 in absolute value in the unit disk K_1. Let B_0 be the subclass of B consisting of all $\omega \in B$ with $\omega(0) = 0$. In what follows we assume that f and F are functions regular in K_1.

We start with familiar definitions of subordination and majorization.

Definition 1. We say that f is subordinate to F, if there exists $\omega \in B$ such that $f = F \circ \omega$; then we write $f \prec F$.

Definition 2. We say that f is majorized by F, if there exists $\varphi \in B$ such that $f = \varphi F$; then we write $f < F$.

Both concepts are well known and many results point out an analogy between them. Aiming at a unification of results involving these notions M. S. Robertson [1] introduced a more general notion of quasisubordination.

Definition 3. We say that f is quasisubordinate to F, if there exists a function g regular in K_1 such that

$$f < g \text{ and } g \prec F;$$

then we write $f <\prec F$.

Obviously $f <\prec F$, if there exist $\omega \in B_0$, $\varphi \in B$, such that

$$f = \varphi(F \circ \omega). \tag{1}$$

Here and in the sequel the brackets in functional notation indicate the order of operations. Thus f in (1) has the form: $f(z) = \varphi(z)F(\omega(z))$.

In special case when F is the identity function id, majorization and subordination are equivalent by Schwarz's Lemma and $f \prec \mathrm{id} \Leftrightarrow f < \mathrm{id}$ $\Leftrightarrow f = \omega$ with $\omega \in B_0$. Moreover, $f < \omega \prec \mathrm{id}$, i.e. $f << \mathrm{id}$.

Evidently $\varphi(z) \equiv 1$ and $\omega = \mathrm{id}$ in (1) yield subordination and majorization, resp.

As pointed out by the latter author, there is another way of obtaining a simultaneous generalization of subordination and majorization indicated by following

Definition 4. [2]. We say that f is quasimajorized by F, if there exists a function h regular in K_1 such that

$$f \prec h \text{ and } h < F;$$

then we write $f \prec< F$.

Obviously $f \prec< F$, if there exist $\omega_1 \in B_0$, $\varphi_1 \in B$, such that

(2) $$f = (\varphi_1 \circ \omega_1)(F \circ \omega_1) = (\varphi_1 F) \circ \omega_1 .$$

In [2] the latter author proved the following

Lemma 1. *If* $f \prec< F$, *then* $f < \prec F$.

He also put the question whether the converse of Lemma 1 is true.In this communication we answer this question in the negative.

2. A counterexample

In what follows we need the following, well-known

Lemma 2. *If* $\psi(z) = a_0 + a_1 z + a_2 z_0 + \ldots \in B$, *then*

(3) $$|a_k| \leqslant 1, \; k = 0, 1, 2, \ldots$$

If $|a_k| = 1$, *then* $\psi(z) = \eta z^k$ *with* $|\eta| = 1$.

This lemma is an immediate consequence of a well-known inequality:

$$\sum_{k=0}^{\infty} |a_k|^2 \leqslant 1.$$

Suppose that

(4) $$F(z) = z + A_2 z^2 + A_3 z^3 + \ldots, \; z \in K_1,$$

and consider

(5) $$f(z) = zF(z^2) = z^3 + A_2 z^5 + A_3 z^7 + \ldots .$$

Obviously (1) holds with $\varphi(z) = z$, $\omega(z) = z^2$, thus $f \prec\prec F$. We shall prove that quasimajorization $f \prec< F$ holds only if $F = \mathrm{id}$. Hence the case of $F \neq \mathrm{id}$ and $f(z) = zF(z^2)$ leads to a function f which satisfies $f \prec\prec F$, while the relation $f \prec < F$ does not hold.

Suppose, on the contrary, that there exist $\varphi_1 \in B$ and $\omega_1 \in B_0$ such that (2) holds. If

$$\omega_1(z) = c_1 z + c_2 z^2 + \ldots, \tag{6}$$

$$\varphi_2(z) = \varphi_1(\omega_1(z)) = b_0 + b_1 z + b_2 z^2 + \ldots, \tag{7}$$

then by (4) and (5), the condition (2) takes the form

$$\begin{aligned} z^3 + A_2 z^5 + A_3 z^7 + \ldots &= (b_0 + b_1 z + b_2 z^2 + \ldots) \times \\ &\times [c_1 z + c_2 z^2 + \ldots + A_2(c_1 z + c_2 z^2 + \ldots)^2 + \ldots] \\ &= b_0 c_1 z + [b_0(c_2 + A_2 c_1^2) + b_1 c_1] z^2 + \\ &+ [b_0(c_3 + 2A_2 c_1 c_2 + A_3 c_1^3) + b_1(c_2 + A_2 c_1^2) + b_2 c_1] z^3 + \ldots \end{aligned}$$

By equating corresponding coefficients we obtain the following system of equations:

$$\begin{aligned} 0 &= b_0 c_1, \\ 0 &= b_0(c_2 + A_2 c_1^2) + b_1 c_1, \\ 1 &= b_2(c_3 + 2A_2 c_1 c_2 + A_3 c_1^3) + b_1(c_2 + A_2 c_1^2) + b_2 c_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{8}$$

The first equation implies one of the following possibilities:

(i) $b_0 = 0,\ c_1 \neq 0$;

(ii) $b_0 \neq 0,\ c_1 = 0$;

(iii) $b_0 = 0,\ c_1 = 0$.

We start with the discussion of the case (i). The second equation in (8) yields $b_1 = 0$ and this gives, in view of the third equation in (8), $b_2 c_1 = 1$. By Lemma 2 we see that

$$\varphi_2(z) = \eta z^2,\ \omega_1(z) = \eta z, \text{ where } |\eta| = 1. \tag{9}$$

The equality $zF(z^2) \equiv \eta z^2 F(\eta z)$, where $F(z)$ has the form (4), implies $A_2 = A_3 = \ldots = 0$, or $F = \mathrm{id}$. (ii). The second equation in (8) gives $b_0 c_2 = 0$ and consequently $c_2 = 0$. Thus the third equation in (8) takes the form $b_0 c_1 = 1$. By Lemma 2 we see that $\varphi_2(z) = \eta$, $\omega_1(z) = \eta z^3$. Again $zF(z^2) \equiv \eta F(\eta z^3)$ holds for $F = \mathrm{id}$ only.
(iii). The third equation in (8) has the form $b_1 c_2 = 1$ and by Lemma 2 we obtain $\varphi_2(z) = \eta z$, $\omega_1(z) = \eta z^2$. On the other hand, $\eta z \equiv \varphi_1(\eta z^2)$ by (7) which is impossible since φ_2 is even and odd while not vanishing identically.

Thus we have proved that for any F given by (4) the function $zF(z^2)$ that is quasisubordinate to F is quasimajorized by F, if $F = \mathrm{id}$.

In our counterexample quasisubordinate function f has a zero of order three at the origin. It would be interesting to find possibly a corresponding counterexample with $f'(0) \neq 0$. Also the relation between quasisubordination and quasimajorization in case of univalent functions f and F remains an open question.

REFERENCES

[1] Robertson, M. S., *Quasisubordinate functions*, Mathematical essays dedicated to A. J. Macintyre, Ohio Univ. Press, Athens, Ohio (1970), 311-330.

[2] Stankiewicz, J., *Quasisubordination and quasimajorization of analytic functions*, Ann. Univ. Mariae Curie-Skłodowska, Sectio A (to appear).

STRESZCZENIE

W pracy tej podany jest pewien kontrprzykład na to, że pojęcia quasipodporządkowania i quasimajoryzacji wprowadzone odpowiednio w pracy [1] i [2], nie są sobie równoważne.

РЕЗЮМЕ

В данной работе представлен контрпример на то, что понятие квасиподчинения и квазимажорации введено соответствующим образом в работе [1] и [2] не являются эквивалентными.