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Coefficient Regions for Starlike Polynomials

Obszar zmienności współczynników dla wielomianów gwiaździstych

Область изменения коэффициентов звёздных полиномов

1. Introduction

Let T_n, P_n, and P_n^* denote the classes of polynomials

$$p_n(z) = z + a_2 z^2 + \ldots + a_n z^n$$

which are typically real, univalent, and starlike univalent in $|z| < 1$. Rogosinski [7] and Hummel [5] have completely determined the coefficient regions for typically real and starlike functions respectively in $|z| < 1$; however their determination for T_n and P_*^n would have a number of useful applications. We note also the recent important work of T. J. Suffridge on the coefficient regions for starlike functions [8], which depends on the approximation of starlike functions by polynomials; this is closely related to the corresponding results for P_n^*.

In this note we discuss the coefficient regions for polynomials in T_3 and P_3^*; special cases of our results may be compared with the following observation of W. E. Kirwan:

Lemma 1. *Suppose* $f(z) = z - \sum_{n=2}^{\infty} c_n z^n$, $c_n \geq 0$. *Then the necessary and sufficient condition that* $f(z)$ *be univalent, starlike, or typically real in* $|z| < 1$ *is that* $\sum_{n=2}^{\infty} nc_n \leqslant 1$.

The sufficiency is an immediate consequence of a result of Alexander [1], and the necessity follows since $f'(z)$ cannot vanish in $-1 < z < 1$.

[1] The second author was supported by N.S.F. grant GP 5714.

Notice that $z+a_2z^2$ is starlike univalent in $|z|<1$ if and only if $|a_2|\leqslant 1/2$, and typically real if and only if $-1/2\leqslant a_2\leqslant 1/2$.

Our fundamental tool will be:

Lemma 2. *Let λ_1 and λ_2 be real. Then the necessary and sufficient conditions that $1+\lambda_1\cos\theta+\lambda_2\cos 2\theta$ be non-negative for $0\leqslant\theta\leqslant 2\pi$ are:*
(a) $|\lambda_1|\leqslant 1+\lambda_2$ if $-1\leqslant\lambda_2\leqslant 1/3$, *and*
(b) $|\lambda_1|\leqslant\sqrt{[8\lambda_2(1-\lambda_2)]}$ *if* $1/3\leqslant\lambda_2\leqslant 1$.

Proof. Putting $c=\cos\theta$, the result follows at once by examining the behaviour of

$$f(c)=1-\lambda_2+\lambda_1 c+2\lambda_2 c^2$$

and its derivative in the range $-1\leqslant c\leqslant 1$.

2. Coefficient regions for T_3

First of all, we note the following:

Lemma 3. *Suppose $p_n(z)=z+a_2z^2+\ldots+a_nz^n$, where all the a_k are real, and*

$$\sin\theta\cdot\operatorname{Im}p(e^{i\theta})\geqslant 0\quad \textit{for } 0\leqslant\theta\leqslant 2\pi.$$

Then $p_n(z)\in T_n$.

This follows from the definition of the class T, and the fact that $T_n\subset T$. We use this in the proof of

Theorem 1. *Suppose $p(z)=z+a_2z^2+a_3z^3$, where a_2 and a_3 are real. Then the necessary and sufficient conditions that $p(z)\in T_3$ are:*

(a) $$|a_2|\leqslant\tfrac{1}{2}(1+3a_3)\ \textit{if}\ -\tfrac{1}{3}\leqslant a_3\leqslant\tfrac{1}{5},\ \textit{and}$$

(b) $$|a_2|\leqslant 2\sqrt{[a_3(1-a_3)]}\ \textit{if}\ \tfrac{1}{5}\leqslant a_3\leqslant 1.$$

In particular:

(c) *if* $-\frac{1}{3}\leqslant a_3\leqslant\frac{1}{3}$, $p(z)\in T_3$ *if and only if it also* $\in P_3$; *and*

(d) *if* $p(z)\in T_3$, $|a_2|\leqslant 1$, *with equality only for* $z\pm z^2+\frac{1}{2}z^3$ *(which* $\notin P_3$, *by* [2, Theorem 2]).

Proof. The results follow from Lemmata 2 and 3, after some computation.

3. Coefficient regions for P_3^*, with a_2 and a_3 real

It is known [6] that a function is starlike univalent in $|z|<1$ if and only if so are all its de la Vallée Poussin means; starting from $K(z)=z/(1-z)^2$, this shows that $z+\frac{4}{5}z^2+\frac{1}{5}z^3\in P_3^*$. Further, if $z+a_2z^2+a_3z^3\in P_3^*$, then $|a_3| \leqslant \frac{1}{3}$, with equality only if $a_2 = 0$, [3]. This might have suggested that $|a_2| \leqslant \frac{4}{5}$ for P_3^*. However we will show that, even for real a_2 and a_3, a_2 may be as large as $0.85\ldots$; this may be compared with the sharp inequality $|a_2| \leqslant \sqrt{\frac{8}{9}} = 0.94\ldots$ for P_3.

Suppose $p(z)/z$ does not vanish in $|z| \leqslant 1$. Then the condition that $p(z) = z + a_2z^2 + a_3z^3 \in P_3^*$ (for real a_2 and a_3) is that

$$\operatorname{Re} \frac{e^{i\theta}p'(e^{i\theta})}{p(e^{i\theta})} \geqslant 0 \text{ for } 0 \leqslant \theta \leqslant 2\pi,$$

which reduces to $P_2(\theta) \geqslant 0$, where

$$P_2(\theta) = 1 + 2a_2^2 + 3a_3^2 + a_2(3 + 5a_3)\cos\theta + 4a_3\cos 2\theta.$$

This may be compared with Lemma 2, with

$$\lambda_1 = a_2(3 + 5a_3)/(1 + 2a_2^2 + 3a_3^2) \text{ and } \lambda_2 = 4a_3/(1 + 2a_2^2 + 3a_3^2).$$

Doing this at once leads to impossible complication; consequently we use a little geometrical intuition to cut the Gordian knot.

Suppose the radius of starlikeness of $p(z)$ is unity, and let D be the domain of variation of $w = zp'(z)/p(z)$ for $|z| \leqslant 1$. Then D is symmetric about the real axis, since a_2 and a_3 are real; and so either (A) ∂D meets the imaginary axis in two distinct points, one above and one below the real axis, or (B) ∂D passes through the origin.

Case A. Here $P_2(\theta) = L_0 + L_1\cos\theta + L_2\cos 2\theta$ has at least one real zero, and that where $\theta \neq 0, \pi$; hence $P_2(\theta) = 2L_2(A + \cos\theta)(B + \cos\theta)$ for some $A, -1 < A < 1$, and some real B. But then $P_2(\theta) \geqslant 0$ only if $A = B$, and so

$$\begin{aligned} P_2(\theta) &= 2L_2(A + \cos\theta)^2 \\ &= L_2[(2A^2 + 1) + 4A\cos\theta + \cos 2\theta]. \end{aligned}$$

We can now apply Lemma 2 with

$$\lambda_1 = 4A/(2A^2 + 1) \text{ and } \lambda_2 = 1/(2A^2 + 1).$$

Here $\frac{1}{3} < \lambda_2 \leqslant 1$, and so we must have $\lambda_1^2 = 8\lambda_2(1 - \lambda_2)$; putting this in terms of a_2 and a_3, we find the condition

$$a_2^2 = \frac{32a_3(1 - 3a_3)}{9 - 25a_3} \quad \text{for } \tfrac{1}{5} < a_3 \leqslant \tfrac{1}{3}.$$

Case B. Here $p'(z)$ has at least one zero on $|z| = 1$ and one in $|z| \geqslant 1$. If both are on $|z| = 1$, we already know that $p(z) = z \pm \frac{1}{3}z^3$, [3]. In the other case, since a_2 and a_3 are real, both zeros must be real, and so $p'(z)$

is either of the form $(1+z)(1+Bz)$ or $(1-z)(1-Bz)$ for some B, $-1 < B < 1$. In fact we restrict ourselves to the case

$$p'(z) = (1+z)(1+Bz)$$

from which we may deduce the other; and, by Lemma 1, we may assume that $B > 0$. Hence we deal with

$$p(z) = z+\tfrac{1}{2}(B+1)z^2+\tfrac{1}{3}z^3 \quad (0 < B < 1).$$

In the corresponding non-negative trigonometric polynomial, we have

$$\lambda_2 = \frac{8B}{5B^2+6B+1} \text{ and } \lambda_1 = \frac{5B^2+14B+9}{5B^2+6B+9} = 1+\lambda_2.$$

Then $\frac{1}{3} \leqslant \lambda_2 \leqslant 1$ if $\frac{3}{5} \leqslant B < 1$, and so $P_2(\theta) \geqslant 0$ and $p(z) \epsilon P_3^*$ only if $\lambda_1^2 \leqslant 8\lambda_2(1-\lambda_2)$, which is satisfied only if $\lambda_2 = \frac{1}{3}$ and $B = \frac{3}{5}$. Finally, $0 < \lambda_2 < \frac{1}{3}$ if $0 < B < \frac{3}{5}$, and then $p(z) \epsilon P_3^*$ only if $|\lambda_1| < 1+\lambda_2$, which is also satisfied.

This completes the proof of the necessity part of

Theorem 2. *Suppose $p(z) = z+a_2z^2+a_3z^3$, where a_2 and a_3 are real. Then the necessary and sufficient conditions that $p(z)$ have radius of starlikeness unity are:*

(a) *if $-\frac{1}{3} \leqslant a_3 \leqslant \frac{1}{5}$, $a_2 = \pm\frac{1}{2}(1+3a_3)$; in particular $z \pm \frac{4}{5}z^2+\frac{1}{5}z^3 \epsilon P_3^*$;*

(b) *if $\frac{1}{5} \leqslant a_3 \leqslant \frac{1}{3}$, $a_2^2 = 32a_3(1-3a_3)/(9-25a_3)$; and moreover*

(c) $\max\limits_{p \epsilon P_3^*} |a_2| = \frac{4}{25}(3\sqrt{6}-2)$, *and this is attained only for*

$$z \pm \tfrac{4}{25}(3\sqrt{6}-2)z^2+\tfrac{1}{25}(9-\sqrt{6})z^3.$$

On the other hand, both (a) and (b) imply $|a_2| < 1+a_3$, and so

$$|a_2(1-a_3)| < 1-a_3^2.$$

Hence, by the Cohn Rule [2, Lemma C], $p(z)/z$ cannot vanish in $|z| \leqslant 1$. The sufficiency part of the theorem then also follows from the above discussion.

The coefficient region (a_2, a_3) of Theorem 2 is convex. On the other hand, we now establish

Theorem 3. *The coefficient region* $(\mathrm{Re}\, a_2, \mathrm{Im}\, a_2, \mathrm{Re}\, a_3, \mathrm{Im}\, a_3)$ *for polynomials $z+a_2z^2+a_3z^3$ in P_3^* is not convex.*

Proof. Suppose $V(z) = z+\frac{4}{5}z^2+\frac{1}{5}z^3$ (which ϵP_3^*, by Theorem 2). It follows from Lemma 2 that $\mathrm{Re}\, V'(\frac{9}{10}e^{i\theta})$ vanishes for some θ_0 in

$[0, 2\pi]$. Let $z_0 = \frac{9}{10}e^{i\theta_0}$. Now let

$$V_1(z) = e^{2i\theta_0}V(e^{-2i\theta_0}z), \text{ and}$$

$$V_*(z) = \tfrac{1}{2}[V(z)+V_1(z)].$$

Then $V_*'(z_0) = 0$, since

$$V_1'(z_0) = V'(\bar{z}_0) = \overline{V'(z_0)}.$$

Consequently $V_*(z) \notin P_3^*$, and the result follows.

4. Coefficient regions for P_3^*, with a_3 real

We now consider the class of starlike cubic polynomials

$$p(z) = z + a_2z^2 + Bz^3$$

where $0 < B < \frac{1}{3}$, $a_2 = re^{i\varphi} = u + iv$, whose radius of starlikeness is unity.

Suppose $p(z)/z$ does not vanish in $|z| \leqslant 1$; by the Cohn Rule [2, Lemma o] a necessary and sufficient condition for this is that

$$1-B^2 > |a_2 - \bar{a}_2B|.$$

Under this assumption, $p(z) \in P_3^*$ if and only if

$$0 \leqslant \underset{0<\theta<2\pi}{\operatorname{Re}} \; [1+2re^{i(\varphi+\theta)}+3Be^{2i\theta}][1+re^{-i(\varphi+\theta)}+Be^{-2i\theta}]$$

$$= 1+2r^2+3B^2+4B\cos 2\theta+3r\cos(\varphi+\theta)+5rB\cos(\varphi-\theta)$$

$$= Q(r, B, \varphi, \theta), \text{ say.}$$

However, apart from a multiplicative constant, Q must be of the form [4]

$$[1+\cos(\theta-\theta_1)][1+t\cos(\theta-\theta_2)]$$

for $0 < t < 1$, $0 \leqslant \theta_1, \theta_2 \leqslant 2\pi$. Comparing this with the terms in Q, we deduce that

$$\frac{1+2r^2+3B^2}{1+\frac{1}{2}t\cos 2\theta_1} = \frac{(3+5B)u}{(1+t)\cos\theta_1}$$

$$= \frac{(5B-3)v}{(1-t)\sin\theta_1} = \frac{8B}{t}$$

for $-1 < t < 1$, $0 < \theta_1 < 2\pi$. Hence, after some computation, we may establish

Theorem 4. *Suppose* $p(z) = z + a_2 z^2 + Bz^3$, *where* $0 \leqslant B \leqslant \frac{1}{3}$. *Then the necessary and sufficient conditions that* $p(z)$ *have radius of starlikeness unity are:*

(a) $$1 - B^2 > |a_2| \cdot |B - e^{2i \arg a_2}|; \text{ and}$$

(b) $$a_2 = \pm \frac{8B(1+t)}{(3+5B)t} \cdot \delta(B, t) \pm i \frac{8B(1-t)}{(3-5B)t} \cdot \sqrt{[1 - \delta^2(B, t)]}$$

where $0 \leqslant \delta(B, t) \leqslant 1$ *for* $-1 \leqslant t \leqslant 1$, *and* $\delta^2(B, t) = \dfrac{N}{D}$ *where*

$$N = 8Bt - 4Bt^2 - t^2 - 3B^2t^2 - 128B^2 \left(\frac{1-t}{3-5B}\right)^2,$$

and

$$D = 128B^2 \left\{\left(\frac{1+t}{3+5B}\right)^2 - \left(\frac{1-t}{3-5B}\right)^2\right\} - 8Bt^2.$$

Note 1. We may replace condition (a) by the requirement that $p(z) \in P_3$, and so a_2 and B satisfy [2, Theorem 2].

Note 2. Similar arguments may be used to establish the coefficient region for quartic polynomials in P_4^* with real coefficients.

REFERENCES

[1] Alexander J.W., *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math., 17 (1915), 12-22.

[2] Brannan D.A., *Coefficient regions for univalent polynomials of small degree*, Matematika 14 (1967), 165-169.

[3] ,, , *On Univalent Polynomials*, *Glasgow* Math. J., 11 (1970), 102-107

[4] Fejér L., *Über trigonometrische Polynome*, J. de Crelle, 146 (1916), 53-82.

[5] Hummel J.A., *The coefficient regions of starlike functions*, Pacific J. Math., 7 (1957), 1381-1389.

[6] Pólya G. and Schoenberg I.J., *Remarks on de la Vallée Poussin means and convex conformal maps of the circle*, Pacific J. Math., 8 (1958), 295-334.

[7] Rogosinski W.W., *Über positive harmonische Entwicklungen und typischreele Potenzreihen*, Math. Z., 35 (1932), 93-121.

[8] Suffridge T.J., *Starlike functions as limits of polynomials*, Advances in Complex Function Theory Maryland 1973/74, Berlin-Heidelberg Nev York 1976, 164-203.

STRESZCZENIE

W pracy tej autorzy rozważają następujący problem: jakie warunki muszą spełniać współczynniki a_2, a_3 wielomianu $P(z) = z + a_2 z^2 + a_3 z^3$, żeby jego promień gwiaździstości był równy jedności lub żeby wielomian ten był typowo-rzeczywisty w kole jednostkowym.

РЕЗЮМЕ

В данной работе авторы решают следующую проблему: Какие условия должны выполнить коэффициенты a_2, a_3 многочлена $P(z) = z + a_2 z^2 + a_3 z^3$, чтобы его радиус звёздности равнялся единству или чтобы многочлен тот являлся типично-реальный в единичном круге.