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Analytical Treatment of Isometries of Hyperbolic Space

Analityczne podejście do izometrii przestrzeni hiperbolicznej

Аналитический подход к изометриям гиперболического пространства

A model of hyperbolic Lobachevskian stereometry which is a direct generalization of a model of H. Poincaré was been presented in the paper [2]. A group of motions of the metric under investigation is just a group of complex homographies. A basic Riemannian space is $\{x \in R^3, x^3 > 0\}$. A hyperbolic plane is either a hemisphere which has a center at the boundary $x^3 = 0$, or a vertical half-plane $Ax^1 + Bx^2 = 0$. A hyperbolic straight line is an intersection of two such planes. In this paper we prove some theorems on motions and on their invariants. Then we investigate the generating of hyperbolic isometries by symmetries.

Let C and H denote respectively a field of complex numbers and that of quaternions. Let c denote a multiplicative group of those $z \in C$ for which $|z| = 1$. Thus c acts on H by the rule.

$$(z, h) \mapsto z^{-1} h z \tag{1}$$

Denote by N the orbit space with respect to the above action. There is proved in [2], that N is homeomorphic to a closure of the manifold of circles in Euclidean plane. In fact, if we present any quaternion h in the form $h = h' + h''j$, where $h', h'' \in C$ and j is the "third unity" in H, then we have

$$e^{-ia}(h' + h''j)e^{ia} = h' \quad (h'' e^{2ia})j.$$

We see that the orbit of h may be identified with an Euclidean circle with a center $(\mathrm{re}\, h', \mathrm{im}\, h')$ and with a radius $|h''|$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a complex non-singular matrix. The group L of such matrices acts on H in the following manners

(2) $$(A, h) \mapsto (ah+b)(ch+d)^{-1},$$

or

(3) $$(A, h) \mapsto (a\bar{h}+b)(c\bar{h}+d)^{-1}.$$

Observe that transformation (3) is a product of a homography (2) and the action $h \to \bar{h}$.

For any $e^{ia} \in C$ we have

$$\big(a(e^{-ia}he^{ia})+b\big)\big(c(e^{-ia}he^{ia})+d\big)^{-1} = e^{-ia}(ah+b)(ch+d)^{-1}e^{ia}$$

Thus we conclude that the transformations (2) and (3) induces some associative left action of L on N, [2].

Denote by A^* a transformation which corresponds to the matrix A, and by h^* an orbit of h by the action (1). Then we define x^1, x^2, x^3, (resp. $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3$) by the decomposition $h = h' + h''j$, and $x^1 = \operatorname{re} h'$, $x^2 = \operatorname{im} h'$, $x^3 = |h''|$ (resp. $A^*h = \tilde{h}' + \tilde{h}''j$ and $\tilde{x}^1 = \operatorname{re}\tilde{h}'$, $\tilde{x}^2 = \operatorname{im}\tilde{h}'$, $\tilde{x}^3 = |\tilde{h}''|$).

Proposition 1. *The result of (2) may be written in local coordinates as follows:*

$$A^* h^* = [\tilde{x}^1, \tilde{x}^2, \tilde{x}^3]$$

where

$$\tilde{x}^1 = \frac{1}{m}\Big(\operatorname{re}(a\bar{c})\big((x^1)^2+(x^2)^2+(x^3)^2\big) + \operatorname{re}(a\bar{d}+b\bar{c})x^1 + \\ + \operatorname{re}\big((a\bar{d}-b\bar{c})i\big)x^2 + \operatorname{re}(b\bar{d})\Big),$$

$$\tilde{x}^2 = \frac{1}{m}\Big(\operatorname{im}(a\bar{c})\big((x^1)^2+(x^2)^2+(x^3)^2\big) + \operatorname{im}(a\bar{d}+b\bar{c})x^1 + \\ + \operatorname{im}\big((a\bar{d}-b\bar{c})i\big)x^2 + \operatorname{im}(b\bar{d})\Big),$$

$$\tilde{x}^3 = \frac{1}{m}|ad-bc|x^3,$$

$$m = |c|^2\big((x^1)^2+(x^2)^2+(x^3)^2\big) + (c\bar{d}+\bar{c}d)x^1 + \\ + \big((c\bar{d}-\bar{c}d)i\big)x^2 + |d|^2.$$

A similar result holds for (3). Proof by a direct computing. The following proposition may be also proved by a direct computing.

Proposition 2. *A Jacobian determinant of the above considered mapping*

$$[x^1, x^2, x^3] \underset{A^*}{\mapsto} [\tilde{x}^1, \tilde{x}^2, \tilde{x}^3]$$

is equal to $(|ad-bc|/m)^3$.

Proposition 3. *The above mapping A^* is an isometry of the space $L = \operatorname{int} N$ which is provided with the following Riemanian metric*

$$ds^2|_{(x^1,x^2,x^3)} = \left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right)(\varrho/x^3)^2$$

where ϱ is some positive constant.

Proof: Let $t \mapsto [v^1(t), v^2(t), v^3(t)]$ be a parametrisation of a curve in L. A^* sends this curve into another one, $t \mapsto [\bar{v}^1(t), \bar{v}^2(t), \bar{v}^3(t)]$ where the components $\bar{v}^1, \bar{v}^2, \bar{v}^3$ may be computed be means of proposition 1. Then the length of the curve described by $\bar{v}$ is equal to

$$\int_{t_1}^{t_2} \sqrt{\left((\bar{v}^{1'}(t))^2 + (\bar{v}^{2'}(t))^2 + (\bar{v}^{3'}(t)\right)^2 (\varrho/\bar{v}^3(t))^2}\, dt$$

$$= \int_{t_1}^{t_2} \sqrt{m^{-4}(m\,|ad - bc|)^2\left((v^{1'}(t))^2 + (v^{2'}(t))^2 + (v^{3'}(t))^2\right)\frac{\varrho^2 m^2}{|ad - bc|^2 (v^3(t))^2}}\, dt$$

$$= \int_{t_1}^{t_2} \sqrt{\left((v^{1'}(t))^2 + (v^{2'}(t))^2 + (v^{3'}(t))^2\right)(\varrho/v^3(t))^2}\, dt.$$

Thus the right hand member yields the length of the origin curve, q. e. d. The transformation (3) is also an isometry as a product (2) and the isometry $h \mapsto \bar{h}$.

The following theorem is a corollary from the above propositions.

Theorem 4. *There exists a homomorphism of the group of complex homographies into the group of isometries of 3-dimensional hyperbolic space.*

In what follows we shall deal with those isometries which are performed by complex homographies. We shall call them L-isometries. We begin with the following

Theorem 5. *Each L-isometry sends hyperbolic planes to such planes.*

Proof: A general form of an equation of a hyperbolic planes is

$$k\left((x^1)^2 + (x^2)^2 + (x^3)^2\right) + 2mx^1 + 2nx^2 + p = 0 \tag{4}$$

where $k \geq 0$. If we replace x^1, x^2, x^3 respectively by $\bar{x}^1, \bar{x}^2, \bar{x}^3$ which are computed by means of Proposition 1. Then we obtain

$$\bar{k}\left((\bar{x}^1)^2 + (\bar{x}^2)^2 + (\bar{x}^3)^2\right) + 2\bar{m}\bar{x}^1 + 2\bar{n}\bar{x}^2 + \bar{p} = 0 \tag{5}$$

where

$$\tilde{k} = k|a|^2 + 2m\,\mathrm{re}(a\bar{c}) + 2n\,\mathrm{im}(a\bar{c}) + p|c|^2$$

$$\tilde{m} = \tfrac{1}{2}k(a\bar{b} + \bar{a}b) + m\cdot\mathrm{re}(a\bar{d} + b\bar{c}) + n\cdot\mathrm{im}(a\bar{d} + b\bar{c}) + \\ + \tfrac{1}{2}p(c\bar{d} + \bar{c}d),$$

(6) $$\tilde{n} = \tfrac{1}{2}k(a\bar{b} - \bar{a}b)i + m\cdot\mathrm{re}((a\bar{d} - b\bar{c})i) + n\cdot\mathrm{im}((a\bar{d} - b\bar{c})i) \\ + \tfrac{1}{2}p(cd - \bar{c}d)i,$$

$$\tilde{p} = k|b|^2 + 2m\cdot\mathrm{re}(b\bar{d}) + 2n\cdot\mathrm{im}(b\bar{d}) + p|d|^2.$$

If $\tilde{k} = 0$ then we have the hyperbolic plane which is represented by a vertical half-plane. If $\tilde{k} \neq 0$ then we have to show that the equation (5) is an equation of a hemisphere, because that we have

$$\tilde{m}^2 + \tilde{n}^2 > \tilde{p}\tilde{k}$$

This follows by a direct evaluation.

Since each straight line is an intersection of two planes, we have

Corollary 6. *Each Ł-isometry sends straight lines to straight lines.*

It is proved in [2] that if there are given the two distinct points a and b in Ł then there exists an Ł-isometry which sends them respectively to points with coordinates $(0, 0, 1)$ and $(0, 0, c)$, where $c > 1$. We prove some stronger theorem now.

Theorem 7. *Let π be a hyperbolic plane, k be a straight line $\subset \pi$ and Q be a point on k. Thus there exist at least four Ł-isometries which send π to a plane $x^2 = 0$ so that k is sent to a line $x^1 = x^2 = 0$ and Q is sent to* $(0, 0, 1)$.

Proof: We make use from the known fact, that for every circle on the Z-plane there exist homographies which send this circle to an "unproper" circle $x^2 = 0$. We apply this theorem to a case where the circle is a set of improper points of Ł-plane. Let us chose any such homography, h_0. Then all other ones are of the form $h_1 \circ h_0$ where either $h_1\colon z \mapsto (az + b)(cz + d)^{-1}$ or $h_1\colon z \mapsto (a\bar{z} + b)(c\bar{z} + d)^{-1}$. Now we check $\begin{bmatrix} a, & b \\ c, & d \end{bmatrix}$ so that h_1 sends the improper points of $h_0(k)$ to 0 and to ∞ respectively and that $h_0(h_1(Q)) = i$. This is also performable in two ways, [1]. In view of an isomorphism of the group of hyperbolic motions with the group of homographies our theorem is valid.

By an analogy with Euclidean geometry we distinguish here some subgroups of isometries. The following theorem may be proved by simple calculations.

Theorem 8. *A hyperbolic isometry which sends the two distinct points to themselves is an identity mapping on the straight line joining these points.*

An analogical theorem is valid for a triple of points and a corresponding plane.

By an analogy to Euclidean geometry we shall distinguish here some special subgroups of isometries. The following theorem may be proved by a direct calculation.

Theorem 9. *If some isometry sends some pair of distinct points to itself then this izometry is an identity mapping of the straight line which contains these points.*

An analogical theorem is valid for a triple of points which determines a unique hyperbolic plane.

Definition 10. *A homography of the form* (2) *which leaves fixed a given point P is called a rotation around P.*

The following theorem may be proved a by direct calculation

Theorem 11. *Let $P = (x_0^1, x_0^2, x_0^3)$, where $x_0^1 = \operatorname{re} p$, $x_0^2 = \operatorname{im} p$, $x_0^3 = |q| = |r|$. Then a rotation around P is represented by a homography with a matrix*

$$\begin{bmatrix} cp + \bar{c}\bar{p}\dfrac{\bar{q}}{\bar{r}} + \bar{d}\dfrac{\bar{q}}{\bar{r}}, & -\bar{c}\dfrac{\bar{q}}{\bar{r}}(|p|^2 + |q|^2) + dp - \bar{d}p\dfrac{\bar{q}}{\bar{r}} \\ c \quad , & d \end{bmatrix}$$

where $|cd\bar{p} - c\bar{d}p + |d|^2 + |c|^2(|p|^2 + |q|^2)| = 1$

An equation

$$\left(cp + \bar{c}\bar{p}\frac{\bar{q}}{\bar{r}} + \bar{d}\frac{\bar{q}}{\bar{r}}\right)x - \bar{c}\frac{\bar{q}}{\bar{r}}(|p|^2 + |q|^2) + dp - \bar{d}p\frac{\bar{q}}{\bar{r}} = x(cx + d) \tag{8}$$

has either one or two complex solutions. That means that a rotation around P may have someone fixed point except P. In view of theorem 8 we conclude the following

Corollary 12. *If a determinant of the equation* (8) *is $\neq 0$, then there exist two distinct points of the rotation in question. Thus the unique straight line through these points may be considered as an axis of the rotation.*

An axial symmetry is a special rotation in the hyperbolic 3-space.

Theorem 13. *Each axial symmetry is represented by a convenable homography. If its matrix is $\begin{bmatrix} -1, 2p \\ 0, 1 \end{bmatrix}$ then the axis is a straight line* *which is represented in N by equations* $x^1 = \operatorname{re} p$, $x^2 = \operatorname{im} p$. *If the corresponding matrix is* $\begin{bmatrix} \frac{m+n}{m-n}, & \frac{-2mn}{m-n} \\ \frac{2}{m-n} & \frac{-(m+n)}{m-n} \end{bmatrix}$ *then the corresponding axis is represended in N by a semicircle which touches the boundary of N in points* $(\operatorname{re} m, \operatorname{im} m, 0)$ *and* $(\operatorname{re} n, \operatorname{im} n, 0)$.

Definition 14. If the stable point P is unproper then the corresponding homography will be called a translation. A central symmetry is — by a definition — such an isometry which sends each vector v at the stable point P to the vector $-v$.

Theorem 15. *Each isometry represented by mappings* (3) *with matrices of the form*

$$\begin{bmatrix} p, & -|p|^2 - |q|^2 \\ 1, & -\overline{p} \end{bmatrix}$$

is a central symmetry with respect to the point $(\operatorname{re} p, \operatorname{im} p, |q|)$.

Let us investigate the isometries (3) in details.

Definition 16. An isometry which sends some prefixed plane to itself and sends each vector at π to the opposite one will be called a plane symmetry.

Theorem 17. *A mapping*

$$h \mapsto \frac{q-p}{\overline{q}-\overline{p}}\,\overline{h} + \frac{p\overline{q}-\overline{p}q}{\overline{q}-\overline{p}} \tag{9}$$

represents a symmetry with respect to the plane, which map in N is a half-plane touching the improper points $(\operatorname{re} p, \operatorname{im} p, 0)$ *and* $(\operatorname{re} q, \operatorname{im} q, 0)$. *Thus the mapping*

$$h \mapsto \left(\frac{m}{r}\,\overline{h} + r - \frac{|m|^2}{r}\right)\left(\frac{1}{r}\,\overline{h} - \frac{\overline{p}}{r}\right)^{-1} \tag{10}$$

is a plane symmetry with respect to a plane which map in N is a hemisphere with a center $(\operatorname{re} m, \operatorname{im} m, 0)$ *and with its radius* r.

A proof is easy and purely computational.

A composition of the two symmetries of the form (9) is an isometry represented by a homography

$$h \mapsto \frac{q'-p'}{\overline{q}'-\overline{p}'}\cdot\frac{\overline{q}-\overline{p}}{q-p}\,h + \frac{(q'-p')(\overline{p}q - p\overline{q}) + (q-p)(p'\overline{q}' - \overline{p}'q')}{(\overline{q}'-\overline{p}')(q-p)} \tag{11}$$

and we see that the determinant of the corresponding matrix, A, is equal to $\frac{q'-p'}{\overline{q}'-\overline{p}'}\cdot\frac{\overline{q}-\overline{p}}{q-p}$. Thus $|\det A|=1$.

A main special case is if $\frac{q'-p'}{\overline{q}'-\overline{p}'}=\frac{q-p}{\overline{q}-\overline{p}}$, i.e. the both planes are parallel. Thus the composition takes the form $h\mapsto h+c$ where c depends on p, q, p', q'. The corresponding isometry may be viewed as a translation. In other cases we obtain a rotation the axis of which is a straight line common for both planes.

If we compose two planar symmetries of the form (10) then we obtain the mapping

$$(12)\qquad h\mapsto\left(\frac{1}{r'r}\left(m'\overline{m}+(r')^2-|m'|^2\right)h+\frac{m'}{r'r}\left(r^2-|m|^2\right)-\right.$$

$$\left.-\frac{m}{r'r}\left((r')^2-|m'|^2\right)\right)\cdot\left(\frac{1}{r'r}\left(\overline{m}-\overline{m}'\right)h+\frac{1}{r'r}\left(\overline{m}'m+r^2-|m|^2\right)\right)^{-1}$$

so that the corresponding determinant is equal to 1. (12) is an identity if the both planes coincide. If they have not any common line in proper then the composition of the symmetries yields some translation.

If we perform a composition of the two mappings, one being of the form (9) and the other of the form (10) then we obtain a homography

$$(13)\qquad h\mapsto\left(\frac{m'}{r'}\frac{\overline{q}-\overline{p}}{q-p}h+\frac{m'(\overline{p}q-p\overline{q})}{r'(q-p)}+r'-\frac{|m'|^2}{r'}\right)\cdot$$

$$\cdot\left(\frac{\overline{q}-\overline{p}}{r'(q-p)}h+\frac{\overline{p}q-p\overline{q}}{r'(q-p)}-\frac{\overline{m}'}{r'}\right)^{-1}$$

Here also we have $|\det A|=1$. (13) represents either a rotation or a translation.

Corollary 18. *A composition of any two planar symmetries is either the identity, either a rotation or a translation. In particular, a composition of the two planar symmetries with respect to planes which are perpendicular one to another is an axial symmetry.*

Theorem 19. *A composition of three symmetries with respect to the three planes which are perpendicular one to another and intersect in one point P is a symmetry with respect to P.*

Proof: We shall prove the theorem in a case when the two planes are represented by half-planes in N and the third one is represented by a hemisphere. (The proofs of other cases are analogous). In the just case the planar symmetries are of the form

$$h \mapsto \frac{q-p}{\bar{q}-\bar{p}}\,\bar{h} + \frac{p\bar{q}-\bar{p}q}{\bar{q}-\bar{p}},$$

$$h \mapsto -\frac{q-p}{\bar{q}-\bar{p}}\,\bar{h} + \frac{p(\bar{q}-\bar{p})+\bar{p}(q-p)}{\bar{q}-\bar{p}}$$

and

$$h \mapsto (p\bar{h} + |q|^2 - |p|^2)(\bar{h}-\bar{p})^{-1}.$$

The point of intersection is $P = (\mathrm{re}\,p,\ \mathrm{im}\,p,\ |q|)$. A composition of these symmetries yields

$$h \mapsto (p\bar{h} - (|p|^2 + |q|^2))(\bar{h}-\bar{p})^{-1}.$$

In view of theorem 14 this represents a central symmetry.

The following theorem follows by the above deduced formulas of compositions of symmetries.

Theorem 20. *Every hyperbolic isometry may be obtained by a composition of some planar symmetries. In other words the planar symmetries generate the group of hyperbolic isometries.*

Theorem 21. *If an isometry j is a result of a composition of an even* (resp. odd) *sequence of planar symmetries then there does not exist any odd (resp. even) sequence of planar symmetries, such that j is a result of their composition.*

Proof: We have seen that a result of an even (resp. odd) composition of planar symmetries is represented by a homography of the form (2) (resp. by a mapping like (3)). There does not exist an isometry except a planar symmetry $h \mapsto \bar{h}$ which transforms a mapping of the form (2) to a mapping of the form (3), and this ends the proof.

The above theorem gives a reason to distinguish even and odd isometries. There hold

Proposition 22: *The set of even isometries is a subgroup in the group of isometries.*

Proposition 23. *Each rotation around a point P is an even isometry for which P is a stable point. The set of rotations around a point P forms a subgroup in the subgroup of isometries for which P is stable.*

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STRESZCZENIE

W pracy podano szereg fundamentalnych twierdzeń o grupie izometrii trójwymiarowej przestrzeni Łobaczewskiego, wychodząc od uogólnionego modelu Poincaré'go. W modelu tym rozważana grupa wyraża się przez zespolone homografie pierwszego i drugiego rodzaju, zachowujące lub zmieniające orientację. Wyznaczono różne podgrupy i zbadano ich tranzytywność.

РЕЗЮМЕ

В работе дается ряд фундаментальных теорем о группе изометрии трехмерного пространства Лобачевского, выходя из обобщенной модели Поинкарэ. В этой модели изучается группа выраженная комплексными дробно линейными отображениями первого и второго рода, которые соблюдают или изменяют ориентацию. Определены разные подгруппы и исследовалась их транзитивность.