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Some Remarks on the Wave Operator in a Curvilinear Coordinate System

Pewne uwagi o operatorze falowym w krzywoliniowym układzie współrzędnych

Некоторые замечания о волновом операторе в криволинейных координатах

In this paper we shall study some properties of the differential operator of second order, the characteristic form of which has a signature $(+, -, \ldots, -)$ (it is called wave operator), acting in a suitable chosen coordinate system.

I. Definitions and notions.

Let M be a smooth (class C^∞) real manifold of dimension $n+1$. If $x \in M$, then by $T_x(M)$ and $T_x^*(M)$ we denote the tangent and the cotangent spaces of M at the point x respectively.

Consider the differential operator with the variable coefficients, the action of which on function is described by the formula

$$(1) \qquad (P(D)u)(x) = \sum_{i,k=0}^{n} a^{ik}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_k},$$

where $(x_0, \ldots, x_n)$ are the coordinates in some local coordinate system on M. We assume that $a^{ik}(x) = a^{ki}(x)$.

The characteristic quadratic form of $P(D)$ at the point $x \in M$ is defined by the formula

$$(2) \qquad F_x(f, f) = \sum_{i,k=0}^{n} a^{ik}(x) \langle f, e_i(x)\rangle \langle f, e_k(x)\rangle; \; f \in T_x^*(M)$$

where $e_s(x) = \left(\frac{\partial}{\partial x_s}\right)_x \in T_x(M)$ and the symbol $\langle f, a\rangle$ denotes the value of the form $f \in T_x^*(M)$ at the vector $a \in T_x(M)$. The operator $P(D)$ will be called wave operator if its characteristic form F_x has the signature

$(+, -, \ldots, -)$ for any $x \in M$. Everywhere throughout this paper it is assumed that $P(D)$ is the wave operator.

The characteristic cone Γ_x at the point $x \in M$ is defined by formula

$$\Gamma_x = \{f \in T_x^*(M) : F_x(f, f) > 0\}. \tag{3}$$

It consists of two connected parts, one of which we denote by Γ_x^+. The future light cone K_x^+ at $x \in M$ is defined by the formula

$$K_x^+ = \{a \in T_x(M) \colon \langle f, a\rangle > 0, \text{ for every } f \in \Gamma_x^+\}. \tag{4}$$

Remark 1. Let $\big(f_0(x), \ldots, f_n(x)\big)$ be the basis in $T_x^*(M)$, biorthonormal to a basis $\big(e_0(x), \ldots, e_n(x)\big)$ in $T_x(M)$ and let $A^{ik}(x)$ be the inverse matrix to $a^{ik}(x)$. It is easy to see that the following equalities hold

$$\begin{aligned} \Gamma_x &= \Big\{f = \beta_0 f_0(x) + \ldots + \beta_n f_n(x) : \sum_{i,k=0}^{n} a^{ik}(x)\beta_i\beta_k > 0\Big\}, \\ K_x &= K_x^+ \cup (-K_x^+) \cup \{0\} \\ &= \Big\{a = a_0 e_0(x) + \ldots + a_n e_n(x) : \sum_{i,k=0}^{n} A^{ik}(x) a_i a_k \geqslant 0\Big\}. \end{aligned} \tag{5}$$

By (5), one can see that Γ_x and K_x are in one-to-one correspondence (up to a constant positive multiplier) with the coefficients of $P(D)$ in the coordinate system $(x_0, \ldots, x_n)$.

Remark 2. In the sequel we'll need a continuous family of half-cones K_x^+. The choosing of such family will be possible under some additional assumptions on M.

We say that the hyperplane P in $T_x(M)$ has the space-like orientation if P lies outside K_x^+ and has time-like one, if P intersects interior K_x^+. Let P be a hyperplane in $T_x(M)$ and let $0 \neq p \in T_x^*(M)$ be a form vanishing on P. One can easy prove the following

Lemma 1.

(a) $\{$*the orientation of* P *is time-like*$\} \Leftrightarrow \{p \notin \Gamma_x\}$

(b) $\{$*the orientation of* P *is space-like*$\} \Leftrightarrow \{p \in \Gamma_x\}$.

Let N be a smooth submanifold of dimension n of the manifold M. We say that the orientation of N at the point x is time-like (space-like) if the orientation of the hyperplane $T_x(N)$ is time-like (space-like).

II. The equation $P(D)u = 0$.

Assume that there is a local coordinate system $(x_0, \ldots, x_n)$ given on M, satisfying the following conditions:

1° $e_0(x) \in \operatorname{Int} K_x^+$, 2° the orientation of the hyperplane $P(x) = \mathrm{lin}(e_1(x), \ldots, e_n(x))$ is space-like, for any $x \in M$, where $e_i(x) = \left(\frac{\partial}{\partial x_i}\right)_x$ (Fig. 1)

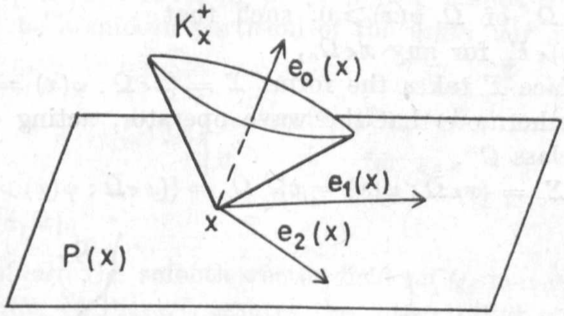

Fig. 1

Lemma 2. *In the above mentioned coordinate system there is $a^{00}(x) > 0$ and the form $\left(-\sum_{i,k=1}^{n} a^{ik}(x)\, a_i a_k\right)$ is positive-defined.*

Proof. Fix $x \in M$. Let $(f_0(x), \ldots, f_n(x))$ be a biorthonormal basis in $T_x^*(M)$ with respect to the basis $(e_0(x), \ldots, e_n(x))$. Each form which is different to 0 and vanishes on $P(x)$, has the form $af_0(x)$, $a \neq 0$. Since the orientation of $P(x)$ is space-like, thus by Lemma 1, $f_0(x) \in \Gamma_x$. Hence $a^{00}(x) = F_x(f_0(x), f_0(x)) > 0$. Consider the sequence $a_1, \ldots, a_n$; $\sum_{i=1}^{n} a_i^2 > 0$, and the form $f(x) = \sum_{i=1}^{n} a_i f_i(x)$. The form $f(x)$ vanishes on $e_0(x)$, thus the hyperplane on which vanishes $f(x)$ contains $e_0(x)$, hence intersects $\mathrm{Int}\, K_x^+$. Hence, by Lemma 1, $f(x) \notin \Gamma_x$ and it shows that $\sum_{i,k=1}^{n} a^{ik}(x)\, a_i a_k = F_x(f(x), f(x)) < 0$. So, in the coordinate system satisfying 1°, 2°, the equation $P(D)u = 0$ can be written in the form

$$\frac{\partial^2 u(x)}{\partial x_0^2} + \sum_{k=1}^{n} b^k(x) \frac{\partial^2 y(x)}{\partial x_0 \partial x_k} - \sum_{i,k=1}^{n} b^{ik}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_k} = 0 \tag{6}$$

where $b^{ki}(x) = b^{ik}(x)$ and the form $\sum_{i,k=1}^{n} b^{ik}(x)\, a_i a_k$ is positive-defined.

Construction. Now we shall give some sufficient conditions for the possibility of the construction of a coordinate system satisfying 1°–2° conditions and the following one:
3° in a such coordinate system, the coordinates of points from $\overline{M}$ compose the set $\langle 0, \infty) \times \Omega_1$, Ω_1 being a bounded domain in R^n.

Assumptions. Let Ω be a domain in R^{n+1}, the boundary of which consists of two parts: the space-like oriented, smooth, compact surface Σ and the time-like oriented, smooth surface (or finitely many of surfaces) σ. Assume, that there exists a functional ψ of class C^2 defined on some neighbourhood Ω_0 of $\overline{\Omega}$, $\psi(x) \geqslant 0$, such that

(i) $\operatorname{grad}\psi(x) \in \Gamma_x$ for any $x \in \Omega_0$,

(ii) the surface Σ takes the form: $\Sigma = \{x \in \overline{\Omega} : \psi(x) = 0\}$

Assume furthermore that the wave operator, acting on Ω_0 has the coefficients of class C^∞.

Let us denote $\Sigma_c = \{x \in \overline{\Omega} : \psi(x) = c\}$, $\Omega_c = \{(x \in \overline{\Omega} : \psi(x) > c\}$, (Fig. 2)

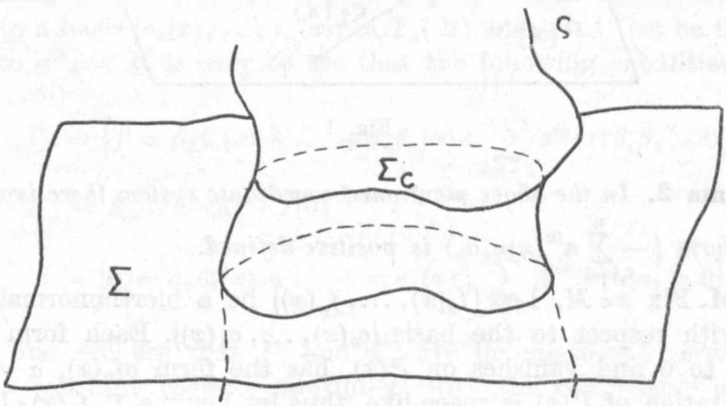

Fig. 2

Remark 3. Condition (i) arises from the paper of Hörmander (cf. [1], p. 108). In other words condition (i) means that the surfaces Σ_c are space-like oriented.

Hereafter, by Γ_x^+ we denote this part of Γ_x, such that $\operatorname{grad}\psi(x) \in \Gamma_x^+$. And so, we have a continuous family of half-cones Γ_x^+ and a continuous family of K_x^+ on Ω_0.

We shall construct a coordinate system on $\overline{\Omega}$ satisfying 1°–3°.

The set $Q = \overline{\Omega} \setminus \Sigma_c$, c – some constant < 0, is a smooth manifold with the boundary in σ. We shall construct a smooth vector field on Q such that $e_0(x) \in \operatorname{Int} K_x^+$ and $e_0(x) \in T_x(\sigma)$ for $x \in \sigma$. To do this we shall use partition of the unity.

Let $x \in Q$ and let U be the coordinate neighbourhood of x with the coordinate homeomorphism φ mapping U into $R_+^{n+1} = \{(x_0, \ldots, x_n) \in R^{n+1}: x_n \geqslant 0\}$. By φ_* we denote the map of tangent spaces induced by φ. On $\varphi(U)$ we define a constant field v such that $\varphi_{*,\varphi(x)}^{-1} v \in K_x^+$. If $\varphi(x) \in \partial R_+^{n+1}$, then we choose an arbitrary $v \in \partial R_+^{n+1}$; on the contrary, if $\varphi(x) \notin \partial R_+^{n+1}$, then we take a sufficiently small U such that $\varphi(U) \cap \partial R_+^{n+1} = \emptyset$.

The continuous changing of K_x^+ assures that if U is sufficiently small, then the image of v by φ_*^{-1} still belongs to $\operatorname{Int} K_x^+$ for $x \in U$.
The image of v by φ_*^{-1} composes the required vector field η on U.

Let $\{U_i\}$ be a such covering of Q, $\{\eta_i(x)\}$ – such a vector field on U_i and let $\{\varphi_i\}$ be a smooth partition of the unity subordinates to $\{U_i\}$. We take:

$$\tau_i(x) = \begin{cases} \varphi_i(x)\eta_i(x), & \text{for } x \in U_i \\ 0, & \text{for } x \notin U_i \end{cases}$$

and $e_0(x) = \sum_i \tau_i(x)$.

Thus we obtain the smooth vector field on Q, non-vanishing at any $x \in Q$. The theorem of Piccard assures the integrability of that field.

Let $x \in \bar{\Omega}$, $\psi(x) = c_0$, and let γ be an integral curve passing through x and intersecting σ at the point having the coordinates $(x_0, \ldots, x_n)$. We set a new coordinates of the point x as follows: $(c_0, x_1, \ldots, x_n)$.
The obtained coordinate system satisfies conditions 1°–3°, the coordinates of the points of $\bar{\Omega}$ compose the set $\langle 0, \infty) \times \Sigma$.

Remark 4. Conversely, if a such coordinate system is given, then taking $\psi(x_0, \ldots, x_n) = x_0$, we see that the functional ψ satisfies (i)–(ii).

On the other hand, one can easy construct an example of continuous family of K_x^+ defined on $\bar{\Omega}$ such that there is no functional ψ satisfying (i)–(ii).

REFERENCES

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STRESZCZENIE

W pracy zawarte są pewne uwagi dotyczące operatora falowego $P(D)$, działającego w krzywoliniowym układzie współrzędnych w obszarze $\Omega \subset R^{n+1}$. Jeśli układ współrzędnych jest stosownie dobrany, wtedy

równanie $P(D)u = 0$ można zapisać w postaci:

$$\frac{\partial^2 u}{\partial x_0^2} + \sum_{k=1}^{n} b^k(x) \frac{\partial^2 u}{\partial x_0 \partial x_k} - \sum_{i,k=1}^{n} b^{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} = 0,$$

gdzie forma $\sum_{i,k=1}^{n} b^{ik}(x) a_i a_k$ jest dodatnio określona. Przy pewnych założeniach odnośnie do Ω podaje się metodę konstrukcji takiego układu.

РЕЗЮМЕ

Настоящая работа содержит некоторые замечания, касающиеся волнового оператора действующего в криволинейных координатах в области Ω с R^{n+1}. Если система координат подобрана подходящим способом, тогда уравнение $P(D)_u = 0$ можно записать в виде:

$$\frac{\partial^2 u}{\partial x_0^2} + \sum_{k=1}^{n} b^k(x) \frac{\partial^2 u}{\partial x_0 \partial x_k} - \sum_{i,k=1}^{n} b^{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} = 0,$$

где форма $\sum_{i,k=1}^{n} b^{ik}(x) a_i a_k$ положительно определена. При некоторых предположениях относительно Ω приводится метод построения такой системы.