ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA
VOL. XXX, 9 SECTIO A 1976

Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

MIECZYSŁAW POLAK

On a Queueing System of the Type $(M/M/n)^{\pm}$

O systemie obsługi masowej typu $(M/M/n)^{\pm}$

О системе массового обслуживания типа $(M/M/n)^{\pm}$

1. The random process describing system states. Let $N(t)$ be the number of customers present in the system $(M/M/n)$ at time t. The behaviour $N(t)$ for $t > t_0$ depends only on the value of $N(t_0)$ and does not depend on the behaviour $N(t)$ on $[0, t_0)$. In practice such an independence does not always occur. That is why the research of the queueing systems in which the mentioned independence does not appear, has a great significance both from theoretical and practical points of view. A queueing system of this type is presented in [1].

In this note we are going to consider a queueing system which consists of n serves heaving a bounded number – m of waiting places, and also a system with an infinite number if places in which both interarrival time and service time are distributed exponentially with changeable intensity. Namely, if the last change before the instant t was such that the service has just ended, then intensity of arrival and service at the instant t equals λ^+ and μ^+ respectively; and if that event was the arrival of a customer, then they are λ^- and μ^-.

The considered queueing systems are characterized at every moment t by one of the symbols:

(i) in the case of the system with an infinite number of waiting places

$$N(t) \in \{0^+, 1^{\pm}, 2^{\pm}, \dots, k^{\pm}, \dots\},$$

(ii) in the case of the system with a finite number (m) of waiting places

$$N(t) \in \{0^+, 1^{\pm}, 2^{\pm}, \dots, (m+n-1)^{\pm}, (m+n)^-\}.$$

It is said that at the instant t_0 the system is in the state k^+ if the last change before the instant t_0 was such that the service has been ended, and k^- if the last change just before the instant t_0 was such that the customers have arrived under the condition that in the both cases the number of customers in the system at the moment t_0 equals k.

The system being considered is called the system of the type $(M/M/n)^{\pm}$. Thus it is assumed that $N(t)$ is the homogeneous Markov's chain with transition probabilities as follows:

in the case (i):

$$
\begin{aligned}
&P[0^+ \xrightarrow{\Delta t} 0^+] = 1-\lambda^+\Delta t + o(\Delta t),\\
&P[k^\pm \xrightarrow{\Delta t} k^\pm] = 1-(\lambda^\pm + \mu_k^\pm)\Delta t + o(\Delta t),\\
&P[k^\pm \xrightarrow{\Delta t} (k+1)^-] = \lambda_k^\pm \Delta t + o(\Delta t),\\
&P[k^\pm \xrightarrow{\Delta t} (k-1)^+] = \mu_k^+ \Delta t + o(\Delta t),\\
&P[k^\pm \xrightarrow{\Delta t} (k+r)^-] = o(\Delta t),\ r>1,\\
&P[k^\pm \xrightarrow{\Delta t} (k-r)^+] = o(\Delta t),\ r>1,
\end{aligned}
\tag{1}
$$

where

$$
\mu_k^\pm = \begin{cases} k\mu^\pm, & \text{if } 0 \leqslant k < n,\\ n\mu^\pm, & \text{if } k \geqslant n,\end{cases} \qquad \lambda^\pm = \sum_{k=0}^{\infty} \lambda_k^\pm; \tag{2}
$$

and in the case (ii): the condition (1) holds with

$$
\mu_k^\pm = \begin{cases} k\mu^\pm; & \text{if } 0 \leqslant k < n,\\ n\mu^\pm, & \text{if } n \leqslant k < m+n,\\ n\mu^-, & \text{if } k = m+n.\end{cases} \tag{2'}
$$

If $\lambda^\pm = \lambda$, $\mu^\pm = \mu$ then the considered system is reduced to $(M/M/n)$.

2. The distribution of $N(t)$. Let $P_k^\pm(t) = P[N(t) = k^\pm]$ be the probability of distribution of $N(t)$. With the standard methods applied it may be proved that the Kolmogorov equations for the probabilities $P_k^\pm(t)$ are as follows — in the case (i):

$$
\begin{aligned}
&\frac{dP_0^+(t)}{dt} = -\lambda^+ P_0^+(t) + \mu^+ P_1^+(t) + \mu^- P_1^-(t),\ k=0,\\
&\frac{dP_k^+(t)}{dt} = -(\lambda^+ + k\mu^+)P_k^+(t) + (k+1)\mu^+ P_{k+1}^+(t) + (k+1)\mu^- P_{k+1}^-(t),\\
&\qquad\qquad 0<k<n,\\
&\frac{dP_k^+(t)}{dt} = -(\lambda^+ + n\mu^+)P_k^+(t) + n\mu^+ P_{k+1}^+(t) + n\mu^- P_{k+1}^-(t),\ k>n,
\end{aligned}
\tag{3}
$$

$$\frac{dP_1^-(t)}{dt} = -(\lambda^- + \mu^-)P_1^-(t) + \lambda^+ P_0^+(t),\ k = 1,$$

(4) $$\frac{dP_k^-(t)}{dt} = -(\lambda^- + k\mu^-)P_k^-(t) + \lambda^+ P_{k-1}^+(t) + \lambda^- P_{k-1}^-(t),\ 1 < k \leqslant n,$$

$$\frac{dP_k^-(t)}{dt} = -(\lambda^- + n\mu^-)P_k^-(t) + \lambda^+ P_{k-1}^+(t) + \lambda^- P_{k-1}^-(t),\ k > n,$$

in the case (ii):

$$\frac{dP_0^+(t)}{dt} = -\lambda^+ P_0^+(t) + \mu^+ P_1^+(t) + \mu^- P_1^-(t),\ k = 0,$$

$$\frac{dP_k^+(t)}{dt} = -(\lambda^+ + k\mu^+)P_k^+(t) + (k+1)\mu^+ P_{k+1}^+(t) + (k+1)\mu^- P_{k+1}^-(t),\ 0 < k < n,$$

(3') $$\frac{dP_k^+(t)}{dt} = -(\lambda^+ + n\mu^+)P_k^+(t) + n\mu^+ P_{k+1}^+(t) + n\mu^- P_{k+1}^-(t),\ n \leqslant k < m+n-1,$$

$$\frac{dP_{m+n-1}^+(t)}{dt} = -(\lambda^+ + n\mu^+)P_{m+n-1}^+(t) + n\mu^- P_{m+n}^-(t),$$

$$\frac{dP_1^-(t)}{dt} = -(\lambda^- + \mu^-)P_1^-(t) + \lambda^+ P_0^+(t),\ k = 1,$$

(4') $$\frac{dP_k^-(t)}{dt} = -(\lambda^- + k\mu^-)P_k^-(t) + \lambda^+ P_{k-1}^+(t) + \lambda^- P_{k-1}^-(t),\ 1 < k \leqslant n,$$

$$\frac{dP_k^-(t)}{dt} = -(\lambda^- + n\mu^-)P_k^-(t) + \lambda^+ P_{k-1}^+(t) + \lambda^- P_{k-1}^-(t),\ n < k \leqslant m+n.$$

Theorem. *If* $\frac{\lambda^+\lambda^-}{\mu^+\mu^-} < n^2$ *and* $m = \infty$ *(the case* (i)*), then the stationary probabilities are given by*

(5) $$P_k^+ = \frac{1}{k!}\left(\frac{\lambda^-}{\mu^+}\right)^k \Delta_{k-1};\ P_k^- = \frac{k\mu^+}{\lambda^-}P_k^+,\ 1 \leqslant k \leqslant n,$$

(6) $$P_k^+ = \frac{n^{n-k}}{n!}\left(\frac{\lambda^-}{\mu^+}\right)^k \left(\frac{\lambda^+ + n\mu^+}{\lambda^- + n\mu^-}\right)^{k-n} \Delta_{n-1};\ P_k^- = \frac{n\mu^+}{\lambda^-}P_k^+,\ k > n,$$

and

(7) $$P_0^+ = \left[1 + \sum_{k=1}^{n} \frac{1}{k!}\left(\frac{\lambda^-}{\mu^+}\right)^k \left(1 + \frac{k\mu^+}{\lambda^-}\right)\Delta_{k-1} + \frac{1}{n!}\left(\frac{\lambda^-}{\mu^+}\right)^n \left(1 + \frac{n\mu^+}{\lambda^-}\right)\Delta_{n-1}\frac{\varrho}{1-\varrho}\right]^{-1}$$

where

$$\varrho = \frac{\lambda^-}{n\mu^+}\left(\frac{\lambda^+ + n\mu^+}{\lambda^- + n\mu^-}\right); \ \Delta_k = \prod_{i=0}^{k} \frac{\lambda^+ + i\mu^+}{\lambda^- + (i+1)\mu^-} P_0^+.$$

If $m < \infty$ (the case (ii)), then

$$\text{(5')} \qquad P_k^+ = \frac{1}{k!}\left(\frac{\lambda^-}{\mu^+}\right)^k \Delta_{k-1}; \ P_k^- = \frac{k\mu^+}{\lambda^-} P_k^+, \ 1 \leqslant k \leqslant n,$$

$$\text{(6')} \qquad \begin{aligned} P_k^+ &= \frac{n^{n-k}}{n!}\left(\frac{\lambda^-}{\mu^+}\right)^k \left(\frac{\lambda^+ + n\mu^+}{\lambda^- + n\mu^-}\right)^{k-n} \Delta_{n-1}, \ n < k < m+n, \\ P_k^- &= \frac{n\mu^+}{\lambda^-} P_k^+, \ n < k < m+n; \ P_{m+n}^- = \frac{\lambda^+ + n\mu^+}{n\mu^-} P_{m+n-1}^+, \end{aligned}$$

where

$$\text{(7')} \qquad \begin{aligned} P_0^+ = \Bigg[1 + \sum_{k=1}^{n} \frac{1}{k!}\left(\frac{\lambda^-}{\mu^+}\right)^k \left(1 + \frac{k\mu^+}{\lambda^-}\right)\Delta_{k-1} + \\ \frac{1}{n!}\left(\frac{\lambda^-}{\mu^+}\right)^n \left(1 + \frac{n\mu^+}{\lambda^-}\right)\Delta_{n-1} \frac{1-\varrho^{n-1}}{1-\varrho} + \\ \frac{1}{n!}\left(\frac{\lambda^-}{\mu^+}\right)^n \left(\frac{\lambda^+ + n\mu^+}{n\mu^-}\right)\varrho^{m-1}\Delta_{n-1}\Bigg]^{-1}. \end{aligned}$$

Proof. In the stationary case the equations (3), (4), (3′) and (4′) take the following form — in the case (i):

$$\text{(8)} \qquad \begin{aligned} &-\lambda^+ P_0^+ + \mu^+ P_1^+ + \mu^- P_1^- = 0, \ k = 0, \\ &-(\lambda^+ + k\mu^+)P_k^+ + (k+1)\mu^+ P_{k+1}^+ + (k+1)\mu^- P_{k+1}^- = 0, \ 0 < k < n, \\ &-(\lambda^+ + n\mu^+)P_k^+ + n\mu^+ P_{k+1}^+ + n\mu^- P_{k+1}^- = 0, \ k \geqslant n; \end{aligned}$$

$$\text{(9)} \qquad \begin{aligned} &-(\lambda^- + \mu^-)P_1^- + \lambda^+ P_0^+ = 0, \ k = 1, \\ &-(\lambda^- + k\mu^-)P_k^- + \lambda^+ P_{k-1}^+ + \lambda^- P_{k-1}^- = 0, \ 1 < k \leqslant n, \\ &-(\lambda^- + n\mu^-)P_k^- + \lambda^+ P_{k-1}^+ + \lambda^- P_{k-1}^- = 0, \ k > n; \end{aligned}$$

in the case (ii):

$$\text{(8')} \qquad \begin{aligned} &-\lambda^+ P_0^+ + \mu^+ P_1^+ + \mu^- P_1^- = 0, \ k = 0, \\ &-(\lambda^+ + k\mu^+)P_k^+ + (k+1)\mu^+ P_{k+1}^+ + (k+1)\mu^- P_{k+1}^- = 0, \ 1 \leqslant k < n, \\ &-(\lambda^+ + n\mu^+)P_k^+ + n\mu^+ P_{k+1}^+ + n\mu^- P_{k+1}^- = 0, \ n \leqslant k < m+n-1, \\ &-(\lambda^+ + n\mu^+)P_{m+n-1}^+ + n\mu^- P_{m+n}^- = 0, \ k = m+n-1; \end{aligned}$$

$$-(\lambda^- + \mu^-)P_1^- + \lambda^+ P_0^+ = 0, \; k = 1,$$

$$(9') \quad -(\lambda^- + k\mu^-)P_k^- + \lambda^+ P_{k-1}^+ + \lambda^- P_{k-1}^- = 0, \; 1 < k \leqslant n,$$

$$-(\lambda^- + n\mu^-)P_k^- + \lambda^+ P_{k-1}^+ + \lambda^- P_{k-1}^- = 0, \; n < k < m+n.$$

One can prove, by induction and by simple but tedious evolutions that P_k^+ and P_k^- given by (5) and (6), satisfy the equations (8) and (9), and also that P_k^+ and P_k^- given by (5′) and (6′) satisfy (8′) and (9′).

REFERENCE

[1] Т. Аннаев, *О системе массового обслуживания типа* $(M/M/1)^{\pm}$. *Теория Вероят. и Математ.* Статист. 4 (1971), 27-35.

STRESZCZENIE

W pracy rozpatrujemy n – kanałowy system obsługi masowej z ograniczoną (liczbą m) i nieograniczoną kolejką. Czasy oczekiwania na kolejne zgłoszenia i długości obsługi są zmiennymi losowymi o rozkładzie wykładniczym ze zmienną intensywnością. Mianowicie jeśli ostatnią zmianą do momentu t w systemie była obsługa, to intensywność wejść i obsługi w momencie t jest λ^+ i μ^+ odpowiednio, jeśli natomiast ostatnią zmianą było zgłoszenie, to λ^- i μ^-.

W pracy tej wypisano układy równań różniczkowych opisujących prawdopodobieństwa stanu systemu w momencie t, a także rozwiązano je w przypadku stacjonarnym dla systemu pierwszego i drugiego typu.

РЕЗЮМЕ

В работе рассматриваются системы с n- каналами обслуживания с ограниченной (числом m) и неограниченной очередью. Промежутки между поступлениями и длительностью обслуживания предполагаются экспоненциально распределенными с переменной интенсивностью.

А именно, если последним изменением до момента t в системе было окончание обслуживания, то интенсивность входа и обслуживания в момент t равны λ^+ и μ^+ соответственно, если было поступление требования, то λ^- и μ^-.

В работе выписаны системы дифференциальных уравнений для вероятностей состояний системы в момент t, а также найдены стационарные распределения для обоих типов систем.