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An Equation $dx/dt = f(x, t)$ whose Trivial Solution in Spite of the Strong Stability is not Mean Square Stable under Persistent Random Disturbances from G

Równanie $dx/dt = f(x, t)$, którego rozwiązanie zerowe pomimo mocnej stabilności nie jest średniokwadratowo stabilne względem stale działających losowych zakłóceń z G.

Уравнение $dx/dt = f(x, t)$, которого нулевое решение, несмотря на равномерно асимптотическую устойчивость не устойчиво в квадратном среднем относительно постоянно действующих случайных возмущений из G

Theorem 1 in the paper [1] (5.12. on the page 117 of the monograph [2]) says that if

1. a function $f\colon R^n \times T \to R^n$, $T = \langle 0, \infty)$ is continuous and $f(0, t) = 0$, $t \in T$,
2. there exists a real number $L > 0$ such that

$$\bigwedge_{x,\bar{x}\in R^n} \bigwedge_{t\in T} \|f(x, t) - f(\bar{x}, t)\| \leqslant L \cdot \|x - \bar{x}\|,$$

3. a trivial solution of the differential equation

$$\frac{dx}{dt} = f(x, t) \tag{1}$$

is uniformly asymptotically stable

then the trivial solution of (1) is mean square stable under persistent random disturbances from G i.e.

$$\bigwedge_{t_0\in T} \bigwedge_{\varepsilon>0} \bigvee_{\delta>0} \bigvee_{\eta>0} \Big[\Big(\|x_0\| \leqslant \delta,\ g \in G,\ \sup_{(x,t)\in R^n \times T_0} E\{\|g(x, t, \omega)\|^2\} \leqslant \eta\Big) \Rightarrow \big(E\{\|X_t(\omega)\|^2\} < \varepsilon,\ t \geqslant t_0\big)\Big],$$

where $T_0 = \langle t_0, \infty)$ and G denotes the class of all sample continuous on $R^n \times T$ n-dimensional stochastic processes $g(x, t, \omega)$ for which there exist real functions l and m with finite Lebesgue integrals on each bounded interval in T and a random variable Y with $E\{Y^2\} < \infty$ such that for almost all $\omega \in \Omega$ holds

$$\bigwedge_{x,\bar{x}\in R^n} \bigwedge_{t\in T} \|g(x,t,\omega)-g(\bar{x},t,\omega)\| \leqslant l(t)\|x-\bar{x}\|$$

and

$$\bigwedge_{x\in R^n} \bigwedge_{t\in T} \|g(x,t,\omega)\| \leqslant m(t)[\|x\|+Y(\omega)].$$

X_t denotes the sample solution of the stochastic differential equation

$$\frac{dX_t}{dt} = f(X_t,t)+g(X_t,t,\omega). \tag{2}$$

Here is a counter example showing that this result is false.
Let us consider the function $f(x,t) = -x, f\colon R\times T\to R$ and a differential equation (1) which now has the form

$$\frac{dx}{dt} = -x. \tag{3}$$

Obviously the assumptions 1., 2., 3. of the Theorem 1 of [1] are satisfied. Let us presume that probability space $(\Omega, \mathfrak{A}, P)$ is set $\Omega = \langle 0,1\rangle \subset R$ with the σ-algebra of Borel sets and Lebesgue measure. We ought to show that

$$\bigvee_{t_0\in T} \bigvee_{\varepsilon>0} \bigwedge_{\delta>0} \bigwedge_{\eta>0} \bigvee_{g\in G} \Big[\Big(\|x_0\|\leqslant\delta,\ \sup_{(x,t)\in R^n\times T_0} E\{\|g(x,t,\omega)\|^2\}\leqslant\eta\Big) \wedge \Big(\bigvee_{t\in T_0} E\{\|X_t(\omega)\|^2\}\geqslant\varepsilon\Big)\Big].$$

Let $t_0 = 0$. Let us choose the arbitrary $\varepsilon>0$, $\delta>0$, $\eta>0$. Let us define function g as follows:

$$g(x,t,\omega) = (t+\omega+1)Z\left(\frac{x-t-\omega}{a(t)}\right),$$

$$g\colon R\times T\times\Omega\to R,$$

where

$$Z(y) = \begin{cases} 0, & y\leqslant -1 \\ y+1, & -1<y\leqslant 0 \\ -y+1, & 0<y\leqslant 1 \\ 0, & y>1, \end{cases}$$

$$Z\colon R\to R$$

and

$$a(t) = \frac{\eta}{2(t+2)^2},\quad a\colon T\to R.$$

In view of continuity Z the function g is continuous on $R^n \times T$. We can take

$$l(t) = \frac{t+2}{a(t)}$$

because

$$\|g(x, t, \omega) - g(\bar{x}, t, \omega)\| = \left\|(t+\omega+1)\left[Z\left(\frac{x-t-\omega}{a(t)}\right) - Z\left(\frac{\bar{x}-t-\omega}{a(t)}\right)\right]\right\|$$

$$= |t+\omega+1| \cdot \left\|Z\left(\frac{x-t-\omega}{a(t)}\right) - Z\left(\frac{\bar{x}-t-\omega}{a(t)}\right)\right\| \leqslant (t+2)\,\frac{1}{a(t)}\,\|x-\bar{x}\|.$$

Let us take $m(t) = t+2$ and $Y(\omega) \equiv 1$. Then

$$\|g(x, t, \omega)\| = \left\|(t+\omega+1) Z\left(\frac{x-t-\omega}{a(t)}\right)\right\| \leqslant \|(t+\omega+1)\cdot 1\|$$

$$= t+\omega+1 \leqslant t+2 \leqslant (t+2)[\|x\|+1].$$

It is seen that g belongs to G.

Let us check if $\sup\limits_{(x,t)\in R\times T_0} E\{\|g(x, t, \omega)\|^2\} \leqslant \eta$. It is enough to show that

$$\bigwedge_{(x,t)\in R\times T_0} E\{\|g(x, t, \omega)\|^2\} \leqslant \eta.$$

Let us choose an arbitrary $(x, t) \in R \times T_0$. We have

$$g(x, t, \omega) = (t+\omega+1) Z\left(\frac{\omega - x + t}{a(t)}\right),$$

as

$$Z(y) = Z(-y),\ y \in R.$$

For $\omega \in A = (x-t-a(t), x-t+a(t)) \cap \langle 0, 1\rangle$ we have

$$0 < Z\left(\frac{\omega - x + t}{a(t)}\right) \leqslant 1,$$

and for the others

$$Z\left(\frac{\omega - x + t}{a(t)}\right) = 0.$$

It follows that

$$E\{\|g(x,t,\omega)\|^2\} = \int_{\Omega} [g(x,t,\omega)]^2 d\omega$$

$$= \int_{A} [g(x,t,\omega)]^2 d\omega + \int_{\Omega\setminus A} [g(x,t,\omega)]^2 d\omega$$

$$= \int_{A} (t+\omega+1)^2 \left[Z\left(\frac{\omega-x+t}{a(t)}\right)\right]^2 d\omega$$

$$\leqslant \int_{A} (t+\omega+1)^2 d\omega \leqslant \int_{A} (t+2)^2 d\omega = (t+2)^2 \cdot P(A)$$

$$\leqslant (t+2)^2 \cdot 2a(t) = (t+2)^2 \frac{2\eta}{2(t+2)^2} = \eta.$$

The equation (2) has the form

$$\frac{dX_t}{dt} = -X_t + (t+\omega+1) Z\left(\frac{X_t - t - \omega}{a(t)}\right). \tag{4}$$

Let us consider now sample solution X_t of this equation with the initial conditions $t_0 = 0$, $x_0 = \delta$, which obviously exists, for example because of the theorem 1.2. of [2]. Almost all sample functions of this sample solution (as the solutions of the ordinary differential equations

$$\frac{dX_t(\omega)}{dt} = -X_t(\omega) + (t+\omega+1) Z\left(\frac{X_t(\omega) - t - \omega}{a(t)}\right) \tag{5}$$

obtained from (4) by fixing of ω) are differentiable and hence continuous. Let $\Omega^* \subset \Omega$ be a set all $\omega \in \Omega$ such that sample function $X_t(\omega)$ of the process X_t is not the solution of the equation (5). Obviously $P(\Omega^*) = 0$. Only the solutions of the equation (5) can be the sample functions $X_t(\omega)$, $\omega \in \Omega \setminus \Omega^*$ of the process X_t, and because the Lipschitz conditions with regard to x holds for functions f and g these solutions are unique. Let us fix now any $\omega \in \langle 0, \delta^* \rangle \setminus \Omega^*$, where $\delta^* = \min\{\delta, 1\}$. Then the sample function $X_t(\omega)$ of sample solution X_t of the equation (4) with initial conditions $t_0 = 0$, $x_0 = \delta$ is not smaller than line $x = t + \omega$ (we denote it by p). If it were smaller than p at the point t' then in view of the continuity of this sample function there would exist a point $0 < t'' < t'$ of its coincidence with p.

On the line p function g admits value

$$g(x,t,\omega) = (t+\omega+1) Z\left(\frac{t+\omega-t-\omega}{a(t)}\right) = (t+\omega+1) Z(0) = t+\omega+1$$

$$= x+1$$

and equation (5) has the form

$$\frac{dX_t(\omega)}{dt} = -X_t(\omega)+X_t(\omega)+1 = 1.$$

In view of the uniqueness of the solutions of (5) it means that $X_t(\omega)$ would have to coincide with p for each $t \geqslant t''$. In particular for t' which is impossible.

So, because $\omega \in \langle 0, \delta^*\rangle \setminus \Omega^*$ was arbitrary we see that on this set sample solution X_t of the equation (4) with the initial conditions $t_0 = 0$, $x_0 = \delta$ is not less than $t+\omega$. Hence

$$E\{\|X_t(\omega)\|^2\} = \int_{\Omega} (X_t(\omega))^2 d\omega = \int_{\langle 0,\delta^*\rangle \setminus \Omega^*} (X_t(\omega))^2 d\omega + \int_{(\delta^*,1)\cup \Omega^*} (X_t(\omega))^2 d\omega$$

$$\geqslant \int_{\langle 0,\delta^*\rangle \setminus \Omega^*} (X_t(\omega))^2 d\omega \geqslant \int_{\langle 0,\delta^*\rangle \setminus \Omega^*} (t+\omega)^2 d\omega = \int_{\langle 0,\delta^*\rangle} (t+\omega)^2 d\omega = t^2\delta^* + t(\delta^*)^2 + \frac{1}{3}(\delta^*)^3.$$

The values of this polynomial tend to the infinity if t tends to the infinity. Or there exists $t \in T_0$ such that

$$E\{\|X_t(\omega)\|^2\} \geqslant \varepsilon.$$

REFERENCES

[1] Bunke, H., *On the stability of ordinary differential equations under persistent random disturbances*, Zeitschr. Angew. Math. Mech., 51 (1971), 543-546.

[2] Bunke, H., *Gewöhnliche Differentialgleichungen mit zufälligen Parametern*, Akademie-Verlag, Berlin 1972.

STRESZCZENIE

W pracy podany jest przykład równania różniczkowego zwyczajnego, którego rozwiązanie zerowe jest jednostajnie asymptotycznie stabilne, ale nie jest średniokwadratowo stabilne względem stale działających losowych zakłóceń z G.

РЕЗЮМЕ

В работе представлен пример обыкновенного дифференциального уравнения, которого нулевое решение равномерно асимптотически устойчивое, но не устойчиво в квадратном среднем относительно постоянно действующих случайных возмущений из G.