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**On Some Properties of a Certain Family of Solutions
of the Paratingent Equation**

O kilku własnościach pewnej rodziny rozwiązań równania paratingentowego

Несколько свойств некоторого семейства решений паратингентного уравнения

The present paper is a continuation of the paper [3].

I. Let $\beta < 0$ be a fixed number belonging to the real line R . Let R^n be a n -dimensional Euclidean space with norm $|x| = \max(|x_1|, |x_2|, \dots, |x_n|)$ and \mathcal{C} be the space of all continuous functions $\varphi: R \rightarrow R^n$ with the topology defined by an almost uniform convergence (cf [3]). Having a function $\varphi \in \mathcal{C}$ and with $t \in \langle 0, \infty \rangle \subset R$ by $(Pt\varphi)(t)$ we denote the paratingent of φ at the point t (cf [3]), by $[\varphi]_t$ we denote the function φ which is localized within the interval $\langle \beta, t \rangle$ and by $\|\varphi\|_t$ we denote the best nondecreasing majorant of φ on $\langle \beta, t \rangle$, i.e. $\|\varphi\|_t = \max_{\beta < s < t} |\varphi(s)|$.

\mathfrak{C} denotes the metric space (cf [3]) the elements of which are functions $[\varphi]_t, [\psi]_u$, etc, $\text{Comp}\mathcal{C}$ and $\text{Comp}R^n$ denote the set of all compact non-empty subsets of \mathcal{C} and R^n respectively and, similarly, $\text{Conv}R_n$ denotes the set of all elements of $\text{Comp}R^n$ which are convex.

Having two metric spaces E and E' we say that a mapping $\Gamma: E \rightarrow \text{Comp}E'$ is upper semi-continuous (usc) when for all sequences $\{x_i\} \subset E$, $\{y_i\} \subset E'$ such that $x_i \rightarrow x_0$ and $y_i \in \Gamma(x_i)$, $i = 1, 2, \dots$ there exists a subsequence $\{y_{i_j}\}$ of $\{y_i\}$ which is convergent to y_0 and $y_0 \in \Gamma(x_0)$.

Let $M(t) \geq 0$ and $N(t) \geq 0$ be real-valued continuous functions defined for $t \geq 0$.

We put $A(t) = \int_0^t L(u) du$, where $L(t) = M(t) + N(t)$.

Let $\nu(t) \geq 0$ be a real-valued continuous function such that

$$A(\nu(t)) \leq \alpha^{-1}(A(t) + e^{-1}) \quad \text{for } t \geq 0, \text{ where } 0 < \alpha \leq 1$$

is a fixed number.

Let \mathcal{D} be a compact subset of the space \mathcal{C} and $A \geq \max(1, \sup_{\xi \in \mathcal{D}} \|\xi\|_0)$ be a fixed number.

Let us denote by Φ the class of all functions $\varphi \in \mathcal{C}$ which satisfies the inequality

$$|\varphi(t)| \leq A \exp[e\Lambda(t)] \quad \text{for } t \geq 0.$$

Let $F: \mathcal{C} \rightarrow \text{Conv } R^n$ be a use mapping satisfying the condition

$$F([\varphi]_t) \subset K(\theta, M(t) + N(t)(\|\varphi\|_t)^n) \quad \text{for } t \geq 0,$$

$K(\theta, r)$ denotes an open ball of radius r around θ , where θ is an origin of R^n .

In the present note we give some properties of certain solutions of the paratingent equation

$$(1) \quad (Pt\varphi)(t) \subset F([\varphi]_{\varphi(t)}) \quad \text{for } t \geq 0$$

with

$$(2) \quad x(t) = \xi(t) \quad \text{for } t \leq 0, \text{ where } \xi \in \mathcal{D}.$$

II. In [3] we have proved the following

Theorem 1. *In the class Φ for every $\xi \in \mathcal{D}$ there exists at least one solution of the problem (1) and (2).*

From the proof of this theorem it follows

Corollary. *If $\varphi \in \Phi$ and $(Pt\varphi)(t) \subset F([\varphi]_{\varphi(t)})$ for $t \geq 0$ then $|\varphi(t+h) - \varphi(t)| \leq A \int_t^{t+h} eL(u) \exp[e\Lambda(u)] du$ for $t \geq 0, h > 0$.*

Definitions. *A function $\varphi \in \mathcal{C}$ will be called a trajectory of $F([\varphi]_{\varphi(t)})$ if $(Pt\varphi)(t) \subset F([\varphi]_{\varphi(t)})$ for $t \geq 0$.*

By $E_\Phi(F, \xi)$ we denote the collection of all trajectories φ of $F([\varphi]_{\varphi(t)})$ such that $\varphi \in \Phi$ and $\varphi(t) = \xi(t)$ for $t \leq 0$, where $\xi \in \mathcal{D}$, and by $E_\Phi(F, \mathcal{D})$ we denote the union of $E_\Phi(F, \xi)$ where ξ belongs to \mathcal{D} , i.e.

$$E_\Phi(F, \mathcal{D}) = \bigcup_{\xi \in \mathcal{D}} E_\Phi(F, \xi).$$

By $e_\Phi(F, \xi)$ we denote the union of the graphs of functions belonging to $E_\Phi(F, \xi)$.

By $e_\Phi(F, \xi, T)$ we denote the set $e_\Phi(F, \xi) \cap \{(t, x) : t = T, x \in R^n\}$ and by $e_\Phi(F, \mathcal{D}, T)$ the union of $e_\Phi(F, \xi, T)$, where $\xi \in \mathcal{D}$, i.e.

$$e_\Phi(F, \mathcal{D}, T) = \bigcup_{\xi \in \mathcal{D}} e_\Phi(F, \xi, T).$$

Theorem 2. $E_\phi(F, \xi)$ is a compact nonempty subset of the space \mathcal{C} .

Proof. From Theorem 1 it follows that $E_\phi(F, \xi)$ is nonempty. Now, if $\varphi \in E_\phi(F, \xi)$ then

$$a) \quad |\varphi(t)| \leq A \exp[\Lambda(t)] \quad \text{for } t \geq 0,$$

$$b) \quad |\varphi(t+h) - \varphi(t)| \leq \int_t^{t+h} eL(u) \exp[e\Lambda(u)] du \quad \text{for } t \geq 0, h > 0,$$

$$c) \quad \varphi(t) = \xi(t) \quad \text{for } t \leq 0.$$

Hence immediately we conclude that functions φ belonging to $E_\phi(F, \xi)$ are uniformly bounded and equicontinuous on each compact interval of R . Thus the set $E_\phi(F, \xi)$ is compact in the space \mathcal{C} .

Theorem 3. A mapping $E: \mathcal{D} \rightarrow \text{CompC}$ defined by formula

$$E(\xi) = E_\phi(F, \xi), \quad \xi \in \mathcal{D},$$

is usc on \mathcal{D} .

Proof. Let $\xi_i, \xi \in \mathcal{D}, \varphi_i \in \mathcal{D}, \xi_i \rightarrow \xi$ and $\varphi_i \in E(\xi_i), i = 1, 2, \dots$. Thus we have

$$|\varphi_i(t)| \leq A \exp[e\Lambda(t)] \quad \text{for } t \geq 0,$$

$$\varphi_i(t) = \xi_i(t) \quad \text{for } t \leq 0,$$

and, in view of corollary 1

$$|\varphi_i(t+h) - \varphi_i(t)| \leq A \int_t^{t+h} eL(u) \exp[e\Lambda(u)] du \quad \text{for } t \geq 0, h > 0.$$

Now, we can choose a subsequence $\{\varphi_{i_j}\} \subset \{\varphi_i\}$ which converges to some function $\varphi \in \mathcal{C}$. It is obvious that φ satisfies the inequality $|\varphi(t)| \leq A \exp[e\Lambda(t)]$ for $t \geq 0$ and the equality $\varphi(t) = \xi(t)$ for $t \leq 0$.

On the other hand we have $(P\varphi_{i_j})(t) \subset F([\varphi_{i_j}]_{\eta(t)})$ for $t \geq 0$. Thence from lemma 4 in [3] it follows that $(P\varphi)(t) \subset F([\varphi]_{\eta(t)})$ for $t \geq 0$. Thus $\varphi \in E(\xi)$ which completes the proof.

Now, from theorem 2, 3 and Hukuhara's theorem (cf [1], proposition 12.1) immediately the following results

Theorem 4. $E_\phi(F, \mathcal{D})$ is a compact nonempty subset of the space \mathcal{C} .

Theorem 5. Let $T \geq 0$. Then $e_\phi(F, \xi, T)$ is a compact nonempty subset of R^{1+n} .

Proof. Let $(t_i, x_i) \in e_\phi(F, \xi, T), i = 1, 2, \dots$. In the set $E_\phi(F, \xi)$ there are functions φ_i such that $\varphi_i(t_i) = x_i$. Since $E_\phi(F, \xi)$ is compact, we can choose a subsequence of $\{\varphi_i\}$ convergent to $\varphi_0 \in E_\phi(F, \xi)$. Similarly,

the sequence $\{t_i\}$ being bounded contains a subsequence which converges to some $t_0 \in \langle 0, T \rangle$. Let $\{i_j\}$, $j = 1, 2, \dots$, be a sequence of indices such that conditions $t_{i_j} \rightarrow t_0$ and $\varphi_{i_j} \rightarrow \varphi_0$ hold at the same time. This means that $(t_{i_j}, x_{i_j}) = (t_{i_j}, \varphi_{i_j}(t_{i_j})) \rightarrow (t_0, \varphi_0(t_0))$ and the point $(t_0, \varphi_0(t_0))$ belongs obviously to $e_\phi(F, \xi, T)$. Thus the set $e_\phi(F, \xi, T)$ is compact in the space R^{1+n} .

Theorem 6. *A mapping $e_T: \mathcal{D} \rightarrow \text{Comp} R^{1+n}$ defined by formula*

$$e_\phi(\xi) = e_\phi(F, \xi, T), \quad \xi \in \mathcal{D},$$

is usc on \mathcal{D} .

Proof. Let $\xi_i, \xi \in \mathcal{D}$, $(t_i, x_i) \in e_T(\xi_i)$ and $\xi_i \rightarrow \xi$, $i = 1, 2, \dots$. In view of compactness of $E_\phi(F, \mathcal{D})$ and of boundedness of $\{t_i\}$, there exists a subsequence of $\{\varphi_i\}$ convergent to some $\varphi \in E_\phi(F, \mathcal{D})$ and subsequence of $\{t_i\}$ convergent to some point $t \in \langle 0, T \rangle$. Let $\{i_j\}$, $j = 1, 2, \dots$, be a sequence of indices such that $t_{i_j} \rightarrow t$, $\varphi_{i_j} \rightarrow \varphi$. Then $(t_{i_j}, x_{i_j}) = (t_{i_j}, \varphi_{i_j}(t_{i_j})) \rightarrow (t, \varphi(t))$. At the same time we have

$$(Pt\varphi_{i_j})(t) \subset F([\varphi_{i_j}]_{\eta(t)}) \quad \text{for } t \geq 0$$

and

$$\varphi_{i_j}(t) = \xi_{i_j}(t) \quad \text{for } t \leq 0.$$

Then obviously $\varphi(t) = \xi(t)$ for $t \leq 0$ and in view of lemma 4 in [3] $(Pt\varphi)(t) \subset F([\varphi]_{\eta(t)})$ for $t \geq 0$. This means that $\varphi \in E_\phi(F, \xi)$. Consequently $(t, x) = (t, \varphi(t)) \in e_\phi(F, \xi, T) = e_T(\xi)$. The proof of upper semi-continuity of e_T is completed.

Now, from theorem 5, 6 and Hukuhara's theorem (cf [1] Proposition 12.1) we obtain

Theorem 7. *Let $T \geq 0$. Then $e_\phi(F, \mathcal{D}, T)$ is a compact nonempty subset of R^{1+n} .*

Remark. If $0 \leq \nu(t) \leq t$, then $E_\phi(F, \xi)$ is the set of all solutions of the problem (1) and (2) and then the set is sometimes called the emission of initial function ξ according to equation (1). This case has been studied in detail by B. Krzyżowa in [2].

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STRESZCZENIE

Praca poświęcona jest badaniu własności rozwiązań równania paratyngensowego z odchylnym argumentem postaci

$$(1) \quad (Pt_x)(t) \subset F([x]_{\nu(t)}), \quad t > 0,$$

a w szczególności badaniu własności pewnych podzbiorów ustalonych przez trajektorie dla $F([x]_{\nu(t)})$ wyznaczone przez warunek początkowy

$$(2) \quad x(t) = \xi(t), \quad t < 0.$$

Podane są między innymi twierdzenia o zawartości tych podzbiorów.

РЕЗЮМЕ

Настоящая работа посвящена исследованию свойств решений паратингентного уравнения с отклоняющим аргументом вида

$$(1) \quad (Pt_x)(t) \subset F([x]_{\nu(t)}), \quad t > 0.$$

Кроме того исследуются свойства некоторых подмножеств, усталенных траекториями для $F([x]_{\nu(t)})$ определенных начальным условием

$$(2) \quad x(t) = \xi(t), \quad t < 0.$$

Приводятся также другие теоремы о компактности этих подмножеств.

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