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A Note on Extremal Properties for Certain Family of Convex Mappings

Własności ekstremalne pewnej rodziny odwzorowań wypukłych

Экстремальные качества некоторого семейства выпуклых отображений

1. Introduction

Let S^c denote the class of functions $f(z)$ regular and univalent in the unit disk $K_1(K_r = \{z: |z| < r\})$, with $f(0) = 0$, $f'(0) = 1$ and such that the image domain $f(K_1)$ under every f belonging to S^c is convex.

It is well-known that $K_{1/2} \subset f(K_1)$ for any $f \in S^c$. However, if $f(z) \neq z(1-\varepsilon z)^{-1}$, $|\varepsilon| = 1$, there exists R, $1/2 < R < 1$ such that

$$K_R \subset f(K_1). \tag{1.1}$$

Thus, the studying of the class $S^c(R)$, $1/2 < R < 1$, of all functions $f \in S^c$ which satisfy the condition (1.1) seems to be interesting.

In the paper [2] J. Krzyż investigated the class $C(M)$, $M > 1$ of all functions $f(z)$ belonging to S^c and satisfying

$$f(K_1) \subset K_M. \tag{1.2}$$

He determined precise bounds for

$$|f(z)|;\ (1-|z|^2)|f'(z)|;\ |a_2| = \tfrac{1}{2}|f''(0)|\big(f \in C(M)\big), \tag{1.3}$$

using the Hadamard's variational method.

Let $\delta(M)$ denote the so-called Koebe constant for the class $C(M)$, and let $C(R, M)$; $\delta(M) < R < 1 < M < \infty$ denote the subclass of $C(M)$ of all functions $f(z)$ satisfying

$$K_R \subset f(K_1) \subset K_M. \tag{1.4}$$

Adopting the method used in [2] we can also find the estimates of the functionals (1.3) in the class $C(R, M)$.

Hence, we shall also obtain the bounds of the functionals (1.3) in the class $S^c(R) = \bigcup_{M>1} C(R, M)$.

This problem was investigated in my Ph. D. thesis submitted to the Faculty of Mathematics, Physics and Chemistry of Maria Curie-Skłodowska University in Lublin.

2. Main result

Let us put

$$G(z) = \frac{4}{(2+a)^a} \frac{[a(1+z)+2\lambda(z)]^a}{[1+z+\lambda(z)]^2} \frac{z}{(1-z)^a}, \tag{2.1}$$

where $\lambda(z) = [1+(a^2-2)z+z^2]^{1/2}$, $0 < a < 2$,

$$F(z) = \int_0^z \frac{G(\xi)}{\xi} d\xi. \tag{2.2}$$

The function G maps the unit disk K_1 onto a starshaped domain being the union of the disk K_ϱ, where

$$\varrho = 4[(2-a)^{2-a}(2+a)^{2+a}]^{-1/2} \tag{2.3}$$

and the angle $\{w\colon |\operatorname{Arg} w| < a^{\pi/2}\}$, (see [4]).

Examining the behaviour of the boundary of $G(K_1)$ under the transformation (2.2) we find that $F(K_1)$ is a convex circular domain D_F symmetric w.r.t. the real axis whose boundary consists of an arc situated on the boundary of the disk K_R, where

$$R = \int_{-1}^{0} \frac{G(t)}{t} dt \tag{2.4}$$

and two half a lines (or segments) starting from the end points of that arc and tangent to the boundary of K_R.

After some calculations we get the following formula:

$$R = R(a) = 16a \int_q^1 t^a \frac{(2+a)t^2+2-a}{[(2+a)^2t^2-(2-a)^2]^2} dt, \tag{2.5}$$

with

$$q = \sqrt{\frac{2-a}{2+a}}.$$

Clearly, the function $F(z)$ defined by (2.2) belongs to $S^c(R)$ with R given by (2.5).

Moreover, the function $F(z)$ has interesting extremal properties. We now give one of them.

Theorem 1. *Suppose that* $f \in S^c(R)$, $1/2 < R < 1$. *Then*

$$-F(-|z|) \leqslant |f(z)| \leqslant F(|z|).$$

3. Proof of Theorem 1

Let $U = U(R, M)$, $\delta(M) < R < 1 < M$, denote the family of closed convex domains D containing the disk K_R included in the closed circle $\overline{K_M}$ and such that the inner conformal radius $r(0, D) = 1$.

Obviously, for any domain $D \in U$ there exists $f \in C(R, M)$ such that $\overline{f(K_1)} = D$.

Let $g(w, \eta; D)$ denote the classical Green's function of the domain D with the pole η. By the compactness of the family U, there exist two extremal domains $D_0, d_0 \in U$, such that

$$\sup_{D \in U} g(0, \eta; D) = g(0, \eta; D_0), \tag{3.1}$$

$$\inf_{D \in U} g(0, \eta; D) = g(0, \eta; d_0) \text{ provided } |\eta| < R. \tag{3.2}$$

As pointed out by J. Krzyż [2] the problem of determining the extremum of the functional $|f(re^{it})|$ in the class $C(R, M)$ is equivalent to that of finding the domains $D_0, d_0 \in U$ which are satisfying (3.1); (3.2) resp.

In other words, if $\overline{\Phi(K_1)} = D_0$, then

$$|\eta| = \sup_{f, \theta} |f(re^{i\theta})| = \sup_{\theta} |\Phi(re^{i\theta})|$$

and if $\overline{\varphi(K_1)} = d_0$, then

$$|\eta| = \inf_{f, \theta} |f(re^{i\theta})| = \inf_{\theta} |\varphi(re^{i\theta})|, \quad |\eta| < R.$$

Henceforth, we shall find the domains D_0, d_0, using the Hadamard's variational formulas [3] p. 46.

$$\delta g(w, \eta; D) = \frac{1}{2\pi} \int_L \frac{\partial}{\partial n_\xi} g(\xi, w; D) \frac{\partial}{\partial n_\xi} g(\xi, \eta; D) \delta n(s) ds, \tag{3.3}$$

$$\delta \gamma(w; D) = \frac{1}{2\pi} \int_L \left[\frac{\partial}{\partial n_\xi} g(\xi, w, D)\right]^2 \delta n(s) ds, \tag{3.4}$$

where $\gamma(w; D)$ denotes the Robin's constant, L is the boundary of D and $\partial / \partial n_\xi$ is the derivative in the direction of the inward normal $n(s)$

$= tp(s)$. The above given formulas are valid when the boundary L of the convex domain D consists of a finite system of analytic arcs, whereas the function $p(s)$ is bounded and continuous except perhaps, a finite number of points. If the normal vector $n(s) = tp(s)$ is directed outside D then $p(s) > 0$; otherwise $p(s) < 0$.

Taking into consideration the relation between the function $h(w)$ which conformally maps $D \setminus L (D \in U)$, onto the disk K_1 with $h(0) = 0$, and the Green's function of D we bring the formulas (3.3) and (3.4) to a form more convenient for our purposes (see [2]):

$$\delta g(0, \eta; D) = \frac{1}{2\pi} \int_L |h'(w)|^2 X(w) \delta n(s) ds, \tag{3.5}$$

$$\delta \gamma(0; D) = \frac{1}{2\pi} \int_L |h'(w)|^2 \delta n(s) ds, \tag{3.6}$$

where

$$X(w) = \left(1 - |h(w)|^2\right) |h(w) - h(\eta)|^{-2}.$$

We also shall use the following

Lemma 1 [2]. *Suppose that the boundary L of the domain D is a Jordan curve and let the points A, B, C divide L into three arcs which do not degenerate to points. Then we can always choose two arcs: L_1, L_2 that for any arcs l_1, l_2;*

$l_1 \subset L_1$, $l_2 \subset L_2$ the inequality

$$\max_{w \in l_2} X(w) < \min_{w \in l_1} X(w) \tag{3.7}$$

holds.

In order to determine the domain D_0 we introduce the family $U_n \subset U$ of closed convex polygonal domains D_n with at most n vertices. Clearly $\bigcup_{n=1}^{\infty} U_n$ is dense in U. By the compactness of the family U_n there exists an extremal domain $\tilde{D}_n$ such that

$$g(0, \eta; \tilde{D}_n) = \sup g(0, \eta; D_n); \; D_n \in U_n.$$

The same technique as in [2] leads to the following characterisation of the extremal polygonal domain $\tilde{D}_n$.

Lemma 2. *The polygon $\tilde{D}_n$ has exactly n vertices. At most one vertex of $\tilde{D}_n$ situated inside K_M joins two sides neither tangent to the boundary of K_R.*

Let now the sequence $\{D_n\}$, $D_n \in U_n$, be convergent in the sense of kernel convergence to the domain D_0 having the extremal property (3.1) in U.

Then

$$\lim_{n\to\infty} g(0,\eta;D_n) = g(0,\eta;D_0).$$

On the other hand, for each $n \geqslant 3$

$$g(0,\eta;D_n) \leqslant g(0,\eta;\tilde{D}_n) \leqslant g(0,\eta;D_0).$$

This implies that there exists a subsequence $\{\tilde{D}_{n_k}\}$ convergent to $\tilde{D}_0 \epsilon U$ and such that

$$g(0,\eta;\tilde{D}_0) = g(0,\eta;\tilde{D}_0).$$

Let Γ_0 denote the boundary of D_0. According to Lemma 2, Γ_0 consists of the segments and the circular arcs situated on the boundary of K_M or K_R.

We shall use here the method of eliminating those domains which have not the extremal property (3.2). The idea of eliminating is following: Consider a domain $D \epsilon U$ and let D_t, $0 < t < T$, be the domain formed from D by certain deformation of the boundary ∂D where $D_t \to D$ as $t\to 0$. If $D_t \epsilon U$ and $g(0,\eta;D_t) > g(0,\eta;D)$ then the domain D evidently cannot have the extremal property (3.2).

We now prove that Γ_0 cannot contain more than two straight line segments. Suppose contrary to this that Γ_0 contains three segments. Hence $\Gamma_0 = \bigcup_{k=1}^{3} \Gamma_k$ and each Γ_k contains one segment. In view of Lemma 1 we can chose two segments l_1, l_2 such that the condition (3.7) is satisfied. We deform now the boundary of the domain D using the deformation described in [2] by shifting l_1 outside whereas l_2 is turned inside D, so that the domain D_t obtained from D after deformation should belong to the family U. On the arc l_1 we have $\delta n(s) > 0$, whereas $\delta n(s) < 0$ on l_2 and $\delta n(s) = 0$ on the remaining part of the boundary. From the formula (3.6) we get

$$0 = \gamma(0;D_t) - \gamma(0;D) = \frac{1}{2\pi}\int_{l_1+l_2} |h'(w)|^2 \delta n(s)\,ds + o(t),$$

$$\delta n(s) = tp(s).$$

Dividing both sides of the latter equality by t and making $t\to 0$, we obtain

$$\int_{l_1} |h'(w)|^2 p(s)\,ds = \int_{l_2} |h'(w)|^2 [-p(s)]\,ds. \tag{3.8}$$

From (3.7) and (3.8) we have:

$$\int_{l_1} |h'(w)|^2 X(w)\,\delta n(s)\,ds > \int_{l_2} |h'(w)|^2 X(w)[-\delta n(s)]\,ds,$$

which means that $\delta g(0, \eta; D) > 0$ and also

$$\Delta g = g(0, \eta; D_t) - g(0, \eta; D) > 0 \text{ for } 0 < t < T.$$

Using the same reasoning as above and the method of deformation of the convex domain described in [2] we find that Γ_0 is composed of two stright line segments and two arcs situated on δK_R and ∂K_M, resp.

The domain d_0 which minimizes the Green's function can be determined by an analogous argument (cf. [2]). It appears that $d_0 = \tilde{D}_0$ and in both cases the extremal domain is the same irrespective of the choice of the number η. Thus we have the characterisation of the extremal domain $\tilde{D}_0$ apart from rotations about $w = 0$. It is convenient to have the domain $\tilde{D}_0$ symmetric w.r.t. the real axis.

Remark. The solution of the problems (3.1); (3.2) can be extended on the limiting case $M = \infty$. By the above remark the proof of Theorem 1 is complete.

4. Conclusions

Suppose that $f \in C(R, M)$, and $\eta = f(z)$, $D = f(K_1)$. Then $r(\eta; D) = (1-|z|^2)|f'(z)|$. Let us put $\gamma(\eta; D) = \log r(\eta; D)$. In view of (3.4) we obtain the following expression for the variation of the Robin's constant of D:

$$\delta\gamma(\eta; D) = \frac{1}{2\pi}\int_L |\varphi'(w)|^2 X^2(w)\,\delta n(s)\,ds. \tag{4.1}$$

The formula (4.1) is valid if L consists of a finite system of analytic arcs. The function $X^2(w)$ has similar property of monotonity like $X(w)$. By analogous argumentation as in sect. 3 one can prove

Theorem 2. *If $f \in S_R^c$, $D_F = F(K_1)$ (F is defined by* (2.2)) *then*

$$(1-|z|^2)|f'(z)| \leqslant r\big(|f(z)|;\ D_F\big)$$

and for $|f(z)| \leqslant R$

$$r(-|f(z)|;\ D_F) \leqslant (1-|z|^2)|f'(z)|.$$

(cf. [2]).

Let $F(z) = z + A_2 z^2 + \ldots$ be the function given by (2.2). Since F is extremal in the problem $\max|f(re^{it})|$, $f \in S^c(R)$, $|A_2|$ is also extremal in the problem $\sup|a_2|$, $a_2 = \frac{1}{2}|f''(0)|$, $f \in S^c(R)$ (see [1], p. 8).

Thus we have

Theorem 3. *If $f(z) = z + a_2 z^2 + \ldots \in S^c(R)$ then*

$$|a_2| \leqslant \left(\frac{a}{2}\right)^2$$

From (2.5) we have $R(1) = \frac{4}{9} + \frac{16}{27}\ln 2 = 0{,}855\ldots$. If $R > R(1)$ then R corresponds to a with $0 < a < 1$. Hence, the right inequality of theorem 1 as well as (2.1), (2.2) give

Theorem 4. *Suppose that* $R > R(1)$, $f \in S^c(R)$. *Then*

$$|f(z)| \leqslant R\left(\cos a\frac{\pi}{2}\right)^{-1}, \quad z \in K_1.$$

We recall that a, R are connected by (2.5).

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STRESZCZENIE

Niech S^c oznacza klasę funkcji regularnych i jednolistnych w kole jednostkowym K_1 ($K_r = \{z \in \mathbf{C}\colon |z| < r\}$) i takich że dla każdej funkcji $f \in S^c$ $f(0) = f'(0) - 1 = 0$, oraz $f(K_1)$ jest obszarem wypukłym płaszczyzny zespolonej $\mathbf{C}$.

Oznaczmy przez $S^c(R)$ podklasą klasy S^c funkcji f spełniających warunek

$$K_R \subset f(K_1), \qquad \delta(M) < R < 1.$$

W pracy wyznaczono ekstremum funkcjonałów:

$$|f(z)|, \ (1-|z|^2)|f'(z)|, \ |a_2| = \tfrac{1}{2}|f''(0)|, \ f \in S^c(R)$$

(Twierdzenia: 1, 2, 3).

Rozważane ekstrema realizuje funkcja określona wzorem (2.2).

Ponadto wykazano, że jeśli $f \in S^c(R)$ i $R > \frac{4}{9} + \frac{16}{27}\ln 2$, to $|f(z)| < M(R)$, $z \in K_1$

(Twierdzenie 4).

РЕЗЮМЕ

Пусть S^c обозначает класс равномерных и однолистных функций в единичном круге K_1 ($K_r = \{z \in \mathbf{C}: |z| < r\}$) а таких же для каждой функции $f \in S^c f(0) = f'(0) - 1 = 0$, $f(K_1)$ является выпуклым пространством комплексной плоскости $\mathbf{C}$. Обозначим через $S^c(R)$ подкласс S^c функции f выполняя условия

$$K_R \subset f(K_1), \ \delta(M) < R < 1.$$

В работе онределено экстремум функционалов:

$$|f(z)|, \ (1-|z|^2)|f'(z)|, \ |a_2| = \tfrac{1}{2}|f''(0)|, \ f \in S^c(R)$$

(теорема: 1, 2, 3)

Обсуждаемые экстремумы реализует функция представлена формулой (2.2). Показано, что если $f \in S^c(R)$ и $R > \dfrac{4}{9} + \dfrac{16}{27} \ln 2$ тогда $f(z)| < M(R)$, $z \in K_1$ (Теорема 4).