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Grassmannian Connections

Koneksje Grassmannowskie

Грассмановы связности

This paper contains an investigation of a second order G-structure over a manifold which is locally Grassmannian in a sense of T. Hangan. There are introduced canonical forms of the two kinds and their structure equations are derived. Then there are obtained invariant connections.

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Throughout all the paper the standard notations of jet calculus are used and, if necessary, they are slightly modified in consistence with notations used in [1], [2] and [3].

I. The canonical G-structure over a Grassmannian.

Let F denote the field of reals or of complex numbers. We fix the two integers, p and q, both $\geqslant 1$ and we set $p+q = n$. We denote by F^n the topological product of n sheets of F and we provide F with a natural structure of vector space over F. We denote by F^q_p the topological vector space of matrices of the from $[a^i_\alpha]_{\substack{i=1,\ldots,q\\ \alpha=1,\ldots,p}}$ We denote by K the unimodular subgroup of $GL(n, F)$ and we have to consider a subgroup $H \subset K$, viz.

$$H := \{[a^H_I]_{H,I=1,\ldots,n} \,|\, a^j_{q+\alpha} = 0\}$$

Thus the elements of H are matrices of the form

$$\left[\begin{array}{c|c} a^i_j & 0 \\ \hline a^{q+\alpha}_j & a^\alpha_\beta \end{array}\right]_{i,j=1,\ldots,q;\ \alpha,\beta=1,\ldots,p}$$

The conventions on ranges of Greek and of Latin indices just used will remain valid in our following considerations.

1. Proposition: *H is a stationary group of a p-dimensional plane in F^n which is defined by the equations $v^1 = 0, \ldots, v^q = 0$.*

2. Proposition: *The homogeneous space K/H is just a Grassmann space of p-dimensional subspaces in F^m.*

3. Definition: Let M be a pq dimensional manifold with its support space M and its atlas A. Then M is called to be a locally Grassmannian manifold (modelled in F_p^q) if:
i) M is locally modelled in F_p^q.
ii) if (U, h) and (V, k) are the two local charts in A then there exists a matrix

$$\left[\begin{array}{c:c} a^{\mathrm{I}} & a^{\mathrm{II}} \\ \hdashline a^{\mathrm{III}} & a^{\mathrm{IV}} \end{array}\right] \in K$$

where $a^{\mathrm{I}} \in \hom(F^q, F^q)$, $a^{\mathrm{II}} \in \hom(F^p, F^q) = F_p^q$, $a^{\mathrm{IV}} \in \hom(F^p, F^p)$ and $a^{\mathrm{III}} \in \hom(F^q, F^p)$, such that we have for $x \in U \cap V$

$$k(x) = (a^{\mathrm{I}} h(x) + a^{\mathrm{II}})(a^{\mathrm{III}} h(x) + a^{\mathrm{IV}})^{-1} \tag{4}$$

(Cf. [1], [2]).

5. Proposition: *The Grassmann space of p-subspaces in F is a locally Grassmannian manifold.* (Cf. [3]).

Let M be a locally Grassmannian manifold modelled in F_p^q. We assume that the atlas which contains all the local charts for which (ii) of def. 3 holds. Then the r-jets of of the mappings which are reciprocal to modelling functions are the frames of order r on M. The set of these frames is a principal bundle over M. If $r = 1$ then the Jacobian matrices of transition functions (4) constitute the structural group, G, which is a subgroup of $\mathrm{aut}(F_p^q)$.

Generally, we denote by $\mathscr{F}_r M$ the bundle of admissible r-frames on M and we denote by G_r a structure group of $\mathscr{F}_r M$. We call $\mathscr{F}_r M$ the canonical G-structure on M.

We shall write the elements of G in a matrix form like

$$[A_{j\beta}^{i\alpha}]_{i,j=1,\ldots,q;\ \alpha,\beta=1,\ldots,p}$$

The Lie algebra of K (resp. of H, resp. of G) will be denoted by $\mathbf{K}$ (resp. by $\mathbf{H}$, resp. by $\mathbf{G}_r$).

II. The canonical form.

Let us fix r-frame $A = j_0^r f$, f being a diffeomorfism of a neighbourhood of $0 \in F_t^q$ to M. We set $A' = j_0^{r-1} f$. Let X be a vector tangent to $\mathscr{F}_r M$ at A. Thus there exists some real interval, $\Delta = \langle -a, a \rangle$ and a mapping $\Delta \to \mathscr{F}_r M / t \to j_{x|0}^r\ f_t(x)$ such that

$$j_{x|0}^r\ f_0(x) = A \quad \text{and} \quad j_{t|0}^1\ j_{x|}^r\ f_t(x) = X.$$

Thus we have the notation $j^r_{x|0}f_t(x)$ (resp. $j^1_{t|0}f_t(x)$) for the r-jet of the mapping $x \to f_t(x)$ (and, respectively, of the mapping $t \to f_t(x)$), a source being at 0 in F^q_p (resp. 0 in Δ).

We assume that all $f_t(-)$ are admissible so that X is tangent at $\mathscr{F}_r M$. If t is proxime to 0 enough then there exists a mapping $f_0^{-1} \circ f_t$. This is a diffeomorfism of a neighbourhood of 0 in F^q_p into F^q_p. Thus the vector $j^1_{t|0} j^{r-1}_{x|0} f_0^{-1} \circ f_t(x)$ depends linearly on X and it is an element of $F^q_p + \mathbf{G}_{r-1}$, where $\mathscr{G}$ is a vector space tangent to G_{r-1} at its unit element I_G.

The above construction yields a field of linear forms on $\mathscr{F}_r M$ which is called the *canonical form of order* r. Cf. [4], [5], [7], [9], [10], [11].

We denote by I_{i_a} the elements of the natural linear basis in F^q_p. Then we denote by $I^{j_\beta}_{i_a}, \ldots, I^{j_1\beta_1 \ldots j_s\beta_s}_{i_a}$ the natural linear basis in $\mathbf{G}_s$. Now we may. write the following decomposition

$$\omega^{(r)} = \sum_{i,a} \omega^{i_a} \otimes I_{i_a} + \sum \omega^{i_a}_{j_\beta} \otimes I^{j_\beta}_{i_a} + \ldots + \sum \omega^{i_a}_{j_1\beta_1 \ldots j_r\beta_r} \otimes I^{j_1\beta_1, \ldots, j_r\beta_r}_{i_a}$$

The following formulas are consequences from general expressions given in [10] and [11]

$$\text{(6)} \qquad \begin{aligned} \omega^{i_a} &= \sum a^{i_a}_{k_r} dx^k_r \\ \omega^{i_a}_{j_\beta} &= \sum a^{i_a}_{k_r} \Big(dA^{k_r}_{j_\beta} - \sum A^{k_r}_{j_\beta l_\delta} \omega^{l_\delta} \Big) \end{aligned}$$

where $[A^{i_a}_{j_\beta}, A^{i_a}_{j_\beta k_\gamma}]$ are local coordinates in a neighbourhood of A in $\mathscr{F}_2 M$, and the matrix $[a^{i_a}_{j_\beta}]$ is reciprocal with respect to $[A^{i_a}_{j_\beta}]$, i.e. $\sum\limits_{k,r} A^{i_a}_{k_r} a^{k_r}_{j_\beta} = \delta^i_j \delta^\beta_a$.

There hold the following *structure equations*

$$\text{(7)} \qquad \begin{aligned} & d\omega^{i_a} + \sum \omega^{i_a}_{k_r} \wedge \omega^{k_r} = 0 \\ & d\omega^{i_a}_{j_\beta} + \sum \omega^{i_a}_{k_r} \wedge \omega^{k_r}_{j_\beta} + \sum \omega^{i_a}_{j_\beta k_r} \wedge \omega_{k_r} = 0 \end{aligned}$$

8. Remark: The above construction of $\omega^{(r)}$ may be restricted to vectors tangent of the fibres. Thus we obtain a restricted field of canonical forms, $\overline{\omega}^{(r)}$. It follows immediatelly from formula (6) that we have now

$$\text{(6bis)} \qquad \overline{\omega}^{i_a} = 0, \quad \overline{\omega}^{i_a}_{j_\beta} = \sum a^{i_a}_{k_r} dA^{k_\gamma}_{j_\beta}$$

$$\text{(7bis)} \qquad d\overline{\omega}^{i_a} = 0, \quad d\overline{\omega}^{i_a}_{j_\beta} + \sum \overline{\omega}^{i_a}_{k_r} \wedge \overline{\omega}^{k_\gamma}_{j_\beta} = 0$$

Moreover, $\overline{\omega}^{(r)}$ may be defined directly on $\mathscr{F}_{r-1} M$ without performing a projection $A \to A'$ because $\omega^{(r)}$ does not depend on the coordinates of the highest order of the frame.

9. Theorem: *Let* $\varphi = \Sigma\varphi^{i_\alpha}_{j_\beta} \otimes I^{j_\beta}_{i_\alpha}$ *be a form of some linear connection on* M, *more exactly on* $\mathcal{F}_1 M$. *Then there exists a field of linear forms on* $\mathcal{F}_2 M$

$$A \to \sum \Gamma^{i_\alpha}_{k_r j_\beta}(A)\omega^{k_r}_A \otimes I^{j_\beta}_{i_\alpha}$$

such that:

1° *the form* $\Sigma(\omega^{i_\alpha}_{j_\beta} + \Sigma\Gamma^{i_\alpha}_{k_r j_\beta}\omega^{k_r}) \otimes I^{j_\beta}_{i_\alpha}$ *depends only on the projection* A' *of* A,

2° *the following system of differential equations is satisfied*

$$d\Gamma^{i_\alpha}_{k_r j_\beta} + \sum_{l,\lambda} (\Gamma^{i_\alpha}_{l_\lambda j_\beta}\omega^{l_\lambda}_{k_r} + \Gamma^{i_\alpha}_{k_r l_\lambda}\omega^{l_\lambda}_{j_\beta} - \Gamma^{l_\lambda}_{k_r j_\beta}\omega^{i_\alpha}_{l_\lambda}) - \omega^{i_\alpha}_{k_r j_\beta} = \sum_{l,\lambda} \Gamma^{i_\alpha}_{k_r j_\beta, l_\lambda}\omega^{l_\lambda}$$

where $\Sigma\Gamma^{i_\alpha}_{k_r j_\beta, l_\lambda} I^{k_r j_\beta, l_\lambda}_{i_\alpha}$ *is a certain linear form on* $\mathcal{F}_3 M$.

Proof. The system of forms $[\omega^{i_\alpha}, \omega^{i_\alpha}_{j_\beta}]$ constitutes a co-frame at each point A' of $\mathcal{F}_1 M$. Thus the φ-horizontal subspaces tangent to $\mathcal{F}_1 M$ at A' is given by a system of linear equations of the form

$$\omega^{i_\alpha}_{j_\beta} + \sum \Gamma^{i_\alpha}_{k_r j_\beta}\omega^{k_r} = 0$$

where $[\Gamma^{i_\alpha}_{k_r j_\beta}]$ depend on A as $[\omega^{i_\alpha}_{j_\beta}]$ do. Then the statement 2° follows by remark 8. In order to prove 2° we introduce into considerations the curvature tensor, R, of φ and we make from the definition equality

$$d\varphi^{i_\alpha}_{j_\beta} + \sum \varphi^{i_\alpha}_{k_r} \wedge \varphi^{k_r}_{j_\beta} = \sum R^{i_\alpha}_{k_\varkappa l_\lambda j_\beta}\omega^{k_\varkappa} \wedge \omega^{l_\lambda}$$

We substitute $\omega^{i_\alpha}_{j_\beta} + \Sigma\Gamma^{i_\alpha}_{k_r j_\beta}\omega_{k_r}$ in a place of $\varphi^{i_\alpha}_{j_\beta}$ and we obtain the following equalities

$$(*) \quad \sum_{k,r}\Big\{d\Gamma^{i_\alpha}_{k_r j_\beta} + \sum_{l,\lambda}(\Gamma^{i_\alpha}_{l_\lambda j_\beta}\omega^{l_\lambda}_{k_r} + \Gamma^{i_\alpha}_{k_r l_\lambda}\omega^{l_\lambda}_{j_\beta} - \Gamma^{l_\lambda}_{k_r j_\beta}\omega^{i_\alpha}_{l_\lambda}) - \\ - \omega^{i_\alpha}_{k_r j_\beta} - \sum(\Gamma^{i_\alpha}_{l_\lambda h_\sigma}\Gamma^{h_\sigma}_{k_r j_\beta} + R^{i_\alpha}_{l_\lambda k_r j_\beta})\omega^{l_\lambda}\Big\} \wedge \omega^{k_r} = 0$$

In power of the generalized lemma of Cartan+) we conclude that there exist functions on $\mathcal{F}_2 M$, $H^{i_\alpha}_{k_r j_\beta l_\lambda}$, such that $H^{i_\alpha}_{k_r j_\beta l_\lambda} = H^{i_\alpha}_{l_\lambda j_\beta k_r}$ and that the parantheses in (*) are equal to

$$\Gamma^{i_\alpha}_{k_r j_\beta ; l_\lambda} := H^{i_\alpha}_{k_r j_\beta l_\lambda} + \sum_{h,\sigma}\Gamma^{i_\alpha}_{l_\lambda h_\sigma}\Gamma^{h_\sigma}_{k_r j_\beta} + R^{i_\alpha}_{l_\lambda k_r j_\beta}$$

+) The lemma of Cartan generalized by G. F. Łaptev may be formulated as follows: Let $v_1, \ldots, v_a$ be alinearly independent elements of a finite-dimensional vector space V. If the elements $Y_1, \ldots, Y_a$ of the exterior product $\Lambda^p V$ satisfy the equality $\sum_i Y_i v_i = 0$ then there exist in $\wedge^{p-1} V$ some z_{ik} such that $z_{ik} = z_k$ and $Y_i = \sum_{j=1}^{x} z_{ij} v_j$. Cf. [10]. Recently Mr. O. Kowalski in Praha has obtained a profound generalization [8].

III. The canonical form of the second kind

Let us turn to the frame $A = j_0^r f$. If $a \in K$ then we set $a_* A := j_0^r f(a^{-1} y)$ where we denote by ay a result of the action of K on F according to the rule: If

$$g = \left[\begin{array}{c|c} g^{\mathrm{I}} & g^{\mathrm{II}} \\ \hline g^{\mathrm{III}} & g^{\mathrm{IV}} \end{array}\right] \in K$$

then

$$gy = (g^{\mathrm{I}} y + g^{\mathrm{II}})(g^{\mathrm{III}} y + g^{\mathrm{IV}})^{-1} \tag{10}$$

We observe that the action * implies a homomorfism of K into the Lie algebra of vector fields on $\mathscr{F}_r M$. Then a vector field on $\mathscr{F}_r M$ which is generated by a vector $X \in$ K will be denoted by $X^{(r)}$.

We define a canonical form of the second kind, $\theta^{(r)}$, as follows

$$\theta_A^{(r)}(X) := \omega_A^{(r)}(X^{(r)}) \tag{11}$$

The following proposition follows directly from the considerations of the previous section.

12. Proposition: *If $Y \in$ H then $Y^{(r-1)}$ is a vertical field on $\mathscr{F}_{r-1} M$. Consequently we have*

$$\theta_{A'}^{(r-1)}(Y) = \overline{\omega}_{A'}^{(r-1)}(Y_{A'}^{(r-1)}) = \varphi_{A'}^{(r-1)}(Y_{A'}^{(r-1)})$$

for any connection form $\varphi^{(r-1)}$ on $\mathscr{F}_{r-1} M$.

We are going to obtain a local expression for $\theta^{(r)}$. Let $t \to a_t$ describe a curve in K which starts from I_k, such that $X = j_{t|0}^1 a_t$. Then the mapping f^{-1}, such that $A = j_0^r f$, yields some local chart in a neighbourhood of A. The coordinates of A are $(0, [\delta_j^i \delta_\alpha^\beta], 0, \ldots, 0)$. Moreover, this chart defines a local cross-section through A in $\mathscr{F}_r M$. In consistence with the construction of the form $\theta^{(r)}$ we have

$$\theta_A^{(r)}(X) = j_{t|0}^1\, j_{x|0}^{r-1}\, f^{-1}\big((a_t^{-1})_* x\big) = j_{t|0}^1\, j_{x|0}^{r-1}\, f^{-1} a_t^{-1} x \tag{13}$$

We split $\theta^{(r)}$ into components

$$\theta^{(r)} = \sum \theta^{i_\alpha} \otimes \mathrm{I}_{i_\alpha} + \sum \theta_{j\beta}^{i_\alpha} \otimes \mathrm{I}_{i_\alpha}^{j\beta} + \sum \theta_{j_\beta k_r}^{i_\alpha} \otimes \mathrm{I}_{i_\alpha}^{j_\beta k_r} + \ldots \tag{14}$$

We introduce into considerations the canonical left-invariant form $\vartheta = [\vartheta_J^I]$ on K. We perform the computations in details according to (13) and we replace the components of X by the corresponding components of ϑ. Then we obtain the following local expressions of the components of $\theta^{(3)}$:

$$
(15)\quad
\begin{aligned}
\theta^{i_a} &= dx^i_a + \vartheta^i_j x^j_a + \vartheta^i_a - x^i_\beta \vartheta^\beta_j x^j_a - x^i_\beta \vartheta^\beta_a \\
\theta^{i_a}_{j\beta} &= (\vartheta^i_j - x^i_r \vartheta^r_j)\,\delta^\beta_a + (-\vartheta^\beta_a - \vartheta^\beta_k x^k_a)\,\delta^i_j \\
\theta^{i_a}_{j_\beta k_r} &= -\vartheta^r_j \delta^i_k \delta^\beta_a - \vartheta^\beta_k \delta^i_j \delta^r_a \\
\vartheta^{i_a}_{j_1\beta_1 j_2\beta_2 j_3\beta_3} &= 0 \quad \text{etc.}
\end{aligned}
$$

16. Lemma: *The forms*

$$\psi^i_j := \varphi^i_j - x^i_\lambda \varphi^\lambda_j \quad \text{and} \quad \psi^\beta_a := \varphi^\beta_a + \varphi^\beta_k x^k_a$$

satisfy the following identities

$$d\psi^i_j + \psi^i_k \wedge \psi^k_j + \vartheta^{i_\lambda} \wedge \varphi^\lambda_j = 0$$

$$d\psi^\beta_a + \psi^\beta_\mu \wedge \psi^\mu_a + \varphi^\beta_j \wedge \vartheta^{j_a} = 0$$

Proof: We make use from the identities of Maurer and Cartan

$$d\varphi^i_j = -\varphi^i_k \wedge \varphi^k_j - \varphi^i_a \wedge \varphi^a_j,$$

$$d\varphi^\lambda_j = -\varphi^\lambda_k \wedge \varphi^k_j - \varphi^\lambda_a \wedge \varphi^a_j, \qquad \text{etc.}$$

Then we have $d\psi^\lambda_j = d\varphi^i_j - dx^{i_\lambda} \wedge \varphi^\lambda_j - x^{i_\lambda} d\varphi^i_j$ and we replace dx^{i_λ} by its expression which will be obtained from the formula of ϑ^{i_λ}. Then we obtain our lemma by performing some easy reduction.

17. Theorem: *The components of $\theta^{(3)}$ satisfy the following identities*

$$d\vartheta^{i_a} + \sum_{j,\beta} \vartheta^{i_a}_{j_\beta} \wedge \vartheta^{j_\beta} = 0$$

$$d\vartheta^{i_a}_{j_\beta} + \sum \vartheta^{i_a}_{k_r} \wedge \vartheta^{k_r}_{j_\beta} + \sum \vartheta^{i_a}_{j_\beta k_r} \wedge \vartheta^{k_r} = 0$$

$$d\vartheta^{i_a}_{j_\beta k_r} + \sum \vartheta^{i_a}_{j_\beta a_\nu} \wedge \vartheta^{h_\nu}_{k_r} + \sum \vartheta^{i_a}_{h_\nu k_r} \wedge \vartheta^{h_\nu}_{j_\beta} + \sum \vartheta^{i_a}_{h_\nu} \wedge \vartheta^{h_\nu}_{j_\beta k_r} = 0.$$

Proof: By a skew differentiation of (15) and by making use from lemma 16.

In order to investigate better the canonical G-structure on M we have to introduce new coordinates in G_2. We put

$$J^\beta_a = -\sum I^{k_a}_{k_\beta}, \quad J^i_j = -\sum I^{i_k}_{j_k}$$

$$J^i_\lambda = -\sum (I^{i_\varkappa j_\lambda}_{j_\varkappa} + I^{i_\lambda j_\varkappa}_{j_\varkappa}), \quad J^\lambda_i = I_{i_\lambda}$$

and

$$\psi^i_a = \vartheta^{i_a}, \quad \psi^\mu_j = \vartheta^\mu_j$$

Then we may write the decomposition (14) in a following form

$$\theta^{(3)} = \psi^i_a \otimes J^a_i + \psi^i_j \otimes J^j_i + \psi^\beta_a \otimes J^a_\beta + \psi^a_j \otimes \mathbf{J}^j_\mu \tag{18}$$

19. Theorem: *There hold the following structure equations*

$$d\psi^i_a + \psi^i_j \wedge \psi^j_a + \psi^i_\beta \wedge \psi^\beta_a = 0$$
$$d\psi^i_j + \psi^i_k \wedge \psi^k_j + \psi^i_a \wedge \psi^a_j = 0$$
$$d\psi^\beta_a + \psi^\beta_k \wedge \psi^k_a + \psi^\beta_\lambda \wedge \psi^\lambda_a = 0$$
$$d\psi^a_j + \psi^a_k \wedge \psi^k_j + \psi^a_\beta \wedge \psi^\beta_j = 0$$

Moreover, the equality (Σ) $\vartheta^i_i + \vartheta^a_a = 0$ *implies* $\psi^i_i + \psi^a_a = 0$.

Proof: The theorem follows easily by writing out the identities in theorem 17 and by using the formulae (15) in a new form

$$\theta^{i\,\alpha}_{j\,\beta} = \psi^i_j \delta^\beta_\alpha - \psi^\beta_\alpha \delta^i_j,$$

$$\sum \theta^{i\,\alpha}_{j\,\beta} \otimes \mathbf{I}^{j\,\beta}_{i\,\alpha} = \psi^i_j \otimes J^j_i + \psi^\beta_a \otimes J^a_\beta$$

$$\sum \theta^{i\,\alpha}_{j\,\beta k_\gamma} \otimes \mathbf{I}^{j\beta k_\gamma}_{i_\alpha} = \psi^\beta_j \otimes \mathbf{J}^j_\beta$$

Then we are able to prove the main theorem.

20. Theorem: *The structure group of $\mathscr{F}_2 M$ is isomorphic to the stationary group H.*

Proof: In order to obtain the fibre forms we assume $\theta^{i_\alpha} (= \varphi^i_\alpha) = 0$. Thus the Lie algebra the structure group is spanned on the basic vectors $\mathbf{I}^i_j$, $\mathbf{I}^\beta_a$, $\mathbf{I}^j_\beta$. The values of Poisson brackets may be obtained directly from the structure equations of the components of ψ Cf. [17]. These are just consistent with the structure equations of H. Taking into account the identity $\psi^i_i + \psi^a_a = 0$ and the fact that all the groups under consideration are connected we obtain the theorem.

If we restrict our above considerations to the bundle $\mathscr{F}_1 M$ then we obtain the following theorem [2]:

21. Theorem: *The structure group of $\mathscr{F}_1 M$ is izomorfic to $(GL(p, F) \times \times GL(q, F)) \cap SL(p+q, F)$.*

IV. The connections.

From now we assume that K acts on M (at least locally) transitively from the left in such a way that this action is expressed by the generalized homografy (4) in any admissible local coordinates. We denote this left-action by τ and this same notation will be used for an induced left action on

$\mathscr{F}_2 M$. We have to look for connections on the bundle F_2M which are invariant with respect to τ. These will be called the Grassmannian connections.

To begin with we fix a frame $u = j_0^1 f$ and we define a mapping λ: $H \to G_2$ as follows

$$\lambda(h) = j^2_{x|0} f^{-1} \circ \tau_h \circ f(x)$$

We put also

$$\lambda^+ : \mathbf{H} \to \mathbf{G}_2 \ / \ \lambda^+ (j^1_{t|0} h_t) = j^1_{t|0} \lambda(h_t)$$

Then we compute the explicit expression λ^+ in local coordinates which are determined by f^{-1}. We obtain

$$\lambda^+ = (\psi^i_j \otimes J^j_i + \psi^\beta_\alpha \otimes J^\alpha_\beta + \psi^\beta_j \otimes J^j_\beta)_{|u}$$

If we take into account the fact the components ψ^i_j, ψ^α_β, ψ^β_j do not depend on ϑ^i_α then we observe that there holds the following

22. Proposition: *We have*

$$\lambda^+ = \theta^{(2)}|_{x=0,} \ \ \vartheta^i{}_\alpha = 0$$

Now we may apply the theorem of Wang (Cf. [12] [6]) which claimes that:

There holds a $1-1$ correspondence between the invariant connections and the forms Λ: $\mathbf{K} \to \mathbf{G}_2$ such that $(\alpha)\, \Lambda|\mathbf{H} = \lambda^+$, and $(\beta)\, \Lambda \circ adj_h = adj_{\lambda(h)} \Lambda$ This correspondence is given by the relation $\varphi_u \circ \tau^+ = \Lambda$ where φ is a connection form and τ^+: $\mathbf{K} \to T\mathscr{F}_2 M$ is induced by τ.

It follows by the proposition that the condition (α) may be replaced by the following

$$\Lambda = \lambda^+ + \Sigma \psi^k_r \otimes (J^i_j \Gamma^l_{k_r i} + J^\alpha_\beta \Gamma^\beta_{k_r \alpha} + J^j_\beta \Gamma^\beta_{k_r j}) \tag{23}$$

In view of connectedness of K and H we may also reformulate (β) into an infinitesimal form. We take a vector

$$Y = j^1_{t|0} h_t \in \mathbf{H}$$

and we obtain

$$\begin{aligned} \Lambda([Y, X]) &= j^1_{t|0} \Lambda(adj_{h_t} X) = j^1_{t|0} \lambda(h_t) \cdot \Lambda(X) \cdot (\lambda(h_t))^{-1} \\ &= \lambda^+(Y) \cdot \Lambda(X) - \Lambda(X) \cdot \lambda^+(Y) = [\lambda^+(Y), \Lambda(X)] \end{aligned} \tag{24}$$

We substitute the right-hand member of (23) for Λ. In view of the fact that λ^+ is a Lie homomorfism we obtain the following condition which is equivalent to (β):

$$\Gamma([Y, X]) = [\lambda^+(Y), \Gamma(X)] \quad \text{for} \quad Y \in \mathbf{H}, \ X \in \mathbf{K} \tag{25}$$

where $\Gamma = \Lambda - \lambda^+$. A dimension of the vector space $\mathbf{H}+\mathbf{K}$ is equal to $(p^2+q^2+pq)(p+q)^2$ and there are to be determined from (25) $(p^2+q^2+pq)pq$ numbers Γ^{rL}_{kM}. If we write (25) explicitely then we see that it yields a system of linear homogeneous equations which has $(p^2+q^2+pq)^2$ independent solutions. Then the theorem of Wang applied to our case yields the following theorem.

26. Theorem: *There exists a* $(p^2+q^2+pq)^2$ *– parametric family of Grassmannian connections on* M. *These connections are in a* 1–1 *correspondence with the solutions of* (25).

In order to obtain some informations on a curvature R we make use from the following formula (Cf. [6] Ch. X):

$$R(\tilde{X}, \tilde{Z}) = u\circ\big([\Lambda(X), \Lambda(Z)] - \Lambda([X, Z])\circ u^{-1}$$

where X and Z are vectors ϵ K and $\tilde{X}$, $\tilde{Z}$ are their canonical maps at u. An easy transformation of the above formula yields the following

$$R(\tilde{X}, \tilde{Z}) = u\circ([\Gamma(X), \lambda^+(Z)] - [\Gamma(Z), \lambda^+(Y)] + \\ + [\Gamma(X), \Gamma(Z)] - \Gamma([X, Z]])\circ u^{-1}$$

This formula implies immediately the following theorem [3].

26. Theorem. *The canonical Grassmannian connection which corresponds to a case* $\Gamma = 0$ *has the curvature equal to zero.*

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STRESZCZENIE

Praca zawiera konstrukcję formy kanonicznej 1-go i 2-go rodzaju na rozmaitości Grassmanna p-wymiarowego płaszczyzn w n-wymiarowej przestrzeni wektorowej. Na rozmaitości tej, rozpatrywanej jako przestrzeń Kleina mamy naturalną G-strukturę rzędu drugiego. Własności grupy strukturowej otrzymujemy w prosty sposób poprzez własności formy kanonicznej 2-go rodzaju. Specjalizując twierdzenie G. C. Wanga o niezmienniczych koneksjach na rozmaitościach jednorodnych otrzymujemy twierdzenie o istnieniu $(p^2+q^2+pq)^2$-parametrowej rodziny niezmienniczych koneksji na omawianej G-strukturze. Dokładnie jedna z nich, kanoniczna, jest koneksją otrzymaną i zbadaną przez Th. Hangana.

РЕЗЮМЕ

В работе содержится конструкция канонической формы 1-го и 2-го рода на Грассмановом многообразии p-размерных плоскостей n-мерного векторного пространства. На этом многообразии, рассматриваемом как пространство Клейна, получаем соответственно G-структуру второго порядка. Свойства структурной группы получаем из свойств канонической формы 2-го рода. Пользуясь теоремой Ванга об инвариантных связностях на однородных пространствах, получаем теорему о существовании $(p^2+q^2+pq)^2$ — параметрического семейства связностей изучаемой G-структуры. Одна из них, а именно каноническая, является связностью, полученной и изученной Т. Хангановом.