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On a Certain Boundary Value Problem and its Generalized Solution

O pewnym problemie brzegowym i jego rozwiązaniu uogólnionym

О некоторой краевой задаче и ее обобщенном решении

1. The purpose of this paper is to present an application of the Leżański method [1], [2] to generalized solving an ordinary differential equation of the form:

$$\sum_{j=0}^{N}(-1)^j\frac{d^j}{dt^j}f_j(p_j(t)\cdot x^{(j)}(t),t)+a(t)=0 \quad t\in\langle 0,1\rangle \tag{1}$$

with boundary conditions:

$$x^{(s)}(0)=x^{(s)}(1)=0 \quad \text{for } s=0,1,\ldots,N-1 \tag{2}$$

where the real-valued functions $p_j\in C^{(j)}_{\langle 0,1\rangle}$ $j=0,1,\ldots,N$ satisfy the conditions:

$$p_j(t)\geqslant 0 \quad \text{for } t\in\langle 0,1\rangle \tag{3}$$

$$\operatorname{mes}\{t\in\langle 0,1\rangle: p_j(t)=0\}=0 \tag{4}$$

(the symbol "mes" denotes the Lebesgue measure) and the real-valued functions $f_j\in C^{(j)}_{R\times\langle 0,1\rangle}$ $j=0,1,\ldots,N$ fulfil the following assumptions:

$$\beta_j=\int_0^1\frac{1}{p_j(t)}|f_j(0,t)|^2dt<\infty \tag{5}$$

(6) there exists a positive constant μ_j such that

$$|f(s_1,t)-f(s_2,t)|\leqslant\mu_j\cdot|s_1-s_2| \quad \text{for } s_1,s_2\in R \text{ and } t\in\langle 0,1\rangle$$

(7) there exists a positive constant a_j such that

$$[f(s_1,t)-f(s_2,t)]\cdot(s_1-s_2)\geqslant a_j\cdot(s_1-s_2)^2 \quad \text{for } s_1,s_2\in R \text{ and } t\in\langle 0,1\rangle.$$

Moreover, we assume that it is possible to find such real-valued continuous functions $a_0, a_1, \ldots, a_N$ that

(8) $$a(t) = \sum_{j=0}^{N} a_j(t) \quad \text{for} \quad t \in \langle 0, 1\rangle$$

(9) $$\delta_0 = \int_0^1 \frac{|a_0(t)|^2}{p_0(t)} dt < \infty \text{ and } \delta_j = \int_0^1 \frac{1}{p_j(t)} \left| \int_0^t (t-s)^{j-1} a_j(s) ds \right|^2 o\, dt < \infty$$

$$\text{for } j = 1, 2, \ldots, N$$

In the sequel the equation (1) with the boundary condition (2) where the functions $f_j, p_j\ j = 0, 1, \ldots, N$ and $a(\cdot)$ satisfy the conditions (3), (4), (5), (6), (7), (8), (9) shall be called the boundary value problem (1), (2). Later on we shall prove that the boundary value problem (1), (2) has a unique generalized solution.

2. Now we shall quote (see [1], [2]) the fundamental theorems on which our consideration will be based.

Theorem 1. *Let $(M, (\cdot,\cdot)_*)$ be a unitary space and let*

$$\Psi\colon\ M \times M \ni (x, h) \to \Psi(x, h) \in R$$

be a real functional which satisfies the following conditions:

1) $\bigwedge_{x\in M} \Psi(x, \cdot)$ *is a linear functional defined on* M

2) $\bigwedge_{x\in M} \bigvee_{C_x>0} \bigwedge_{h\in M} |\Psi(x, h)| \leqslant C_x \cdot \|h\|_*$

3) $\bigvee_{\mu>0} \bigwedge_{x,y,h\in M} |\Psi(x, h) - \Psi(y, h)| \leqslant \mu \cdot \|x-y\|_* \cdot \|h\|_*$

4) $\bigvee_{a>0} \bigwedge_{x,h\in M} \Psi(x+h, h) - \Psi(x, h) \geqslant a \cdot \|h\|_*^2$

If $(H, (\cdot,\cdot)_*)$ denotes the completion of $(M, (\cdot,\cdot)_*)$ then there exists a unique extension $\tilde{\Psi}$ of Ψ which is defined on $H \times H$ and satisfies analogous to 1), 2), 3), 4) conditions. More over there exists a unique element $\bar{x} \in H$ such that for all $h \in H$ $\Psi(\bar{x}, h) = 0$.

Theorem 2. *Let $(H, (\cdot,\cdot)_*)$ be a Hilbert space and let Ψ be a real-valued functional defined on the set $H \times H$ and satisfying conditions 1), 2), 3), 4) of Theorem 1. If $e\ \in H$ $k = 0, 1, \ldots$ is an orthonormal linearly dense sequence of elements of the space $(H, (\cdot,\cdot)*)$ i.e. the set*

$$\operatorname{lin}\{e_k \in H\colon\ k = 0, 1, \ldots\}$$

is dense in $(H, (\cdot,\cdot)_*)$ and

$$H_n = \operatorname{lin}\{e_0, e_1, \ldots, e_n\}, \quad \Psi_n = \Psi | H_n \times H_n$$

$$n = 0, 1, 2, \ldots$$

then for every $n \in \{0, 1, 2, \ldots\}$ *there exists a unique element* $z_n \in H_n$ *such that for all* $h \in H_n$ $\Psi(z_n, h) = 0$, *moreover the sequence* z_n, $n = 0, 1, \ldots$ *converges to an element* $x \in H$ *which satisfies the condition* $\Psi(x, h) = 0$ *for all* $h \in H$.

We shall also prove the following lemma expressing a property of functionals fulfilling conditions 1), 2), 3), 4) of Theorem 1.

Lemma 1. *Let* M *be a linear space on which there are defined* $N+1$ *scalar products* $(\ldots)_j$, $j = 0, 1, \ldots, N$ *and* $N+1$ *real-valued functionals*

$$\Psi_j\colon\ M \times M \ni (x, h) \to \Psi_j(x, h) \in R \qquad j = 0, 1, \ldots, N$$

satisfying in scalar products $(.,.)_j$ *the conditions* 1), 2), 3), 4) *of* Theorem 1 *with the constants* $C_j(x)$, μ_j, a_j *respectively. If* $d_0, d_1, \ldots, d_N$ *are positive real numbers the functional*

$$\Psi(x, h) = \sum_{j=0}^{N} d_j \Psi_j(x, h) \quad \textit{for } x, h \in M \tag{10}$$

fulfils in the scalar product

$$(x, h)_* = \sum_{j=0}^{N} (x, h)_j \quad \textit{for } x, h \in M \tag{11}$$

the same conditions 1), 2), 3), 4).

Proof. The linearity of the functional Ψ is clear so it remains to prove conditions 2), 3), 4). Let $K = \sup\{d_0, d_1, \ldots, d_N\}$ and $k = \inf\{d_0, d_1, \ldots, d_n\}$; both the numbers are positive. For a fixed $x \in M$ we have

$$|\Psi(x, h)| \leqslant \sum_{j=0}^{N} d_j |\Psi_j(x, h)| \leqslant K \cdot \sum_{j=0}^{N} C_j(x) \cdot \|h\|_j \leqslant$$

$$\leqslant K \cdot \Big(\sum_{j=0}^{N} C_j^2(x)\Big)^{1/2} \cdot \Big(\sum_{j=0}^{N} \|h\|_j^2\Big)^{1/2}$$

for every vector $h \in M$. Putting here

$$C_x = K \cdot \Big(\sum_{j=0}^{N} C_j^2(x)\Big)^{1/2}$$

we obtain condition 2). Denoting $K \cdot \sup\{\mu_j\colon j = 0, 1, \ldots, N\}$ by μ we have

$$|\Psi(x, h) - \Psi(y, h)| \leqslant K \cdot \sum_{j=0}^{N} |\Psi_j(x, h) - \Psi_j(y, h)| \leqslant K \cdot \sum_{j=0}^{N} \mu_j \cdot \|x - y\|_j \cdot \|h\|_j \leqslant$$

$$\leqslant \mu \cdot \Big(\sum_{j=0}^{N} \|x - y\|_j^2\Big)^{1/2} \cdot \Big(\sum_{j=0}^{N} \|h\|_j^2\Big)^{1/2} = \mu \cdot \|x - y\|_* \cdot \|h\|_*$$

for any vectors $x, y, h \in M$. Similarly denoting $k \cdot \inf\{a_j : j = 0, 1, \ldots, N\}$ by a we obtain for any vectors $x, h \in M$

$$\Psi(x+h, h) - \Psi(x, h) \geqslant k \cdot \sum_{j=0}^{N} a_j \cdot \|h\|_j^2 \geqslant a \cdot \|h\|_*^2$$

which concludes the proof.

3. We shall now use Theorem 1 and Theorem 2 as a tool for the investigation of the boundary value problem (1), (2). Let $L^2_{\langle 0,1\rangle}$ denote the set of classes of measurable and square-integrable functions (in the sense of Lebesgue) defined on $\langle 0, 1\rangle$ with the scalar product

$$(x, y) = \int_0^1 x(t) \cdot y(t)\, dt$$

and let N be fixed natural number. Put

$$x \in M \Leftrightarrow x \in C^{(2n)}_{\langle 0,1\rangle} \text{ and } x^{(j)}(0) = x^{(j)}(1) = 0 \quad \text{for} \quad j = 0, 1, \ldots, N-1.$$

The set M is a linear subspace of the space $L^2_{\langle 0,1\rangle}$, moreover M is dense in $L^2_{\langle 0,1\rangle}$ since it contains the set of functions infinitely differentiable and vanishing in boundary strips.

We shall define a sequence of $N+1$ scalar products $(.,.)_j$ $j = 0, 1, \ldots, N$ on the space M. Let $p_j \in C^{(j)}_{\langle 0,1\rangle}$ $j = 0, 1, \ldots, N$ be functions satisfying (3) and (4). Define for $x, y \in M$

$$(12) \qquad (x, y)_j = \int_0^1 p_j(t) \cdot x^{(j)}(t) \cdot y^{(j)}(t)\, dt \qquad j = 0, 1, \ldots, N.$$

It is easy to see that these forms are scalar products defined on M — they are bilinear, symmetric and non-negative. We shall only prove that

$$(x, x)_j = 0 \Rightarrow x = 0 \text{ for } x \in M \text{ and } j = 0, 1, \ldots, N.$$

Really, if for a fixed $j \in \{0, 1, \ldots, N\}$ and $x \in M$ $(x, x)_j = 0$ then owing to the fact that $p_j(t) \cdot [x^{(j)}(t)]^2 \geqslant 0$ for all $t \in \langle 0, 1\rangle$ we obtain

$$p_j(t) \cdot [x^{(j)}(t)]^2 = 0 \quad \text{for almost all } t \in \langle 0, 1\rangle.$$

Hence, by (4) and the continuity of $x^{(j)}$, we get $x^{(j)}(t) = 0$ for all $t \in \langle 0, 1\rangle$. If $j = 0$ that means $x = 0$, if $j > 0$ we gather by $x^{(j-1)}(0) = 0$ that $x^{(j-1)}(t) = 0$ for all $t \in \langle 0, 1\rangle$. Continuing this process, if it is needed, we get after j steps $x = 0$. Then $(.,.)_j$ $j = 0, 1, \ldots, N$ are scalar products.

Let $f_j \in C^{(j)}_{R\times\langle 0,1\rangle}$ $j = 0, 1, \ldots, N$ be functions satisfying (5), (6) and (7). Using these functions we shall define $N+1$ real valued functionals Ψ_j $j = 0, 1, \ldots, N$ on the space M. Put

$$(13) \qquad \Psi_j(x, h) = \int_0^1 f_j\big(p_j(t) \cdot x^{(j)}(t), t\big) \cdot h^{(j)}(t)\, dt,$$

where $j = 0, 1, \ldots, N$ and $x, h \in M$. Obviously all the functionals are linear with respect to h. We shall prove that each functional Ψ_j is bounded in the norm $\|\cdot\|_j$ in the second variable when the first one is fixed. Let us fix a natural number $j \in \{0, 1, \ldots, N\}$ and an element $x \in M$. Then by (6) we have for an arbitrary vector $h \in M$

$$|\Psi_j(x, h)| \leqslant \int_0^1 |f_j(p_j(t)\cdot x^{(j)}(t), t)\cdot h^{(j)}(t)|\,dt \leqslant \int_0^1 |f_j(p_j(t)\cdot x^{(j)}(t), t)-$$

$$-f_j(0, t)|\cdot|h^{(j)}(t)|\,dt + \int_0^1 |f_j(0, t)|\cdot|h^{(j)}(t)|\,dt \leqslant \int_0^1 \mu_j\cdot p_j(t)|x^{(j)}(t)|\cdot|h^{(j)}(t)|\,dt +$$

$$+\int_0^1 |f_j(0, t)|\cdot|h^{(j)}(t)|\,dt \leqslant \mu_j\int_0^1 \left(\sqrt{p_j(t)}\cdot|x^{(j)}(t)|\right)\cdot\left(\sqrt{p_j(t)}\cdot|h^{(j)}(t)|\right)dt +$$

$$+\int_0^1 \left(\frac{1}{\sqrt{p_j(t)}}\,|f_j(0, t)|\right)\cdot\left(\sqrt{p_j(t)}\cdot|h^{(j)}(t)|\right)dt$$

$$\leqslant \mu_j\left(\int_0^1 p_j(t)|x^{(j)}(t)|^2\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^1 p_j(t)|h^{(j)}(t)|^2\,dt\right)^{\frac{1}{2}} +$$

$$+\left(\int_0^1 \frac{|f_j(0, t)|^2}{p_j(t)}\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^1 p_j(t)\cdot|h^{(j)}(t)|^2\,dt\right)^{\frac{1}{2}}$$

hence, by virtue of (5) we get

$$|\Psi_j(x, h)| \leqslant \mu_j\|x\|_j\cdot\|h\|_j + \sqrt{\beta_j}\cdot\|h\|_j = (\mu_j\cdot\|x\|_j + \sqrt{\beta_j})\cdot\|h\|_j.$$

Putting here

$$C_j(x) = \mu_j\cdot\|x\|_j + \sqrt{\beta_j}$$

we obtain

(14) $|\Psi_j(x, h)| \leqslant C_j(x)|\cdot\|h\|_j$ for $x, h \in M$ and $f = 0, 1, \ldots, N$.

Now we shall prove that for each number $j \in \{0, 1, \ldots, N\}$ and vectors $x, y, h \in M$ it holds the inequality

(15) $$|\Psi_j(x, h) - \Psi_j(y, h)| \leqslant \mu_j\cdot\|x-y\|_j\cdot\|h\|_j$$

where μ_j is the number which appears in (6). Let $x, y, h \in M$. By virtue of (6) and the Schwarz inequality we have

$$|\Psi_j(x, h) - \Psi_j(y, h)| \leqslant \int_0^1 |f_j(p_j(t)\cdot x^{(j)}(t), t) - f_j(p_j(t)\cdot y^{(j)}(t), t)|\cdot|h^{(j)}(t)|\,dt$$

$$\leqslant \mu_j\int_0^1 \left(\sqrt{p_j(t)}\cdot|x^{(j)}(t)-y^{(j)}(t)|\right)\cdot\left(\sqrt{p_j(t)}\cdot|h^{(j)}(t)|\right)dt \leqslant$$

$$\leqslant \mu_j\left(\int_0^1 p_j(t)|x^{(j)}(t)-y^{(j)}(t)|^2\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^1 p_j(t)\cdot|h^{(j)}(t)|^2\,dt\right)^{\frac{1}{2}} = \mu_j\|x-y\|_j\cdot\|h\|_j$$

which completes the proof.

Now we verify that for each $j \in \{0, 1, \ldots, N\}$ and vectors $x, h \in M$ it holds the inequality

$$(16) \qquad \Psi_j(x+h, h) - \Psi_j(x, h) \geqslant a_j \cdot \|h\|_j^2$$

where a_j is the constant which appears in (7). Let us take $x, h \in M$. By (7) we obtain

$$\Psi_j(x+h, h) - \Psi_j(x, h) = \int_0^1 \Big[f_j\big(p_j(t)\cdot\big(x^{(j)}(t)+h^{(j)}(t)\big), t\big) - f_j\big(p_j(t)\cdot x^{(j)}(t), t\big)\Big] \times$$
$$\times h^{(j)}(t)\, dt =$$
$$= \int_0^1 \Big[f_j\big(p_j(t)\cdot\big(x^{(j)}(t)+h^{(j)}(t)\big), t\big) - f_j\big(p_j(t)\cdot x^{(j)}(t), t\big)\Big]\cdot p_j(t)\cdot h^{(j)}(t)\cdot \frac{1}{p_j(t)}\, dt$$
$$\geqslant a_j \int_0^1 [p_j(t)\cdot h^{(j)}(t)]^2 \cdot \frac{1}{p_j(t)}\, dt = a_j \cdot \int_0^1 p_j(t)\cdot |h^{(j)}(t)|^2 dt = a_j \cdot \|h\|_j$$

Joining the extreme sides of this sequence of inequalities we get (16).

Let us denote (cf. (10) and (11))

$$(17) \qquad (x, y)_* = \sum_{j=0}^{N} (x, y)_j \quad \text{for } x, y \in M$$

$$(18) \qquad \Psi_*(x, y) = \sum_{j=0}^{N} \Psi_j(x, y) \quad \text{for } x, y \in M$$

Lemma 1 and the inequalities (14), (15), (16) give immediately

Lemma 2. *The functional Ψ_* satiefies conditions* 1), 2), 3), 4) *of* Theorem 1.

Now we consider a real-valued functional defined on M. Define the functional φ_a where $a \in L^2_{\langle 0,1\rangle}$ as follows

$$(19) \qquad \varphi_a(h) = (a, h) = \int_0^1 a(t)h(t)\,dt \quad \text{for } h \in M.$$

The following lemma provides sufficient conditions for the boundedness of φ_a in the norm $\|\cdot\|_*$.

Lemma 3. *Let $a(\cdot)$ be a real-valued continuous function defined on the interval $\langle 0, 1\rangle$. If $a(\cdot)$ satisfies the conditions* (8) *and* (9) *then there exists a constant $\gamma > 0$ such that for all $h \in M$*

$$|\varphi_a(h)| \leqslant \gamma \cdot \|h\|_*$$

Proof. It immediately follows from our assumptions that for $h \in M$

$$\varphi_a(h) = \sum_{j=0}^{N} (a_j, h),$$

therefore we shall estimate the components (a_j, h) $j = 0, 1, \ldots, N$). If $j = 0$ we have by (9)

$$|(a_0, h)| \leqslant \int_0^1 |a_0(t)| \cdot |h(t)|\, dt = \int_0^1 \left(\frac{1}{\sqrt{p_0(t)}} |a_0(t)|\right) \cdot (\sqrt{p_0(t)} \cdot |h(t)|)\, dt \leqslant$$

$$\leqslant \left(\int_0^1 \frac{1}{p_0(t)} |a_0(t)|^2 dt\right)^{\frac{1}{2}} \cdot \left(\int_0^1 p_0(t) \cdot |h(t)|^2 dt\right)^{\frac{1}{2}} = \sqrt{\delta_0} \cdot \|h\|_0.$$

Now, let us take a number $j \in \{1, 2, \ldots, N\}$. By virtue of the continuity of the function $a_j(\cdot)$ we obtain for all $t \in \langle 0, 1 \rangle$

$$a_j(t) = \frac{d^j}{dt^j} \int_0^t dt_1 \int_0^{t_1} \ldots dt_{j-0} \int_0^{t_{j-1}} a_j(s)\, ds = \frac{1}{(j-1)!} \frac{d^j}{dt^j} \left\{\int_0^t (t-s)^{j-1} a_j(s)\, ds\right\}.$$

Using this identity and integrating j-times by parts we get

$$(a_j, h) = (j-1)^j \frac{1}{(j-1)!} \int_0^1 \left(\int_0^t (t-s)^{j-1} a_j(s)\, ds\right) \cdot h^{(j)}(t)\, dt,$$

because $h^{(s)}(0) = h^{(s)}(1) = 0$ for $s = 0, 1, \ldots, N-1$. Now, taking absolute values and applying the Schwarz inequality we have by (9)

$$|(a_j, h)| \leqslant \frac{1}{(j-1)!} \int_0^1 \left(\frac{1}{\sqrt{p_j(t)}} \left|\int_0^t (t-s)^{j-1} a_j(s)\, ds\right|\right) \cdot (\sqrt{p_j(t)} \cdot |h^{(j)}(t)|)\, dt \leqslant$$

$$\leqslant \frac{1}{(j-1)!} \left(\int_0^1 \frac{1}{p_j(t)} \left|\int_0^t (t-s)^{j-1} \cdot a_j(s)\, ds\right|^2 dt\right)^{\frac{1}{2}} \times$$

$$\times \left(\int_0^1 p_j(t) \cdot |h^{(j)}(t)|^2 dt\right)^{\frac{1}{2}} = \frac{\sqrt{\delta_j}}{(j-1)!} \cdot \|h\|_j.$$

Let us denote

$$\vartheta = \sup\left\{\sqrt{\delta_0}, \sqrt{\delta_1}, \ldots, \frac{\sqrt{\delta_N}}{(N-1)!}\right\}$$

then by virtue of the inequalities

$$|(a_0, h)| \leqslant \vartheta \cdot \|h\|_0, \quad |(a_j, h)| \leqslant \vartheta \cdot \|h\|_j \text{ for } j = 1, 2, \ldots, N,$$

we obtain

$$|(a,h)| \leqslant \vartheta \cdot \sum_{j=0}^{N} \|h\|_j \leqslant \vartheta \cdot \sqrt{N+1} \cdot \Big(\sum_{j=0}^{N} \|h\|_j^2\Big)^{1/2} = \vartheta \cdot \sqrt{N+1} \cdot \|h\|_*$$

To end the proof it is enough to put $\gamma = \vartheta \cdot \sqrt{N+1}$.

Now, let us put

$$\Psi(x,h) = \Psi_*(x,h) + \varphi(h) \quad x, h \in M. \tag{20}$$

where a$(\cdot)$ is a function satisfying the conditions (8), (9) and Ψ_* and φ_a are functionals defined by (18) and (19). From Lemma 3 and Lemma 2 it immediately follows the following

Lemma 4 *The functional Ψ defined on the space M by the formula* (20) *satisfies in the norm $\|\cdot\|_*$ conditions* 1), 2, 3), 4) *of* Theorem 1.

Now we shall derive a new representation of the functional Ψ. This new representation will be useful in our further consideration. Let us take two elements $x, h \in M$ and a number $j \in \{1, 2, \ldots, N\}$. Considering that $x^{(s)}(0) = h^{(s)}(0) = x^{(s)}(1) = h^{(s)}(1) = 0$ for $s = 0, 1, \ldots, N-1$ and integrating j-times by parts we obtain

$$\int_0^1 f_j\big(p_j(t)\cdot x^{(j)}(t), t\big)\cdot h^{(j)}(t)\,dt = \sum_{i=0}^{j-1} (-1)^i \cdot h^{(j-i-1)}(t) \cdot \frac{d^i}{dt^i} f_j\big(p_j(t)\cdot x^{(j)}(t), t\big)\Big|_0^1 +$$

$$+ \int_0^1 (-1)^j h(t) \frac{d^j}{dt^j} f_j\big(p_j(t)\cdot x^{(j)}(t), t\big)\,dt = (-1)^j \int_0^1 \frac{d^j}{dt^j} \big\{f_j\big(p_j(t), x^{(j)}(t), t\big)\big\}\, h(t)\,dt$$

hence

$$\Psi(x,h) = \int_0^1 \Big\{\sum_{j=0}^{N} (-1)^j \frac{d^j}{dt^j} f_j\big(p_j(t)\cdot x^{(j)}(t), t\big) + a(t)\Big\} \cdot h(t)\,dt$$

Let us define an operator U by the formula

$$\big(U(x)\big)(t) = \sum_{j=0}^{N} (-1)^j \frac{d^j}{dt^j} f_j\big(p_j(t)\cdot x^{(j)}(t), t\big) + a(t) \quad \text{for } x \in M$$

It is easy to see that U is an operator from the space M to the space of all real-valued continuous functions defined on the interval $\langle 0, 1\rangle$, in particular to the space $L^2_{\langle 0,1\rangle}$. Using the operation U we get the following representation of the functional Ψ

$$\Psi(x,h) = \big(U(x), h\big) \quad \text{for } x, h \in M. \tag{21}$$

Now we shall define a generalized solution of the boundary value problem (1), (2). Let $(H, (.,.)_*)$ denote a completion of the unitary space $(M, (.,.)_*)$ (in the norm $\|\cdot\|_*$ (see (17)) and let $\tilde{\Psi}$ denote the extension of Ψ defined in Theorem 1.

Definition 1. An element $x \in H$ is called a generalized solution of the boundary value problem (1), (2) if *f* it satisfies the condition $\tilde{\Psi}(x, h) = 0$ for every $h \in H$.

From this definition and from Lemma 4 and Theorem 1 we may deduce the following

Theorem 3. *The boundary value problem* (1), (2) *has always a unique generalized solution.*

We shall present two theorems expressing a connection between a generalized solution and a classical one.

Theorem 4. *If a generalized solution of the boundary value problem* (1), (2) *belongs to the space* M *then it is a classical solution of the problem of the class* $C^{(2N)}_{\langle 0,1\rangle}$.

Proof. Let $x \in M$ be a generalized solution of the boundary value problem (1), (2). From the condition $\tilde{\Psi}(x, h) = 0$ for $h \in H$ follows that $\Psi(x, h) = 0$ for every $h \in M$. Hence, by (21) we obtain $(U(x), h) = 0$ for $h \in M$. Since the element $U(x)$ belongs to $L^2_{\langle 0,1\rangle}$ and it is orthogonal to the space M which is dense in $L^2_{\langle 0,1\rangle}$ in the usual norm we conclude $U(x) = 0$. This means that the element x is a solution of the equation (1). The boundary conditions (2) and the relation $x \in C^{(2N)}_{\langle 0,1\rangle}$ are satisfied vacuously because $x \in M$. The theorem is proved.

Theorem 5. *Let* $x \in C^{(2N)}_{\langle 0,1\rangle}$. *If the element* x *is a classical solution of the boundary value problem* (1), (2) *then* x *is a generalized solution of the problem. If there exists a classical solution of the class* $C^{(2N)}_{\langle 0,1\rangle}$ *it is unique.*

Proof. If $x \in C^{(2N)}_{\langle 0,1\rangle}$ is a classical solution of the boundary value problem (1), (2) then $x \in M$ and $U(x) = 0$. From this and from the representation (21) follows the first part of the theorem, the second part is an immediate consequence of the first part and Theorem 3. The proof is ended.

Now we shall prove that there exists an orthonormal and linearly dense sequence in the space $(M, (.,.)_*)$. With the aid of the sequence it is possible to construct a sequence of elements of the space M approximating in the norm $\|\cdot\|_*$ the generalized solution of the boundary value problem (1), (2). We shall take advantage of the following

Remark (see [3] p. 38) Let $B\colon M \to L^2_{\langle 0,1\rangle}$ be a symmetric, positive defined, linear operator. If $x_i \in M$ $i = 1, 2, \ldots$ is such a sequence that the set $\operatorname{lin}\{Bx_i\colon i = 1, 2, \ldots,\}$ is dense in $B(M)$ in the usual norm $\|\cdot\|$ then the set $\operatorname{lin}\{x_i\colon i = 1, 2, \ldots\}$ is dense in M in the scalar product $(.,.)_B$ defined by the formula $(x, y)_B = (Bx, y)$ for $x, y \in M$.

We shall define an operator satisfying the conditions we have specified in the above remark. Let

$$B\colon M \ni x \to (-1)^N \cdot \frac{d^{2N}}{dt^{2N}} x \in L^2_{\langle 0,1\rangle} \tag{22}$$

Integrating N-times by parts and observing that for $x, y \in M$ $x^{(s)}(0) = x^{(s)}(1) = y^{(s)}(0) = y^{(s)}(1) = 0$ $s = 0, 1, \ldots, N-1$ we have

$$(Bx, y) = \int_0^1 x^{(N)}(t) \cdot y^{(N)}(t)\, dt = (x^{(N)}, y^{(N)}) \tag{23}$$

so the operator B is symmetric; we shall also prove that it is positive defined. Let $x \in M$ and $s \in \{0, 1, \ldots, N-1\}$, then for $t \in \langle 0, 1\rangle$ it holds

$$|x^{(s)}(t)| = |x^{(s)}(t) - x^{(s)}(0)| \leqslant \int_0^t |x^{(s+1)}(\tau)|\, d\tau \leqslant \Big(\int_0^t d\tau\Big)^{1/2} \Big(\int_0^t |x^{(s+1)}(\tau)|^2 d\tau\Big)^{1/2}$$

$$\leqslant \sqrt{t} \cdot \Big(\int_0^1 |x^{(s+1)}(t)|^2 dt\Big)^{1/2}$$

hence

$$\int_0^1 |x^{(s)}(t)|^2 dt \leqslant \Big(\int_0^1 |x^{(s+1)}(t)|^2 dt\Big) \cdot \Big(\int_0^1 t\, dt\Big) = 2^{-1} \cdot \int_0^1 |x^{(s+1)}(t)|^2 dt \tag{24}$$

Putting here successively $s = 0, 1, \ldots, N-1$ we get a sequence of inequalities

$$\|x\|^2 = \int_0^1 |x(t)|^2 dt \leqslant 2^{-1} \int_0^1 |x'(t)|^2 dt \leqslant \ldots \leqslant 2^{-N} \int_0^1 |x^{(N)}(t)|^2 dt$$

Joining the extreme sides of this sequence of inequalities we obtain

$$(Bx, x) \geqslant 2^N \cdot \|x\|^2 \quad \text{for } x \in M \tag{25}$$

so the operator B is positive defined.

Now, let us put

$$(x, y)_B = (Bx, y) \text{ for } x, y \in M. \tag{26}$$

We shall show a connection between the scalar products $(.,.)_*$ and $(.,.)_B$

Lemma 5. *There exists a constant $C > 0$ such that for $h \in M$*

$$\|h\|_* \leqslant C \cdot \|h\|_B$$

Proof. Put

$$K = \sup\{\sup\{p_j(t)\colon t \in \langle 0, 1\rangle\}\colon j = 0, 1, \ldots, N\}$$

— K is a positive number because the functions p_j, $j = 0, 1, \ldots, N$ are continuous and we exclude a trivial case when all functions p_j are equal to zero. Let $x \in M$. By the inequality (24) we have

$$\|x\|_* = \sum_{j=0}^{N} \int_0^1 p_j(t)|x^{(j)}(t)|^2 dt \leqslant K \cdot \sum_{j=0}^{N} \int_0^1 |x^{(j)}(t)|^2 dt \leqslant K \cdot \sum_{j=0}^{N} 2^{j-N} \int_0^1 |x^{(N)}(t)|^2 dt$$

Putting here

$$C = \Big(K \cdot \sum_{j=0}^{N} 2^{j-N}\Big)^{1/2}$$

and using (23) and (26) we get the thesis of the lemma.

It follows from the above lemma that a linearly dense set in $(M, (.,.)_B)$ is also linearly dense in the space $(M, (.,.)_*)$ therefore we shall prove

Lemma 6. *The sequence*

$$x_k(t) = t^{N+k}(1-t)^N \ t \in \langle 0, 1\rangle \quad k = 0, 1, \ldots$$

of elements $x_k \in M$ is linearly dense in the space $(M, (.,.)_B)$.

Proof. As it follows from the above Remark it is sufficient to show that the sequence Bx_k $k = 0, 1, \ldots$ is linearly dense in the space $L^2_{\langle 0,1\rangle}$ in the usual norm $\|\cdot\|$. We first find the form of the vectors Bx_k $k = 0, 1, \ldots$. Observing that the derivatives of the function $\{t^{N+k}\}$ of the order $2N-j$, $1 \leqslant j < N-k$ and the derivatives of the function $\{(1-t)^N\}$ of the order $j = N+1, N+2, \ldots, 2N$ are equal to zero we get the formula

$$Bx_k = (-1)^N \sum_{j=m}^{N} \binom{2N}{j} (-1)^j \frac{N!}{(N-j)!} (1-t)^{N-j} \cdot \frac{(N+k)!}{(k+j-N)!} t^{k+j-N} \qquad k = 0, 1, \ldots$$

where $m = \sup(0, N-k)$. Applying the Newton formula we have

$$Bx_k = (-1)^N \sum_{j=m}^{N} \sum_{i=0}^{N-j} \binom{2N}{j}\binom{N-j}{i} (-1)^{N-i} \frac{N!}{(N-j)!} \cdot \frac{(N+k)!}{(k+j-N)!} t^{k-i} \qquad k = 0, 1, \ldots$$

It is easy to see that Bx_k is a polynomial of the order k because the coefficient at t^k is equal to

$$\sum_{j=m}^{N} \binom{2N}{j} \frac{N!}{(N-j)!} \cdot \frac{(N+k)!}{(k+j-N)!}$$

so it doesn't vanish. Consequently we may write the formula

$$Bx_k = \sum_{j=0}^{k} c_j^k t^j \ (c_j^k R,\ j=0,1,\dots,k\ c_k^k \neq 0) \quad k=0,1,\dots$$

where c_j^k $j=0,1,\dots,k$ denotes coefficients of the polynomial Bx_k $k=0,1,\dots$. We shall show that the functions $y_k = \{t^k\}$ $k=0,1,\dots$ belong to the set

$$L = \operatorname{lin}\{Bx_k \in L^2_{\langle 0,1\rangle}\colon\ k=0,1,\dots\}$$

Obviously $y_0 \in L$ since $y_0 = Bx_0$; moreover if k is a natural number and $y_j \in L$ for natural numbers $j<k$ then $y_k \in L$ because

$$y_k = \frac{1}{c_k^k} Bx_k - \sum_{j=0}^{k-1} \frac{c_j^k}{c_k^k} y_j$$

hence $y_k \in L$ for $k=0,1,\dots$. Using the Weierstrass approximation theorem we conclude that the set L is dense in the space $B(M)$ in the scalar product $(.,.)$ which completes the proof.

From the above lemma and Theorem 2 it follows that a generalized solution of the boundary value problem (1), (2) may be approximated by elements of the space M.

Theorem 5. *If $x \in H$ is a generalized solution of the boundary value problem* (1), (2) *then there exists a sequence of elements $z_k \in M$ $k=0,1,\dots$ such that* $\lim\limits_{k\to\infty} \|x - z_k\|_* = 0$.

Proof. The sequence x_k $k=0,1,\dots$ which has been defined in Lemma 6 is linearly dense in the space $(H,(.,.)_*)$ which is a completion of the space $(M,(.,.)_*)$. Realizing the Schmidt orthogonalization of the sequence x_k $k=0,1,\dots$ in the space $(M,(.,.)_*)$ we get the sequence $e_k \in M$ $k=0,1,\dots$ which is orthonormal and linearly dense in the space $(H,(.,.)_*)$. Observing that $H_k = \operatorname{lin}\{e_0, e_1, \dots, e_k\} \subset M$ for every $k=0,1,\dots$ and using Theorem 2 we immediately obtain the thesis of the theorem.

4. A generalized solution of the boundary value problem (1), (2) may belong to the set $H \setminus M$ and in this case it isn't usually even a function. It is interesting when the space H may be considered as a subset of the space $L^2_{\langle 0,1\rangle}$. If such an embedding is possible then every generalized solution of the boundary value problem (1), (2) is a function defined on the interval $\langle 0,1\rangle$. In this passage we show that in order that the embedding be possible it is sufficient that

$$S = \int_0^1 \frac{dt}{p_N(t)} < \infty \tag{27}$$

The above assumption shall be valid in all this passage. We first prove the following

Lemma 7. *If the condition* (27) *is fulfilled then the norms* $\|\cdot\|_*$ *and* $\|\cdot\|_N$ *in the space* M *are equivalent.*

Proof. By (17), it is sufficient to prove that the norm $\|\cdot\|_N$ is not weaker than the norm $\|\cdot\|_j$ for $j = 0, 1, \ldots, N-1$. Let $x \in M$ and $j \in \{0, 1, \ldots, N-1\}$ then

$$\int_0^1 |x^{(j)}(t)|\,dt = \int_0^1 |x^{(j)}(t) - x^{(j)}(0)|\,dt = \int_0^1 \left|\int_0^t x^{(j+1)}(s)\,ds\right| dt \leqslant \int_0^1 |x^{(j+1)}(t)|\,dt$$

Hence for every $x \in M$ and $i \in \{0, 1, \ldots, N-1\}$ we get

$$\int_0^1 |x^{(i)}(t)|\,dt \leqslant \int_0^1 |x^{(N)}(t)|\,dt \tag{28}$$

Using this inequality we obtain for $x \in M$ and $j \in \{0, 1, \ldots, N-1\}$

$$\|x\|_j^2 = \int_0^1 p_j(t)|x^{(j)}(t)|^2\,dt = \int_0^1 p_j(t)|x^{(j)}(t) - x^{(j)}(0)|^2\,dt$$

$$\leqslant \int_0^1 p_j(t) \cdot \left(\int_0^1 |x^{(j+1)}(s)|\,ds\right)^2 dt \leqslant \left(\int_0^1 p_j(t)\,dt\right)\left(\int_0^1 |x^{(N)}(t)|\,dt\right)^2$$

$$= \left(\int_0^1 p_j(t)\,dt\right) \cdot \left(\int_0^1 \frac{1}{\sqrt{p_N(t)}} \cdot \sqrt{p_N(t)} \cdot |x^{(N)}(t)|\,dt\right)^2$$

$$\leqslant \left(\int_0^1 p_j(t)\,dt\right) \cdot \left(\int_0^1 \frac{dt}{p_N(t)}\right) \cdot \left(\int_0^1 p_N(t) \cdot |x^{(N)}(t)|^2\,dt\right) = S \cdot \left(\int_0^1 p_j(t)\,dt\right) \cdot \|x\|_N^2$$

so the norm $\|\cdot\|_N$ is not weaker than the norm $\|\cdot\|_j$. The proof is ended.

As it follows from the above lemma the scalar product $(.,.)_N$ defined on the space M may be extended over the entire space H (the extension shall be denoted by the same symbol $(.,.)_N$), besides the space $(H, (.,.)_N)$ is complete and the norms $\|\cdot\|_*$ and $\|\cdot\|_N$ in H are equivalent.

Now let us consider the operation

$$A\colon M \ni x \to (-1)^N \frac{d_N}{dt^N}\left(p_N \cdot \frac{d^N}{dt^N} x\right) \in L^2_{\langle 0,1\rangle}$$

Integrating by parts we obtain

$$(x, y)_N = (Ax, y) \quad \text{for } x, y \in M \tag{29}$$

so we infer that A is symmetric. Moreover we shall show that A is positive defined. Let $x, y \in M$. Using (27) and (28) we get

$$\|x\|^2 = \int_0^1 |x(t)|^2 dt = \int_0^1 |x(t) - x(0)|^2 dt \leqslant \int_0^1 \left(\int_0^1 |x'(s)|\, ds\right)^2 dt$$

$$\leqslant \left(\int_0^1 |x^{(N)}(t)|\, dt\right)^2 = \left(\int_0^1 \frac{1}{\sqrt{p_N(t)}} \cdot \sqrt{p_N(t)} \cdot |x^{(N)}(t)|\, dt\right)^2$$

$$\leqslant S \left(\int_0^1 p_N(t) |x^{(N)}(t)|^2 dt\right) = S \cdot \|x\|_N^2,$$

hence by (29) we have

$$(Ax, x) \geqslant S^{-1} \|x\|^2 \text{ for } x \in M$$

thus the operator A is positive defined.

Let $(H_A, (.,.)_A)$ denote the Friedrichs space generated by A (obviously $(.,.)_A | M \times M = (.,.)_N$). Since the spaces $(H_A, (.,.)_A)$ and $(H, (.,.)_N)$ are completions of the unitary space $(M, (.,.)_N)$, they are equivalent i.e. there exists a unitary operator V mapping H_A onto H and satisfying the condition $(Vx, Vy)_N = (x, y)_A$ for $x, y \in H_A$ and $V/M = id_M$.

Let us define

$$(x, y)_\square = (Vx, Vy)_* \text{ for } x, y \in H_A,$$

then the spaces $(H_A, (.,.)_\square)$ and $(H, (.,.)_*)$ are equivalent. Consequently we may assume that the functional Ψ is defined on the set $H_A \times H_A$. We shall prove

Theorem 6. *If the condition* (27) *is fulfilled and $x \in H_A$ is a generalized solution of the boundary value problem* (1), (2) *then $x \in C^{(N-1)}_{\langle 0,1\rangle}$ and $x^{(s)}(0) = x^{(s)}(1)$ for $s = 0, 1, \ldots, N-1$.*

Proof. It suffices to prove that every element $x \in H_A$ fulfils the condition $x \in C^{(N-1)}_{\langle 0,1\rangle}$ and $x^{(s)}(0) = x^{(s)}(1)$ for $s = 0, 1, \ldots, N-1$. First we shall prove an auxiliary inequality. Let $x \in M$, $j \in \{0, 1, \ldots, N-1\}$ and $t \in \langle 0, 1\rangle$. By (27) and (28) we have

$$|x^{(j)}(t)| = |x^{(j)}(t) - x^{(j)}(0)| = \left|\int_0^t x^{(j+1)}(s)\, ds\right| \leqslant \int_0^1 |x^{(j+1)}(t)|\, dt \leqslant \int_0^1 |x^{(N)}(t)|\, dt$$

$$= \left(\int_0^1 \frac{1}{\sqrt{p_N(t)}} \cdot \sqrt{p_N(t)} \cdot |x^{(N)}(t)|\, dt \leqslant \left(\int_0^1 \frac{dt}{p_N(t)}\right)^{\frac{1}{2}} \cdot \left(\int_0^1 p_N(t) \cdot |x^{(N)}(t)|^2 dt\right)^{\frac{1}{2}}\right.$$

$$= \sqrt{S} \cdot \|x\|_N$$

whence

(a) $|x^{(j)}(t)| \leqslant \sqrt{S}\cdot\|x\|_N$ for $t \in \langle 0,1\rangle$, $x \in M$, $j \in \{0,\dots,N-1\}$.

Let us take an element $x \in H_A$. It follows from the properties of the Friedrichs space H_A that there exists a sequence $y_k \in M$ $k = 1,2,\dots$ such that

(b) $\lim\limits_{k,l\to\infty}\|y_k - y_l\|_N = 0$ and $\lim\limits_{k\to\infty}\|y_k - x\| = 0$.

Let $j \in \{0,1,\dots,N-1\}$. By (a) and (b) we get

$$\lim_{k,l\to\infty}\sup_{t\in\langle 0,1\rangle}|y_k^{(j)}(t) - y_l^{(j)}(t)| = 0$$

so the sequences $y_k^{(j)}$ $k = 1,2,\dots$ for $j = 0,1,\dots,N-1$ are convergent uniformly on the interval $\langle 0,1\rangle$. Since $y_l^{(s)}(0) = y_l^{(s)}(1) = 0$ for $l = 1,2,\dots$ and $s = 0,1,\dots,N-1$ there exists a function $y \in C^{(N-1)}_{\langle 0,1\rangle}$ such that $y^{(j)}(0) = y^{(j)}(1) = 0$ for $j = 0,1,\dots,N-1$ and the sequence $y_k^{(j)}$ $k = 1,2,\dots$ is convergent uniformly to $y^{(j)}$ for $j = 0,1,\dots,N-1$. From the uniform convergence of the sequence y_k $k = 1,2,\dots$ to y we obtain

$$\lim_{k\to\infty}\|y_k - y\| = 0$$

Hence, by (b) we obtain $x = y$ which completes the proof.

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STRESZCZENIE

W niniejszej pracy rozpatrywane jest równanie różniczkowe zwyczajne (1) z warunkiem brzegowym (2), przy założeniu, że funkcje występujące w tym równaniu spełniają warunki: (3), (4), (5), (6), (7), (8), (9). Dla tego problemu brzegowego sformułowano definicję rozwiązania uogólnionego oraz wykazano jego istnienie i jednoznaczność. Wykazano również pewne związki zachodzące między rozwiązaniem klasycznym i uogólnionym. Ponadto udowodniono, że rozwiązanie uogólnione powyższego problemu brzegowego może być aproksymowane w normie (17) funkcjami klasy $C^{(2N)}_{\langle 0,1\rangle}$ o pochodnych rzędu $0,1,\dots,N-1$ równych zero na końcach przedziału $\langle 0,1\rangle$. W dalszej części pracy podano warunek dostateczny na to aby rozwiązanie uogólnione problemu brzegowego (1), (2) było funkcją klasy $C^{(N-1)}_{\langle 0,1\rangle}$ o pochodnych rzędu $0,1,\dots,N-1$ równych zero na końcach przedziału $\langle 0,1\rangle$.

РЕЗЮМЕ

В работе рассматривается обыкновенно дифференциальное уравнение (1) с краевым условием (2), при предположении, что функции выступающие в этом уравнении исполняют условия (2), (4), (5), (6), (7), (8), (9). Для этой краевой задачи сформулировано определение обобщенного решения и доказано его существование и единственность. Указано также некоторые связи между классическим решением и обобщенным решением. Кроме того доказано, что обобщенное решение этой краевой задачи может быть аппроксимированное в норме (17) при помощи функции класса $C^{(2N)}_{\langle 0,1\rangle}$, которых производные порядка $0, 1, \ldots, N-1$ изчезают в концах нитервала $\langle 0, 1\rangle$. В дальнейшем даны достаточные условия для того, чтобы обобщенное решение краевой задачи (1), (2) было функцией класса $C^{(N-1)}_{\langle 0,1\rangle}$, которой производные порядка $0, 1, \ldots, N-1$ изчезают в концах интервала $\langle 0, 1\rangle$.