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On Properties of Certain Subclasses of Close-to-Convex Functions

O własnościach pewnych podklas funkcji prawie wypukłych

Об свойствах некоторых подклассов почти выпуклых функций

1. Introduction. In this paper we consider the following class of functions introduced by Sakaguchi [7].

Let the function $f(z)$ be analytic in $E(|z|<1)$, with the normalization $f(0) = 0 = f'(0)-1$. Then $f(z)$ is said to be starlike w.r.t. symmetric points in $|z|<1$ if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0 \quad \text{for } |z|<1 \tag{1.1}$$

i.e. the line segment $(f(z)-f(-z))$ turns continuously in one direction as z traverses each circle $|z| = r < 1$.

The class of such functions can be denoted by S_s^*. Obviously, it forms a subclass of close-to-convex functions and hence the functions there in are univalent [3]. Moreover, this class includes the class of convex functions and odd starlike functions w.r.t. the origin [5].

Złotkiewicz [9] considered a class G of normalized analytic functions in E, satisfying (1.1), where the function $(f(z)-f(-z))/2$ is replaced by an odd starlike function $\psi(z)$ in E and proved the following sharp distortion theorems for the class G:
If $f(z) \in G$, then for $|z| = r < 1$

$$(1+r)^{-2} \leqslant |f'(z)| \leqslant (1-r)^{-2} \tag{1.2}$$

$$r(1+r)^{-1} \leqslant |f(z)| \leqslant r(1-r)^{-1} \tag{1.3}$$

It is also known that $f(z) = \log\left(\dfrac{1+z}{\sqrt{1+z^2}}\right) \in G$.

For $z = r$, $|f(z)| = \log\left(\dfrac{1+r}{\sqrt{1+r^2}}\right)$.

By simple calculations, one can see that

$\log\left(\frac{1+r}{\sqrt{1+r^2}}\right) < \frac{r}{1+r}$, which contradicts (1.3).

In this direction, we prove sharp distortion theorems for the subclass S_s^* of G in Section 4.

The inspiring properties [2, 6, 7] of the functions of the class S_s^* lead us to define the order for such functions. By $S_s^*(a)$, we denote the class of functions $f(z) \in S_s^*$, having the additional property

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > a \text{ for } |z|<1,\ 0 \leqslant a < 1/2 \tag{1.4}$$

Here, of course, a is referred to as the order of starlike functions $f(z)$ w.r.t. symmetric points in $|z|<1$ and identify $S_s^*(0) = S_s^*$. We first determine the sharp r.c. for the class $S_s^*(a)$. It is interesting to observe the following:

Remark 1. If $f(z) \in S_s^*(a)$, $0 \leqslant a < 1/2$, then the odd function $\psi(z)$ defined by

$$\psi(z) = \big(f(z)-f(-z)\big)/2 \tag{1.5}$$

belongs to the class $S^*(2a)$ of starlike functions w.r.t. origin of order $2a$ and moreover, $\psi(z) \in S_s^*(a)$.

2. We need the following lemmas:

Lemma A (Singh & Bajpai). *Let*

$$H(z) = \frac{a}{1+z\varphi(z)} - \frac{1}{1+bz\varphi(z)} - \frac{(1-b)z^2\varphi'(z)}{(1+z\varphi(z))(1+bz\varphi(z))} \tag{2.1}$$

where $\varphi(z)$ is analytic and $|\varphi(z)| \leqslant 1$ in $|z|<1$, $-1 \leqslant b < 1$ and $a \geqslant 1$. Then for $|z| = r$, $0 \leqslant r < 1$,

$$\operatorname{Re}\big(H(z)\big) \leqslant \frac{(1-a)+(1-ab)r}{(1-r)(1-br)} \tag{2.2}$$

$$\operatorname{Re}\big(H(z)\big) \geqslant \begin{cases} -\dfrac{(a-1)+(ab-1)r}{(1+r)(1+br)} & \text{for } u_0 \leqslant u_1 \\[2ex] -\dfrac{(1+ab+2b)(1-r^2)+2(1-b)}{(1-b)(1-r^2)} + \\ \quad + \dfrac{2}{1-b}\left(\dfrac{(1+a)(1+b)(1-br^2)}{(1-r^2)}\right)^{1/2} & \text{for } u_0 \geqslant u_1 \end{cases} \tag{2.3}$$

where $u_0 = \dfrac{1}{1-b}\left(\left(\dfrac{(1+b)(1-br^2)}{(1+a)(1-r^2)}\right)^{1/2} - b\right)$ *and* $u_1 = \dfrac{1}{1+r}$.

Lemma 1. *Let a satisfy $0 \leqslant a < 1/2$ and $r(a)$ denote the smallest positive root of the equation, which is unique in $(2-\sqrt{3}, 1]$,*

$$(1-4a)r^3 - 3(1-4a)r^2 + 3r - 1 = 0 \tag{2.4}$$

If $f(z) \in S_s^*(a)$, then for $|z| = r$, $0 \leqslant r < 1$ we have

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \leqslant \frac{1+(3-8a)r+(3-8a)r^2+(4a-1)^2r^3}{(1-r^2)\big(1-(4a-1)r\big)} \tag{2.5}$$

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geqslant \begin{cases} \dfrac{1-2(1-4a)r-2(1-4a)r^2-2(1-8a^2)r^3+(1-4a)r^4}{(1+r)(1+r^2)\big(1+(4a-1)r\big)} & \text{for } 0 \leqslant r \leqslant r(a) \\[2ex] \dfrac{1+(4a-1)r^2}{1+r^2} + \dfrac{1}{1-2a}\big((8aA)^{1/2} - 2a - A\big) & \text{for } r(a) \leqslant r < 1 \end{cases} \tag{2.6}$$

where $A = \dfrac{1+(1-4a)r^2}{1+r^2}$.

The extremal function is of the form

$$f(z) = \int_0^z \frac{1+(4a-1)t}{(1+t)(1+t^2)^{1-2a}}\,dt \quad \text{when } 0 \leqslant r \leqslant r(a) \tag{2.7}$$

and, is otherwise of the form

$$f(z) = \int_0^z \frac{1-4bat+(4a-1)t^2}{(1-2bt+t^2)(1+t^2)^{1-2a}}\,dt \tag{2.8}$$

where b is determined from

$$\frac{1-4bar_0+(4a-1)r_0^2}{(1-2br_0+r_0^2)} = (2aA)^{1/2} \equiv R_0 \tag{2.9}$$

and $r_0 = \dfrac{1}{1-2a}\big((8aA)^{1/2} - 2a - A\big)$.

Proof. Since $f(z) \in S_s^*(a)$, we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > a, \ |z| < 1, \ 0 \leqslant a < 1/2.$$

Consequently,

$$\frac{2zf'(z)}{f(z)-f(-z)} = \frac{1+(4a-1)z\varphi(z)}{1+z\varphi(z)} \tag{2.10}$$

where $\varphi(z)$ satisfies Schwarz's lemma.

Logarithmic differentiation and simplification yield

$$1+\frac{zf''(z)}{f'(z)} = \frac{z\psi'(z)}{\psi(z)} + \frac{1}{1+z\varphi(z)} - \frac{1}{\big(1+(4a-1)z\varphi(z)\big)} - \\ - \frac{2(1-a)z^2\varphi'(z)}{\big(1+z\varphi(z)\big)\big(1+(4a-1)z\varphi(z)\big)} \tag{2.11}$$

where $\psi(z) = \frac{1}{2}\big(f(z)-f(-z)\big) \in S^*(2a)$.

Also,

$$\operatorname{Re}\left(\frac{z\psi'(z)}{\psi(z)}\right) \geqslant \frac{1+(4a-1)r^2}{1+r^2}. \tag{2.12}$$

Now using (2.12) and Lemma A with $a = 1$, $b = 4a-1 < 1$ in (2.11), we have finally

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geqslant \begin{cases} \dfrac{1-2(1-4a)r-2(1-4a)r^2-2(1-8a^2)r^3+(1-4a)^2r^4}{(1+r)(1+r^2)\big(1+(4a-1)r\big)} & \text{for } u_0 \leqslant u_1 \\[2ex] \dfrac{1+(4a-1)r^2}{1+r^2} + \dfrac{1}{1-2a}\left(\sqrt{8aA}-2a-A\right) & \text{for } u_0 \geqslant u_1 \end{cases} \tag{2.13}$$

where $u_0 = \frac{1}{2(1-2a)}\big((2aA)^{1/2}+(1-4a)\big)$ and $u_1 = \frac{1}{1+r}$.

The two inequalities of (2.13) become equal for such values of a, for which $u_0 = u_1$

i.e. $$\left(\frac{1-(1-4a)r}{1+r}\right)^2 = \frac{2a\big(1+(1-4a)r^2\big)}{1-r}$$

i.e. $$g(a,r) \equiv (1-4a)r^3-3(1-4a)r^2+3r-1 = 0$$

$g(a,r)$ is a strictly increasing function of r, $0 \leqslant r < 1$, for each a, $0 \leqslant a < 1/2$

$$g(a, 2-\sqrt{3}) = 2(1-2a)(5-3\sqrt{3}) < 0$$

$$g(a,1) = 8a \geqslant 0.$$

Thus $g(a,r)$ has a unique root $r(a)$ in $(2-\sqrt{3}, 1]$. The proof is now complete.

3. Radius of Convexity for the class $S_s^*(a)$

Theorem 1. *Let $f(z) \in S_s^*(a)$, $0 \leqslant a < 1/2$ and $r(a)$ the root, unique in $(2-\sqrt{3}, 1]$ of the equation (2.3). Then $f(z)$ is convex of order β, $0 \leqslant \beta < 1$ for $|z| < r_0$, where r_0 is the smallest positive root of the equation*

$$(1-\beta)-2(1-4a+2a\beta)r-2(1-4a+2a\beta)r^2-2(1-8a^2+2a\beta)r^3+ \\ +(1-4a)(1-4a+\beta)r^4 = 0 \qquad (3.1)$$

if $0 \leqslant r_0 \leqslant r(a)$
and, of the equation

$$(1-2a)\left(1+(4a-1)r^2\right)+(1+r^2)\left(\sqrt{8aA}-2a-A-\beta(1-2a)\right) = 0 \qquad (3.2)$$

if $r(a) \leqslant r_0 < 1$

$$\textit{where } A = \frac{1+(1-4a)r^2}{1-r^2}.$$

This result is sharp.

Proof. Since $f(z) \in S_s^*(a)$, we see from Lemma 1 that

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}-\beta\right) \geqslant \frac{1-2(1-4a)r-2(1-4a)r^2-2(1-8a^2)r^3+(1-4a)^2r^4}{(1+r)(1+r^2)\left(1+(4a-1)r\right)}-\beta$$

$$= \frac{(1-\beta)-2(1-4a+2a\beta)r-2(1-4a+2a\beta)r^2 -2(1-8a^2+2a\beta)r^3+(1-4a)(1-4a+\beta)r^4}{(1+r)(1+r^2)\left(1+(4a-1)r\right)} \qquad (3.3)$$

if $0 \leqslant r \leqslant r(a)$, and

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}-\beta\right) \geqslant \frac{1+(4a-1)r^2}{1+r^2}+\frac{1}{1-2a}\left(\sqrt{8aA}-2a-A-\beta(1-2a)\right)$$

if $r(a) \leqslant r < 1$ (3.4)

Therefore, $\mathrm{Re}\left(1+\dfrac{zf''(z)}{f'(z)}-\beta\right) \geqslant 0$ if

$$(1-\beta)-2(1-4a+2a\beta)r-2(1-4a+2a\beta)r^2-2(1-8a^2+2a\beta)r^3+ \\ +(1-4a)(1-4a\beta+\beta)r^4 \geqslant 0 \qquad (3.5)$$

and

$$(1-2a)\left(1+(4a-1)r^2\right)+(1+r^2)\left(\sqrt{8aA}-2a-A-\beta(1-2a)\right) \geqslant 0 \qquad (3.6)$$

(3.5) is valid only when $0 \leqslant r \leqslant r_0 \leqslant r(a)$ and (3.6) is valid only when $r(a) \leqslant r \leqslant r_0 < 1$.

The equality sign in (3.3) is attained for the function given by (2.7) and that in (3.4) for the function determined by (2.8) and (2.9). By taking $\alpha=\beta=0$ in the theorem above, we arrive at the following: —

Corollary 1. *If $f(z)\in S_s^*(0)$, then $f(z)$ is convex in $|z|<r_0$, where $r_0=\frac{1}{2}\left((1+\sqrt{5})-\sqrt{2(1+\sqrt{5})}\right)$. The function $f(z)=\log\left(\frac{1+z}{\sqrt{1+z^2}}\right)$ shows that this value is best possible.*

The above Corollary can be compared with the corresponding result of Zlotkiewicz [9].

Remark 2. We can replace the condition (1.4) by

$$\operatorname{Re}\left(\frac{zf'(z)}{\varphi(z)}\right)>\alpha,\ |z|<1,\ 0\leqslant\alpha<1 \tag{3.7}$$

where $\varphi(z)$ is an odd starlike function of order β, $0\leqslant\beta<1$ there and then apply Lemma A to determine the r.c. as usual. We recall that the sharp r.c. for the class of close-to-convex functions of order α and type β [4] has been recently found out by Silverman [8], as an application of a theorem of Zmorovič [10]. But, Lemma A helps us to look into similar type of problem with a different angle and to have simple and shorter proofs.

4. The following **distortion theorems** can be obtained for the class $S_s^*(\alpha)$.

Theorem 2. *If $f(z)\in S_s^*(\alpha)$, $0\leqslant\alpha<1/2$, then for $|z|=r$, $0\leqslant r<1$, we have*

$$\text{(A)}\qquad \frac{1+(4\alpha-1)r}{(1+r)(1+r^2)^{1-2\alpha}}\leqslant|f'(z)|\leqslant\frac{1-(4\alpha-1)r}{(1-r)(1-r^2)^{1-2\alpha}} \tag{4.1}$$

$$\text{(B)}\qquad \int_0^r\frac{1+(4\alpha-1)t}{(1+t)(1+t^2)^{1-2\alpha}}\,dt\leqslant|f(z)|\leqslant\int_0^r\frac{1-(4\alpha-1)t}{(1-t)(1-t^2)^{1-2\alpha}}\,dt \tag{4.2}$$

The extremal function corresponding to the left and right side inequalities are attained respectively for

$$f(z)=\int_0^z\frac{1+(4\alpha-1)t}{(1+t)(1+t^2)^{1-2\alpha}}\,dt \tag{4.3}$$

$$f(z)=\int_0^z\frac{1-(4\alpha-1)t}{(1-t)(1-t^2)^{1-2\alpha}}\,dt \tag{4.4}$$

Corollary 1. *If $f(z)\in S_s^*(0)$, then for $|z|=r$, $0<r<1$*

(A) $$\frac{1-r}{(1+r)(1+r^2)}\leqslant|f'(z)|\leqslant\frac{1}{(1-r)^2} \tag{4.5}$$

(B) $$\log\frac{1+r}{\sqrt{1+r^2}}\leqslant|f(z)|\leqslant\frac{r}{1-r} \tag{4.6}$$

The equality sign in left and right hand inequalities respectively are attained for the functions

$$f(z)=\log\frac{1+z}{\sqrt{1+z^2}} \tag{4.7}$$

$$f(z)=\frac{z}{1-z}. \tag{4.8}$$

Corollary 2. *The disc $|\omega|<\frac{1}{2}\log 2$ is always covered by the map of $|z|<1$ of any function $\omega=f(z)$ belonging to $S_s^*(0)$. The result is sharp* i.e. *the constant $\frac{1}{2}\log 2$ cannot be replaced by any larger number, as the extremal function (4.7) shows. This Corollary can be expressed as "$\frac{1}{2}\log 2$ – Theorem".*

Proof. We demonstrate that the proof of (4.1) is an easy consequence of the following aspects:

(i) $f(z)\in S_s^*(a)$ implies that $\mathrm{Re}\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)>2a$, $|z|<1$, $0\leqslant a<1/2$ and the function $\varphi(z)=\frac{1}{2}\big(f(z)-f(-z)\big)\in S^*(2a)$.

(ii) The sharp bounds for $|\varphi(z)|$, where $\varphi(z)$ is an odd starlike function of order $2a$, $0\leqslant a<1/2$, as follows:

$$\frac{r}{(1+r^2)^{1-2a}}\leqslant|\varphi(z)|\leqslant\frac{r}{(1-r^2)^{1-2a}};\ |z|=r,\ 0<r<1$$

(iii) If $p(z)$ is an analytic function in $|z|<1$, with $p(0)=1$, that satisfies $\mathrm{Re}\,p(z)>2a$ there, then the domain of values of $p(z)$ is the circle with the line segment from $\frac{1+(4a-1)|z|}{1+|z|}$ to $\frac{1-(4a-1)|z|}{1-|z|}$ as a diameter $(0\leqslant a<1/2)$.

The other parts follow in the usual manner.

We note that the Corollary 1 can also be obtained from (2.5) and (2.6) on using classical approach.

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STRESZCZENIE

W pracy tej wyznaczono dokładną wartość promienia wypukłości oraz podano twierdzenia o zniekształceniu dla funkcji gwiaździstych względem punktów symetrycznych rzędu a, które stanowią podklasę funkcji prawie wypukłych.

РЕЗЮМЕ

В этой работе получено точную оценку радиуса выпуклости а также теоремы об искажению для звездообразных функций относительно симметрических точек порядка a, которые являются подклассом почти выпуклых функций.