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On Some Properties of a Certain Family of Solutions of the Paratingent Equation

O kilku własnościach pewnej rodziny rozwiązań równania paratyngensowego

Несколько свойств некоторого семейства решений паратингентного уравнения

The present paper is a continuation of the paper [3].

I. Let $\beta < 0$ be a fixed number belonging to the real line R. Let R^n be a n-dimensional Euclidean space with norm $|x| = \max(|x_1|, |x_2|, \ldots, |x_n|)$ and $\mathscr{C}$ be the space of all continuous functions $\varphi: R \to R^n$ with the topology defined by an almost uniform convergence (cf [3]). Having a function $\varphi \in \mathscr{C}$ and with $t \in \langle 0, \infty) \subset R$ by $(Pt\varphi)(t)$ we denote the paratingent of φ at the point t (cf [3]), by $[\varphi]_t$ we denote the function φ which is localized within the interval $\langle \beta, t\rangle$ and by $\|\varphi\|_t$ we denote the best nondecreasing majorant of φ on $\langle \beta, t\rangle$, i.e. $\|\varphi\|_t = \max\limits_{\beta \leq s \leq t} |\varphi(s)|$.

$\mathfrak{C}$ denotes the metric space (cf [3]) the elements of which are functions $[\varphi]_t$, $[\psi]_u$, etc, $\mathrm{Comp}\,\mathscr{C}$ and $\mathrm{Comp}\,R^n$ denote the set of all compact non-empty subsets of $\mathscr{C}$ and R^n respectively and, similary, $\mathrm{Conv}\,R_n$ denotes the set of all elements of $\mathrm{Comp}\,R^n$ which are convex.

Having two metric spaces E and E' we say that a mapping Γ: $E \to \mathrm{Comp}\,E'$ is upper semi-continuous (usc) when for all sequences $\{x_i\} \subset E$, $\{y_i\} \subset E'$ such that $x_i \to x_0$ and $y_i \in \Gamma(x_i)$, $i = 1, 2, \ldots$ there exists a subsequence $\{y_{i_j}\}$ of $\{y_i\}$ which is convergent to y_0 and $y_0 \in \Gamma(x_0)$.

Let $M(t) \geq 0$ and $N(t) \geq 0$ be real-valued continuous functions defined for $t \geq 0$.

We put $\Lambda(t) = \int\limits_0^t L(u)\,du$, where $L(t) = M(t) + N(t)$.

Let $\nu(t) \geq 0$ be a real -valued continuous function such that

$$\Lambda(\nu(t)) \leq a^{-1}(\Lambda(t) + e^{-1}) \quad \text{for } t \geq 0, \text{ where } 0 < a \leq 1$$

is a fixed number.

Let $\mathscr{D}$ be a compact subset of the space $\mathscr{C}$ and $A \geqslant \max(1, \sup_{\xi \in \mathscr{D}} \|\xi\|_0)$ be a fixed number.

Let us denote by Φ the class of all functions $\varphi \in \mathscr{C}$ which satisfies the inequality

$$|\varphi(t)| \leqslant A \exp[eA(t)] \quad \text{for } t \geqslant 0.$$

Let $F\colon \mathfrak{C} \to \operatorname{Conv} R^n$ be a usc mapping satisfying the condition

$$F([\varphi]_t) \subset K(\theta, M(t) + N(t)(\|\varphi\|_t)^a) \quad \text{for } t \geqslant 0,$$

$K(\theta, r)$ denotes an open ball of radius r around θ, where θ is an origin of R^n.

In the present note we give some properties of certain solutions of the paratingent equation

$$(1) \qquad (Ptx)(t) \subset F([x]_{\tau(t)}) \quad \text{for } t \geqslant 0$$

with

$$(2) \qquad x(t) = \xi(t) \quad \text{for } t \leqslant 0, \text{ where } \xi \in \mathscr{D}.$$

II. In [3] we have proved the following

Theorem 1. *In the class Φ for every $\xi \in \mathscr{D}$ there exists at least one solution of the problem* (1) *and* (2).

From the proof of this theorem it follows

Corollary. *If $\varphi \in \Phi$ and $(Pt\varphi)(t) \subset F([\varphi]_{\tau(t)})$ for $t \geqslant 0$ then* $|\varphi(t+h) - \varphi(t)| \leqslant A \int_t^{t+h} eL(u) \exp[eA(u)]\,du$ *for* $t \geqslant 0$, $h > 0$.

Definitions. *A function $\varphi \in \mathscr{C}$ will be called a trajectory of $F([x]_{\tau(t)})$ if $(Pt\varphi)(t) \subset F([\varphi]_{\tau(t)})$ for $t \geqslant 0$.*

By $E_\Phi(F, \xi)$ we denote the collection of all trajectories φ of $F([x]_{\tau(t)})$ such that $\varphi \in \Phi$ and $\varphi(t) = \xi(t)$ for $t \leqslant 0$, where $\xi \in \mathscr{D}$, and by $E_\Phi(F, \mathscr{D})$ we denote the union of $E_\Phi(F, \xi)$ where ξ belongs to $\mathscr{D}$, i.e.

$$E_\Phi(F, \mathscr{D}) = \bigcup_{\xi \in \mathscr{D}} vE_\Phi(F, \xi).$$

By $e_\Phi(F, \xi)$ we denote the union of the graphs of functions belonging to $E_\Phi(F, \xi)$.

By $e_\Phi(F, \xi\, T)$ we denote the set $e_\Phi(F, \xi) \cap \{(t, x)\colon t = T, x \in R^n\}$ and by $e_\Phi(F, \mathscr{D}, T)$ the union of $e_\Phi(F, \xi, T)$, where $\xi \in \mathscr{D}$, i.e.

$$e_\Phi(F, \mathscr{D}, T) = \bigcup_{\xi \in \mathscr{D}} e_\Phi(F, \xi, T).$$

Theorem 2. *$E_\Phi(F, \xi)$ is a compact nonempty subset of the space $\mathscr{C}$.*

Proof. From Theorem 1 it follows that $E_\Phi(F, \xi)$ is nonempty. Now, if $\varphi \in E_\Phi(F, \xi)$ then

a) $$|\varphi(t)| \leqslant A\exp[\Lambda(t)] \quad \text{for } t \geqslant 0,$$

b) $$|\varphi(t+h) - \varphi(t)| \leqslant \int_t^{t+h} eL(u)\exp[e\Lambda(u)]du \quad \text{for } t \geqslant 0, h > 0,$$

c) $$\varphi(t) = \xi(t) \quad \text{for } t \leqslant 0.$$

Hence immediately we conclude that functions φ belonging to $E_\Phi(F, \xi)$ are uniformly bounded and equicontinuous on each compact interval of R. Thus the set $E_\Phi(F, \xi)$ is compact in the space $\mathscr{C}$.

Theorem 3. *A mapping $E: \mathscr{D} \to \mathrm{Comp}\, C$ defined by formula*

$$E(\xi) = E_\Phi(F, \xi), \quad \xi \in \mathscr{D},$$

is usc on $\mathscr{D}$.

Proof. Let ξ_i, $\xi \in \mathscr{D}$, $\varphi_i \in \mathscr{D}$, $\xi_i \to \xi$ and $\varphi_i \in E(\xi_i)$, $i = 1, 2, \ldots$. Thus we have

$$|\varphi_i(t)| \leqslant A\exp[e\Lambda(t)] \quad \text{for } t \geqslant 0,$$

$$\varphi_i(t) = \xi_i(t) \quad \text{for } t \leqslant 0,$$

and, in view of corollary 1

$$|\varphi_i(t+h) - \varphi_i(t)| \leqslant A\int_t^{t+h} eL(u)\exp[e\Lambda(u)]du \quad \text{for } t \geqslant 0,\ h > 0.$$

Now, we can choose a subsequence $\{\varphi_{i_j}\} \subset \{\varphi_i\}$ which converges to some function $\varphi \in \mathscr{C}$. It is obvious that satisfies φ the inequality $|\varphi(t)|$ $A\exp[e\Lambda(t)]$ for $t \geqslant 0$ and the equality $\varphi(t) = \xi(t)$ for $t \leqslant 0$.
On the other hand we have $(Pt\varphi_{i_j})(t) \subset F([\varphi_{i_j}]_{\varkappa(t)})$ for $t \geqslant 0$. Thence from lemma 4 in [3] it follows that $(Pt\varphi)(t) \subset F([\varphi]_{\varkappa(t)})$ for $t \geqslant 0$. Thus $\varphi \in E(\xi)$ which completes the proof.

Now, from theorem 2, 3 and Hukuhara's theorem (cf [1], proposition 12.1) immediately the following results

Theorem 4. *$E_\Phi(F, \mathscr{D})$ is a compact nonempty subset of the space $\mathscr{C}$.*

Theorem 5. *Let $T \geqslant 0$. Then $e_\Phi(F, \xi, T)$ is a compact nonempty subset of R^{1+n}.*

Proof. Let $(t_i, x_i) \in e_\Phi(F, \xi, T)$, $i = 1, 2, \ldots$. In the set $E_\Phi(F, \xi)$ there are functions φ_i such that $\varphi_i(t_i) = x_i$. Since $E_\Phi(F, \xi)$ is compact, we can choose a subsequence of $\{\varphi_i\}$ convergent to $\varphi_0 \in E_\Phi(F, \xi)$. Similarly, the sequence $\{t_i\}$ being bounded contains a subsequence which converges to some $t_0 \in \langle 0, T\rangle$. Let $\{i_j\}$, $j = 1, 2, ..,$ be a sequence of indices such that conditions $t_{i_j} \to t_0$ and $\varphi_{i_j} \to \varphi_0$ hold at the same time. This means that $(t_{i_j}, x_{i_j}) = (t_{i_j}, \varphi_{i_j}(t_{i_j})) \to (t_0, \varphi_0(t_0))$ and the point $(t_0, \varphi_0(t_0))$ belongs obviously to $e_\Phi(F, \xi, T)$. Thus the set $e_\Phi(F, \xi, T)$ is compact in the space R^{1+n}.

Theorem 6. *A mapping* e_T: $\mathscr{D} \to \operatorname{Comp} R^{1+n}$ *defined by formula*

$$e_\Phi(\xi) = e_\Phi(F, \xi, T), \quad \xi \in \mathscr{D},$$

is usc on $\mathscr{D}$.

Proof. Let ξ_i, $\xi \in \mathscr{D}$, $(t_i, x_i) \in e_T(\xi_i)$ and $\xi_i \to \xi$, $i = 1, 2, \ldots$. In view of compactness of $E_\Phi(F, \mathscr{D})$ and of boundedness of $\{t_i\}$, there exists a subsequence of $\{\varphi_i\}$ convergent to some $\varphi \in E_\Phi(F, \mathscr{D})$ and subsequence of $\{t_i\}$ convergent to some point $t \in \langle 0, T\rangle$. Let $\{i_j\}$, $j = 1, 2, \ldots$, be a sequences of indices such that $t_{i_j} \to t$, $\varphi_{i_j} \to \varphi$. Then $(t_{i_j}, x_{i_j}) = (t_{i_j}, \varphi_{i_j}(t_{i_j})) \to (t, \varphi(t))$. At the same time we have

$$(Pt\varphi_{i_j})(t) \subset F([\varphi_{i_j}]_{\nu(t)}) \quad \text{for } t \geqslant 0$$

and

$$\varphi_{i_j}(t) = \xi_{i_j}(t) \quad \text{for } t \leqslant 0.$$

Then obviously $\varphi(t) = \xi(t)$ for $t \leqslant 0$ and in view of lemma 4 in [3] $(Pt\varphi)(t) \subset F([\varphi]_{\nu(t)})$ for $t \geqslant 0$. This means that $\varphi \in E_\Phi(F, \xi)$. Consequently $(t, x) = (t, \varphi(t)) \in e_\Phi(F, \xi, T) = e_T(\xi)$. The proof of upper semi-continuity of e_T is completed.

Now, from theorem 5, 6 and Hukuhara's theorem (cf [1] Proposition 12.1) we obtain

Theorem 7. *Let* $T \geqslant 0$. *Then* $e_\Phi(F, \mathscr{D}, T)$ *is a compact nonempty subset of* R^{1+n}.

Remark. If $0 \leqslant \nu(t) \leqslant t$, then $E_\Phi(F, \xi)$ is the set of all solutions of the problem (1) and (2) and then the set is sometimes called the emission of initial function ξ according to equation (1). This case has been studied in detail by B. Krzyżowa in [2].

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STRESZCZENIE

Praca poświęcona jest badaniu własności rozwiązań równania paratyngensowego z odchylonym argumentem postaci

(1) $$(Ptx)(t) \subset F([x]_{\nu(t)}), \quad t \geqslant 0,$$

a w szczególności badaniu własności pewnych podzbiorów usłanych przez trajekorie dla $F([x]_{\nu(t)})$ wyznaczone przez warunek początkowy

(2) $$x(t) = \xi(t), \quad t \leqslant 0.$$

Podane są między innymi twierdzenia o zawartości tych podzbiorów.

РЕЗЮМЕ

Настоящая работа посвящена исследованию свойств решений паратингентного уравнения с отклоняющим аргументом вида

(1) $$(Ptx)(t) \subset F([x]_{\nu(t)}), \quad t \geqslant 0.$$

Кромс того исследуются свойства некоторых подмножеств, усталенных траекториями для $F([x]_{\nu(t)})$ определенных начальным условием

(2) $$x(t) = \xi(t), \quad t \leqslant 0.$$

Приводятся также другие георемы о компактности этих подмножеств.

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