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Some Problems for Linearly Invariant Families*

Pewne problemy dla rodzin liniowo-niezmienniczych

Некоторые проблемы для линейно-инвариантных семейств

1. Introduction. Notations

One of the most interesting family of locally univalent functions

$$(1) \quad \varphi(z) = z + b_2 z^2 + \dots$$

which are analytic in the unit disk $K = \{z: |z| < 1\}$ is the class V_k ($k \geq 2$) of functions with bounded boundary rotation at most $k\pi$ in K . Namely, we say that $\varphi \in V_k$ if for every $r \in [0, 1)$

$$\int_0^{2\pi} \left| \operatorname{Re} \left(1 + re^{i\theta} \frac{\varphi'''(re^{i\theta})}{\varphi'(re^{i\theta})} \right) \right| d\theta \leq k\pi.$$

It is well known that $\varphi \in V_k$ if and only if

$$\varphi'(z) = \exp \left\{ - \int_0^{2\pi} \log(1 - ze^{-it}) d\mu(t) \right\},$$

where μ is real valued function with bounded variation on $[0; 2\pi]$ with

$$\int_0^{2\pi} d\mu(t) = 2, \quad \int_0^{2\pi} |\mu(t)| \leq k.$$

Another very useful necessary and sufficient condition for φ to be an element of the class V_k has been done by Brannan [1].

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Lemma 1. [1]. A function φ belongs to V_k if and only if there exist two normalized starlike functions s_1, s_2 such that

$$(2) \quad \varphi'(z) = \left[\frac{s_1(z)}{z} \right]^{\frac{k+2}{4}} / \left[\frac{s_2(z)}{z} \right]^{\frac{k-2}{4}}, \quad z \in K.$$

The condition (2) can be written in the following form

$$(3) \quad \varphi'(z) = [g_1'(z)]^{\frac{2+k}{4}} [g_2'(z)]^{\frac{2-k}{4}}, \quad z \in K,$$

where g_1, g_2 are two normalized convex functions. In this way we may say that the class S^c of convex normalized functions generates the class V_k by the formula (3). In the same manner we can define another class of functions using instead of S^c for example the class S of normalized univalent functions in K . Looking at this problem somewhat more generally, we consider the following class \mathcal{F} .

Definition 1. We say that $F \in \mathcal{F}$ if F is analytic in K and its derivative F' has the form

$$(4) \quad F'(z) = \prod_{j=1}^n [f_j'(z)]^{a_j}, \quad a_j \text{ is real, } \sum_{j=1}^n a_j = 1, \quad z \in K,$$

where $f_j \in \mathfrak{M}_j$, $j = 1, 2, \dots, n$, and \mathfrak{M}_j is linearly invariant family in the sense of Pommerenke [10].

For some known families \mathfrak{M}_j we will determine here the region

$$(5) \quad D(z, a) = \left\{ w \in C : w = \log \frac{F'(z)}{F'(a)}, \quad F \in \mathcal{F} \right\}, \quad D(z, 0) = D(z),$$

for fixed $z, a \in K$. As a corollary we obtain the sharp estimates for $|F'(z)|$ and $|\arg F'(z)|$.

Moreover, we will determine the region $D(z)$ for the class of β -close-to- V_k functions defined below.

The fact that \mathcal{F} is linearly invariant family for which we know $\max |\arg F'(z)|$, $z \in K$, allow us to find the radii of univalence and close-to-convexity of \mathcal{F} . In particular we show that if f is univalent ($f \in S$) and $a \in [0; 1]$ then the integral $F(z) = \int_0^z (f'(\xi))^a d\xi$ is univalent at least in the disk $|z| < 0,81$.

Definition 2. Let $\beta \geq 0$. An analytic function f of the form

$$(6) \quad f(z) = z + a_1 z^2 + \dots, \quad z \in K$$

is β -close-to- V_k function if there exist a real number a , and a function $\varphi \in V_k$ such that

$$(7) \quad \left| \arg e^{ia} \frac{f'(z)}{\varphi'(z)} \right| \leq \beta \frac{\pi}{2}, \quad z \in K.$$

The class of β -close-to- V_k functions which we will denote by $LV(\beta, k)$ has been considered in [3].

In particular, we have

$LV(\beta, 2) = L\beta$ = the class of β -close-to-convex functions;

$LV(1, 2) = L$ = the class of close-to-convex functions;

$LV(0, k) = V_k$; $LV(0, 2) = S^c$, e.g. [5], [7], [8], [13] respectively.

We will need the class \mathcal{P} of all analytic functions in K of the form

$$(8) \quad p(z) = e^{i\delta} + p_1 z + p_2 z^2 + \dots \quad (\delta \text{ is real})$$

which satisfy the condition

$$\operatorname{Re} p(z) > 0, \quad z \in K.$$

Finally, let us define the function which will play further an important role:

$$(9) \quad F_{\beta, k}(z) = \int_0^z \frac{(1 - \xi e^{i\theta_2})^{\beta + \frac{k}{2} - 1}}{(1 - \xi e^{i\theta_1})^{\beta + \frac{k}{2} + 1}} = \frac{2}{(2\beta + k)(e^{i\theta_1} - e^{i\theta_2})} \left[\left(\frac{1 - ze^{i\theta_2}}{1 - ze^{i\theta_1}} \right)^{\beta + \frac{k}{2}} - 1 \right],$$

$$\theta_1, \theta_2 \in [0, 2\pi], \quad e^{i\theta_1} \neq e^{i\theta_2}.$$

2. Statement of results

Let us denote

$$(10) \quad D_j(z) = \{w: w = \log f'_j(z), f_j \in \mathfrak{M}_j\}, \quad j = 1, 2, \dots, n,$$

and

$$(11) \quad \eta = \left(\frac{1 - |a|^2}{1 - \bar{a}z} \right)^2, \quad \xi = \frac{z - a}{1 - \bar{a}z}, \quad z, a \in K.$$

We have

Theorem 1. If $F \in \mathcal{F}$. then

$$(12) \quad D(z, a) = \bigoplus_{j=1}^n D_j(\xi) \cup \{\log \eta\},$$

where \bigoplus denotes the geometric sum of sets.

From Theorem 1 it follows that in order to find the set $D(z, a)$ it is sufficient to determine the sets $D_j(\xi)$. The set $D(z)$ is known in the case of the classes S [12], L [7] and S^c [13]. Here we will find this set for the class $LV(\beta, k)$.

Theorem 2. The set $D(z) = \{w: w = \log f'(z), f \in LV(\beta, k)\}$ is a closed and convex set whose boundary has the equation

$$(13) \quad w(t) = \log \frac{(1 - re^{i\theta_2})^{t-1}}{(1 - re^{i\theta_1})^{t+1}}, \quad t \in [0; 2\pi], \quad r = |z| < 1,$$

where $\gamma = \beta + \frac{k}{2}$ and

$$(14) \quad \theta_1 = \theta_1(t) = t - \arcsin(r \sin t), \quad \theta_2 = \theta_2(t) = \pi + t + \arcsin(r \sin t).$$

The functions corresponding to the boundary points of $D(z)$ have the form (9) with θ_1, θ_2 given by (14).

Putting $\beta = 0$ in Theorem 2 we have

Theorem 2'. The set $D(z) = \{w: w = \log \varphi'(z), \varphi \in V_k\}$ is a closed and convex set whose boundary has the equation

$$w(t) = \log \frac{(1 - re^{it\theta_2})^{k/2-1}}{(1 - re^{it\theta_1})^{k/2+1}}, \quad t \in [0; 2\pi], \quad r = |z| < 1,$$

where θ_1 and θ_2 are given by (14). The functions corresponding to the boundary points of $D(z)$ have the form (9) with $\beta = 0$.

By putting $\beta = 0$ and $k = 2$ in Theorem 2 we obtain the result for close-to-convex functions [7].

From Theorem 2 we can get exact estimates for $|f'(z)|$ and $|\arg f'(z)|$ if $f \in LV(\beta, k)$.

Theorem 3. If $f \in LV(\beta, k)$ then for $|z| = r < 1$

$$(15) \quad |\arg f'(z)| \leq (2\beta + k) \arcsin r \quad [3],$$

$$(16) \quad \frac{(1-r)^{\gamma-1}}{(1+r)^{\gamma+1}} \leq |f'(z)| \leq \frac{(1+r)^{\gamma-1}}{(1-r)^{\gamma+1}}.$$

The extremal function has the form (9) with θ_1, θ_2 given by (14) with appropriate t .

So far as the set $D(\xi)$ has been determined for $f \in LV(\beta, k)$ we can find the set $D(z, a)$ for the following class \mathcal{F} .

Theorem 4. Let

$$\mathcal{F}_{LV} = \left\{ F: F'(z) = \prod_{j=1}^n [f'_j(z)]^{a_j}, f_j \in LV(\beta_j, k_j) \right\}.$$

Then $D(z, a)$ is a closed and convex set whose boundary has the equation

$$(17) \quad w(t) = \log \eta \prod_{j=1}^n \frac{(1 - |\xi| e^{it\theta_2})^{(\gamma_j-1)a_j}}{(1 - |\xi| e^{it\theta_1})^{(\gamma_j+1)a_j}}, \quad t \in [0, 2\pi],$$

where $\gamma_j = \beta_j + k_j/2$ and η, ξ are given by (11) and

$$(18) \quad \theta_1 = \theta_1(t) = t - \arcsin(|\xi| \sin t).$$

$$\theta_2 = \theta_2(t) = \pi + t + \arcsin(|\xi| \sin t).$$

The functions F corresponding to the boundary of $D(z, a)$ are given by

$$(19) \quad \frac{F'(z)}{F'(a)} = \eta \prod_{j=1}^n \frac{(1 - \xi e^{i\theta_2})^{(\gamma_j-1)a_j}}{(1 - \xi e^{i\theta_1})^{(\gamma_j-1)a_j}}.$$

Putting in Theorem 4 $n = 1$ and $k = 2$ or $\beta = 0$ we obtain

Corollary 1. The boundary of

$$D(z, a) = \left\{ w : w = \log \frac{f'(z)}{f'(a)}, f \in L_\beta \right\} \text{ has the equation}$$

$$w(t) = \log \eta \frac{(1 - |\xi| e^{i\theta_2})^\beta}{(1 - |\xi| e^{i\theta_1})^{\beta+2}}, \quad t \in [0; 2\pi].$$

Corollary 2. The boundary of

$$D(z, a) = \left\{ w : w = \log \frac{\varphi'(z)}{\varphi'(a)}, \varphi \in V_k \right\} \text{ has the equation}$$

$$w(t) = \log \eta \frac{(1 - |\xi| e^{i\theta_2})^{k/2-1}}{(1 - |\xi| e^{i\theta_1})^{k/2+1}}, \quad t \in [0; 2\pi].$$

Moreover, from Theorem 4 it follows

Corollary 3. If $F \in \mathcal{F}_{LV}$ then

$$(20) \quad \left| \arg \frac{F'(z)}{F'(a)} \eta^{-1} \right| \leq \left(2 \sum_{j=1}^n \gamma_j \right) \arcsin |\xi|,$$

$$(21) \quad |\eta| \frac{(1 - |\xi|)^{\sum_{j=1}^n (\gamma_j-1)a_j}}{(1 + |\xi|)^{\sum_{j=1}^n (\gamma_j+1)a_j}} \leq \left| \frac{F'(z)}{F'(a)} \right| \leq |\eta| \frac{(1 + |\xi|)^{\sum_{j=1}^n (\gamma_j-1)a_j}}{(1 - |\xi|)^{\sum_{j=1}^n (\gamma_j+1)a_j}}.$$

Theorem 5. The radius of univalence of $LV(\beta, k)$ is equal to $\tan \frac{\pi}{2\beta + k}$.

The extremal function is given by (9) with appropriate θ_1 and θ_2 .

Corollary 4. The radius of univalence of \mathcal{F}_{LV} is equal to $\tan \frac{\pi}{\sum (2\beta_j + k_j)}$.

Theorem 6. The radius of κ -close-to-convexity of $LV(\beta, k)$ is the unique root of the equation

$$(22) \quad 2 \operatorname{arccot} w - (2\beta + k) \operatorname{arccot} \left[\left(\beta + \frac{k}{2} \right) w \right] = -\kappa\pi$$

where $w = (1 - r^2) [(2\beta + k)^2 r^2 - (1 - r^2)^2]^{-1/2}$.

The result is sharp.

Corollary 5. $LV(\beta, k) \subset L_{\beta+k/2-1}$, in particular $V_k \subset L_{k/2-1}[2]$.

Remark 1. If $2\beta + k \leq 4$ then $LV(\beta, k)$ consists only of univalent close-to-convex functions.

An interesting example of the class \mathcal{F} is the following one:

$$(23) \quad \mathcal{F}_S = \left\{ F: F(z) = \int_0^z [f'(\xi)]^\alpha [g'(\xi)]^{1-\alpha} d\xi, f, g \in S \right\}.$$

The considerations of properties of the class \mathcal{F}_S are close to the problem concerning univalence of the integral $\int_0^z (f'(\zeta))^\alpha d\zeta$. We will prove.

Theorem 7. If $\alpha \in [0, 1]$ then every function $F \in \mathcal{F}_S$ is univalent at least in the disk $|z| < r_u$ where

$$(24) \quad r_u = \frac{e^{\pi/2}}{1 + \sqrt{1 + e^\pi}}, \quad (r_u > 0,81).$$

Remark 2. The integral $\int_0^z (f'(\zeta))^\alpha d\zeta$, $f \in S$, $\alpha \in [0, 1]$ is univalent at least for $|z| < r_u$.

3. Lemmas

Lemma 1'. A function $f \in LV(\beta, k)$ if and only if there exist a function $p \in \mathcal{P}$ and functions $g_1, g_2 \in S^c$ such that

$$(25) \quad f'(z) = e^{-iz} p^\beta(z) [g'_1(z)]^{\frac{2+k}{4}} [g'_2(z)]^{\frac{2-k}{4}}, \quad z \in K.$$

The proof of Lemma 1' follows from Lemma 1 and formula (7).

Lemma 2. For every $\beta \geq 0$ and $k \geq 2$ the class $LV(\beta, k)$ is linearly invariant family of order $\gamma = \beta + \frac{k}{2}$.

Proof. Let β, k be admissible and $f \in LV(\beta, k)$. From the definition 2 $f'(z) \neq 0$, $z \in K$, so in order to prove the linear invariance of $LV(\beta, k)$ we should show that for arbitrary $a \in K$

$$(26) \quad F(z) = \frac{f\left(\frac{z+a}{1+\bar{a}z}\right) - f(a)}{(1-|a|^2)f'(a)} \in LV(\beta, k).$$

Robertson has proved [11] that V_k is linearly invariant family which implies

$$(27) \quad \Phi(z) = \frac{\varphi\left(\frac{z+a}{1+\bar{a}z}\right) - \varphi(a)}{(1-|a|^2)\varphi'(a)} \in V_k \text{ if } \varphi \in V_k.$$

From (26) and (27) we obtain

$$\frac{F'(z)}{\Phi'(z)} = \frac{\varphi'(a)}{f'(a)} \cdot \frac{f'\left(\frac{z+a}{1+\bar{a}z}\right)}{\varphi'\left(\frac{z+a}{1+\bar{a}z}\right)}$$

Because $\left| \arg e^{ia} \frac{f'(a)}{\varphi'(a)} \right| \leq \beta \frac{\pi}{2}$, $a \in K$, we can choose \tilde{a} such that

$$\left| \arg e^{i\tilde{a}} \frac{F'(z)}{\Phi'(z)} \right| \leq \beta \frac{\pi}{2}, z \in K, \text{ and hence } F \in LV(\beta, k).$$

From (7) we can write

$$(28) \quad e^{ia} \frac{f'(z)}{\varphi'(z)} = p^\beta(z) = (e^{i\delta} + p_1 z + \dots)^\beta$$

where $p \in \mathcal{P}$. Comparing the coefficients in (28) we find that

$$(29) \quad a_2 = b_2 + \frac{1}{2} \beta p_1 e^{-i\delta}$$

Taking into account that $|b_2| \leq \frac{k}{2}$ and $|p_1| \leq 2$ we find that the order of family $LV(\beta, k)$ is equal to $\gamma = \sup |a_2| = \beta + \frac{k}{2}$.

Lemma 3. *The class \mathcal{F} is linearly invariant family.*

Proof. Let $F \in \mathcal{F}$ be given by (4). Then $f_j \in \mathfrak{M}_j$ as well as

$$(30) \quad F_j(z) = \frac{f_j\left(\frac{z+a}{1+\bar{a}z}\right) - f_j(a)}{(1-|a|^2)f'_j(a)} \in \mathfrak{M}_j, \quad j = 1, 2, \dots, n.$$

If we put $\psi(z) = \frac{F\left(\frac{z+a}{1+\bar{a}z}\right) - F(a)}{(1-|a|^2)F'(a)}$, then

$$\begin{aligned} \psi'(z) &= \frac{F'\left(\frac{z+a}{1+\bar{a}z}\right)}{(1+\bar{a}z)^2 F'(a)} = \frac{1}{(1+\bar{a}z)^2} \prod_{j=1}^n \left[\frac{f'_j\left(\frac{z+a}{1+\bar{a}z}\right)}{f'_j(a)} \right]^{a_j} = \\ &= \frac{1}{(1+\bar{a}z)^2} \prod_{j=1}^n [(1+\bar{a}z)^2 F'_j(z)]^{a_j} = \prod_{j=1}^n [F'_j(z)]^{a_j} \end{aligned}$$

which completes the proof.

Remark 3. It is easy to observe that if $f_j(z) \in \mathfrak{M}_j$, then $e^{-i\theta} f_j(e^{i\theta} z) \in \mathfrak{M}_j$, for arbitrary real θ . This implies that if γ_j is the order of \mathfrak{M}_j , then the order of \mathcal{F} is equal to $\gamma = \sum_{j=1}^n |a_j| \gamma_j$.

Remark 4. The result of Pommerenke [10] implies that radius of convexity of the family \mathcal{F} is equal to

$$r_c = \left(\sum_{j=1}^n |a_j| \gamma_j + \sqrt{\left(\sum_{j=1}^n |a_j| \gamma_j \right)^2 - 1} \right)^{-1}.$$

4. Proofs of theorems

Proof of Theorem 1. Let $F \in \mathcal{F}$. Then from (4) and (30) we have

$$\begin{aligned} \frac{F'(z)}{F'(a)} &= \prod_{j=1}^n \left[\frac{f'_j(z)}{f'_j(a)} \right]^{a_j} = \prod_{j=1}^n \left[\left(\frac{1-|a|^2}{1-\bar{a}z} \right)^2 F'_j \left(\frac{z-a}{1-\bar{a}z} \right) \right]^{a_j} = \\ &= \eta \prod_{j=1}^n [F'_j(\xi)]^{a_j}, \quad \xi \in K. \end{aligned} \quad (32)$$

Now, we see that

$$\log \frac{F'(z)}{F'(a)} = \log \eta + \sum_{j=1}^n a_j \log F'_j(\xi),$$

which implies (12) because F_j is ranging independently over \mathfrak{M}_j , $j = 1, 2, \dots, n$.

It is worthwhile to mention that convexity of D_j , $j = 1, 2, \dots, n$ (which occurs for example if \mathfrak{M}_j is one of the classes $S^c, L, S, LV(\beta, k)$) implies convexity of $D(z, a)$.

Proof of Theorem 2. First of all it is convenient to observe that $D(ze^{i\theta}) = D(|z|) = D$, $|z| = r$. The set D is closed because the class $LV(\beta, k)$ is compact. The convexity of D follows from the fact that for $\lambda \in [0, 1]$ and $f_1, f_2 \in LV(\beta, k)$ the function

$$f_\lambda(z) = \int_0^z [f'_1(t)]^\lambda \cdot [f'_2(t)]^{1-\lambda} dt \in LV(\beta, k).$$

Let $f \in LV(\beta, k)$. From Lemma 1' we have

$$(31) \quad f'(r) = e^{-ia} p^\beta(r) [\varphi'_1(r)]^{\frac{2+k}{k}} [\varphi'_2(r)]^{\frac{2-k}{4}},$$

$0 < r < 1$, is real, where $p \in \mathcal{P}$, $\varphi_1, \varphi_2 \in S^c$.

It is well-known that functions corresponding to the boundary points of $\{w: w = p(r), p \in \mathcal{P}\}$ have the form

$$p(r) = \frac{e^{ir} - re^{ir}}{1 - re^{ir}}, \quad \tau \in [0; 2\pi],$$

as well as the functions corresponding to the boundary points of $\{w: w = \varphi_j'(r), \varphi_j \in S^c, j = 1, 2\}$ have the form

$$\varphi_j(r) = \frac{r}{1 - re^{i\theta_j}}, \quad j = 1, 2, [13].$$

From the definition of the class $LV(\beta, k)$ it follows that $e^{ia} = e^{i\delta\beta}$. This fact and the facts cited above imply together with (31) that the function f corresponding to the boundary points of D has the following form

$$(32) \quad f'(r) = \left(\frac{1 - \varepsilon_1 r}{1 - \varepsilon_2 r} \right)^\beta (1 - \varepsilon_3 r)^{-\frac{2+k}{2}} (1 - \varepsilon_4 r)^{-\frac{2-k}{2}},$$

$$\varepsilon_j = e^{i\theta_j}, \quad \theta_j \in [0, 2\pi], \quad j = 1, 2, 3, 4.$$

The convexity of D implies that finding the boundary of D is equivalent to determining the maximum of the function

$$(33) \quad \operatorname{Re}[e^{-it} \log f'(r)] = \operatorname{Re} \left\{ e^{-it} \left[\beta \log(1 - re^{i\theta_1}) - \beta \log(1 - re^{i\theta_2}) - \frac{2+k}{2} \log(1 - re^{i\theta_3}) - \frac{2-k}{2} \log(1 - re^{i\theta_4}) \right] \right\}$$

with respect to $\theta_j \in [0, 2\pi]$, for fixed $t \in [0, 2\pi]$. The number t denotes the angle between the imaginary axis and supporting line to D . Moreover, we may observe from (24) that D is symmetric with respect to the real axis because the image of the circle $\xi = 1 - ze^{i\theta}, \theta \in [0, 2\pi], z \in K$, under the mapping $w = \log \zeta$ is a convex curve symmetric with respect to the real axis.

So, we may assume that $t \in [0; \pi]$.

It is easy to check that the function

$$\chi(\theta) = \operatorname{Re}\{e^{-it} \log(1 - re^{i\theta})\}$$

attains its minimum for $\theta = \theta_1(t)$ and maximum for $\theta = \theta_2(t)$, where θ_1 and θ_2 are given by (14). Substituting (14) into (33) we obtain (13).

Proof of Theorem 3. Taking into account the convexity and symmetry with respect to the real axis of $D(z)$ we see that $\max |f'(z)|$ is attained for $t = 0$, $\min |f'(z)|$ for $t = \pi$ and $\max |\arg f'(z)|$ for $t = \pi/2$. Putting these special values of t into (14) and (32) we get (15) and (16).

Proof of Theorem 4. This result follows immediately from Theorems 1 and 2.

Proof of Theorem 5. In order to find the radius of univalence r_u of linearly invariant family of it is sufficient to find the largest disk $|z| < r_0$ in

which every function from $LV(\beta, k)$ is different from zero because then we have [10]

$$(34) \quad r_u = \frac{r_0}{1 + \sqrt{1 - r_0^2}}$$

The reasoning as in [10, Satz 2.6] implies that if $f \in LV(\beta, k)$ and $f(r_0) = 0$ then $\arg f'(r_0) = \pm 2\pi$. Since from (15) we have $|\arg f'(z)| \leq (2\beta + k) \arcsin |z|$ we conclude that $r_0 \geq \sin \frac{2\pi}{2\beta + k} = \varrho$. By considering the function $F_{\beta, k}$ given by (9) with $\theta_1 = \arccos \varrho$, $\theta_2 = -\arccos \varrho$ we find that $F_{\beta, k}\left(\sin \frac{2\pi}{2\beta + k}\right) = 0$. Thus $r_0 \leq \varrho = \sin \frac{2\pi}{2\beta + k}$ which complete the proof after using formula (34).

Proof of Theorem 6. The estimate of $|\arg f'(z)|$, $f \in LV(\beta, k)$ help us to find the radius of κ -close-to-convexity, $\kappa \geq 0$, of $LV(\beta, k)$. Goodman [5] has proved that $f \in L_\kappa$ if for each $r \in (0, 1)$ and for each pair θ_1, θ_2 , $0 \leq \theta_1 < \theta_2 \leq 2\pi$

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left(1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right) d\theta \geq -\kappa\pi$$

Using this condition and the method originated in [6] (see also [3]) one can prove Theorem 6. Corollary 5 follows from the fact that the left hand side of the equation (22) is strictly decreasing function of $r \in (0, 1)$ and its value for $r = 1$ is equal to $\frac{\pi}{2}(2 - 2\beta - k)$.

Proof of Theorem 7. Let $F \in \mathcal{F}_S$ be given by (23) and $a \in [0, 1]$. Then

$$(35) \quad |\arg F'(z)| \leq \begin{cases} 4 \arcsin |z|, & |z| \leq \frac{1}{\sqrt{2}} \\ \pi + \log \frac{|z|^2}{1 - |z|^2}, & |z| \geq \frac{1}{\sqrt{2}} \end{cases}$$

The estimate (35) is exact and the sign of equality holds if $f = g = F_0 \in S$, where F_0 is extremal function for $\max |\arg F'(z)|$ [4, p. 116] within class S . Let $r_0 = \inf_{F \in \mathcal{F}_S} \sup_{r > 0} \{r : F(z) \neq 0 \text{ for } 0 < |z| < r\}$ The fact that \mathcal{F}_S is linearly invariant family allow us to apply the method of Pommerenke [10, Satz 2.6] like in the Proof of Theorem 5. In this manner we find that r_0 is the root of the equation $\log \frac{r^2}{1 - r^2} = \pi$ which implies (24) after using (34). It may be checked that $r_u(S) > 0.81$.

From (35) and from linearly invariance of \mathcal{F}_S follows that the exact radius of close-to-convexity $r_L(\mathcal{F}_S) = r_L(S) \in (0,80; 0,81)$ [6].

If $g(z) \equiv z$ then the class \mathcal{F}_S reduces to

$$(36) \quad \left\{ F: F(z) = \int_0^z (f'(\xi))^a d\xi, f \in S \right\},$$

which is no more linearly invariant.

The problem of finding $\max |a|$ for which (36) consists of univalent functions has been considered by many authors. Recently Pfaltzgraff [9] proved that (36) is univalent for $|a| \leq \frac{1}{2}$ (a may be a complex number).

Since the class (36) for $a \in (0, 1)$ is a subclass of \mathcal{F}_S given by (23) we obtain from Theorem 7 that the radius of univalence for (36) is at least 0,81.

REFERENCES

- [1] Brannan D. A., *On functions of bounded boundary rotation* I, Proc. Edinburgh Math. Soc. Ser II, 16 (1969), 339-347.
- [2] Brannan D. A., Clunie J. G. and Kirwan W. E., *On the coefficient problem for functions of bounded boundary rotation*. Annales Academiae Scientiarum Fennicae, Series A (1973), 2-18.
- [3] Campbell D. M., Ziegler M. R., *The argument of the derivative of linear invariant families of finite order and the radius of close-to-convexity*, Ann. Univ. M. Curie-Skłodowska, Sec. A, 28 (1974), 5-22.
- [4] Goluzin G. M., *Geometric theory of functions of a complex variable*. Vol. 26 Amer. Math. Soc., Providence, R. I., 1969.
- [5] Goodman A. W., *On close-to-convex functions of higher order*. Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae, Sectio Math., 15 (1972), 17-30.
- [6] Krzyż J., *The radius of close-to-convexity within the family of univalent functions*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys., 10 (1962), 201-104.
- [7] —, *Some remarks on close-to-convex functions*, ibidem, 12 1964, 25-28.
- [8] Lewandowski Z., *Sur l'identité de certaines classes de fonctions univalentes* I, Ann. Univ. M. Curie-Skłodowska, Sectio A, 12 (1958), 131-146.
- [9] Pfaltzgraff J. A., *Univalence of the integral $f'(z)^{\lambda}$* . Bull. London Math. Soc. 7 (1975), 254-256.
- [10] Pommerenke Ch., *Linear - invariante Familien analytischer Funktionen* I, Math. Annalen, 155 (1964), 108-154.
- [11] Robertson M. S., *Coefficients of functions with bounded boundary rotation*, Canad. J. Math. 21 (1969), 1477-1482.
- [12] Schaeffer A. C., Spencer D. C., *The coefficient regions of schlicht functions*, New York 1950.
- [13] Strohhäcker E., *Beiträge zur Theorie der schlichten Funktionen*, Math. Z. 37 (1933), 356-380.

STRESZCZENIE

Niech \mathcal{F} oznacza rodzinę funkcji holomorficznych F w kole $K = \{z: |z| < 1\}$ danych wzorem

$$F'(z) = \prod_{j=1}^n [f'_j(z)]^{a_j}, \quad z \in K,$$

gdzie a_j jest liczbą rzeczywistą, $\sum_{j=1}^n a_j = 1$ a funkcje f_j należą do ustalonej rodziny \mathfrak{M}_j , która jest liniowo-niezmiennecka w sensie Pommerenke [10].

W pracy wyznaczono obszar zmienności

$$D(z, a) = \left\{ w: w = \log \frac{F'(z)}{F'(a)}, \quad F \in \mathcal{F} \right\}, \quad z, a \in K$$

dla kilku znanych rodzin \mathfrak{M}_j .

W szczególności wyznaczono obszar wartości $D(z)$

$$D(z) = \{w: w = \log f'(z)\}$$

dla ustalonego $z \in K$ i funkcji f zmieniającej się w klasie funkcji β -prawie- V_k (definicja 2).

Podano również promień jednolistności i prawie-wypukłości pewnych rodzin \mathcal{F} , a jako wniosek otrzymano, że jeśli f jest funkcją jednolistną, to całka

$$F(z) = \int_0^z (f'(t))^a dt, \quad a \in [0, 1]$$

jest jednolistna przynajmniej w kole $|z| < 0,81$.

РЕЗЮМЕ

Пусть F обозначает класс голоморфных функций в кругу $K = \{z: |z| < 1\}$ данных формулой

$$F'(z) = \prod_{j=1}^n [f'_j(z)]^{a_j}, \quad z \in K,$$

где a_j вещественное число, $\sum a_j = 1$ и функция f_j принадлежит к фиксированному семейству \mathfrak{M}_j , которое есть линейно-инвариантное в смысле Поммеренке [10].

В этой работе определено область изменения

$$D(z, a) = \left\{ w: w = \log \frac{F'(z)}{F'(a)}, \quad F \in \mathcal{F} \right\}, \quad z, a \in K$$

для некоторых известных семейств \mathfrak{M}_j .

В частности определено область изменения

$$D(z) = \{w: w = \log f'(z)\}$$

для фиксированного $z \in K$, когда функция f изменяется в класс β — почти — V_k (определение 2).

Кроме того поданы радиусы одnolistnosti и почти-вypuklosti некоторых семейств \mathcal{F} . В следствии получен результат, что для одnolistnoj функции f интеграл

$$F(z) = \int_0^z (f(t))^a dt, \quad a \in [0, 1]$$

одnolistny по крайней mере w круге $|z| < 0,81$.

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