ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. XXVIII, 9 SECTIO A 1974

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Some Classes of Functions Subordinate to Linear Transformation and their Applications

Pewne klasy funkcji regularnych podporządkowanych transformacji ułamkowo-liniowej i ich zastosowania

Некоторые классы регулярных функций, подчиненных дробно-линейной трансформате, и их приложения

1. Let Ω denote the class of functions $\omega(z)$ regular in the unit disc $K_1(K_r = \{z\colon |z| < r\})$, and such that

$$\omega(0) = 0, \ |\omega(z)| < 1 \text{ for } z \in K_1.$$

The function f is said to be subordinate to a regular function F in K_1 if there exists a function $\omega \in \Omega$ such that $f(z) = (F \circ \omega)(z)$. If f is subordinate to F in K_1 then we write $f \dashv F$.

Many authors have investigated extremal problems in some classes of regular functions which can be defined in a homogeneous form by the subordination. For instance, if P is the class of functions p such that $p(z)$ is regular in K_1 and $p(0) = 1$, re $p(z) > 0$ for $z \in K_1$ then we have $P = \left\{p : p(z) \dashv \frac{1+z}{1-z}\right\}$. W. Janowski [8], investigated the class $P(A, B)$, $-1 \leqslant A \leqslant 1$, $-A < B \leqslant 1$, which can be defined as follows: $P(A, B) = \left\{p : p(z) \dashv \frac{1+Az}{1-Bz}\right\}$. D. Shaffer [12], [13], has studied the class $P_{a,n}(0 \leqslant a < 1)$, which can be defined as follows: $P_{a,n} = \left\{p : p(z) = 1 + +c_n z^n + c_{n+1} z^{n+1} + \ldots \dashv \frac{1+z}{1-(1-2a)z}\right\}$,

In this paper we shall investigate the class

$$P_n(A, B) = \left\{p : p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots \dashv \frac{1+Az}{1-Bz}\right\},$$

where n is any natural number and A, B are any fixed complex numbers such that $|A| \leqslant 1$, $|B| \leqslant 1$.

Some special cases of $P_n(A, B)$ are the classes which were investigated among others by Caratheodory [1], Jakubowski [4] – [7], Janowski [8], Libera [9], Mac Gregor [11], Shaffer [12], [13], Szynal [14].

2. Let us denote by D_z a region of variability of the complex functional $p(z)$, where $z \in K_1$, $|z| = r$, and p ranges over the class $P_n(A, B)$.

Theorem 2.1. *Let us put*

$$H(z) = \frac{1+Az}{1-Bz}. \tag{2.1}$$

Then

$$D_z = H(\overline{K}_{r^n}). \tag{2.2}$$

Proof. By the definition of the class $P_n(A, B)$ we have $p(z) = H(\omega(z))$,

$$\omega(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots \in \Omega. \tag{2.3}$$

In view of the general Schwarz lemma $|\omega(z)| \leqslant |z|^n$ and therefore $p(z) = H(\omega(z)) = H(\zeta)$ for some ζ, $|\zeta| \leqslant r^n$. This implies that $D_z \subset H(\overline{K}_{r^n})$. On the other hand, the function

$$p(z) = H(\eta z^n) \tag{2.4}$$

belongs to the class $P_n(A, B)$ for every η, $|\eta| \leqslant 1$, and therefore $H(\overline{K}_{r^n}) \subset D_z$. Hence the equality (2.2) holds.

Remark. Theorem 2.1 remains true if the function H is replaced by any arbitrary regular and univalent function F and $P_n(A, B)$ is replaced by a class $\{p\colon p(z) = c_0 + c_n z^n + c_{n+1} z^{n+1} + \ldots \prec F(z)\}$.

The theorem 2.1 may be written in the following form:

Theorem 2.2. *If* $p \in P_n(A, B)$ *and* $|z| = r < 1$, *then*

$$|p(z) - s| \leqslant R, \text{ where}$$

$$s = \frac{1 + A\overline{B} r^{2n}}{1 - |B|^2 r^{2n}}, \quad R = \frac{|A+B| r^n}{1 - |B|^2 r^{2n}}. \tag{2.5}$$

This follows from the principle of subordination (see [10], p. 164) by the fact $H(\overline{K}_{r^n}) = \{w\colon |w - s| \leqslant R\}$.

Thus we have

Theorem 2.3. *If* $p \in P_n(A, B)$ *and* $|z| = r < 1$, *then*

$$\frac{1-|A+B|r^n+\frac{1}{2}(A\bar{B}+\bar{A}B)r^{2n}}{1-|B|^2r^{2n}} \leqslant \operatorname{re} p(z) \leqslant \frac{1+|A+B|r^n+\frac{1}{2}(A\bar{B}+\bar{A}B)r^{2n}}{1-|B|^2r^{2n}},$$

$$\frac{|1+A\bar{B}r^{2n}|-|A+B|r^n}{1-|B|^2r^{2n}} \leqslant |p(z)| \leqslant \frac{|1+ABr^{2n}|+|A+B|r^n}{1-|B|^2r^{2n}},$$

$$\frac{\frac{1}{2i}(A\bar{B}-\bar{A}B)r^{2n}-|A+B|r^n}{1-|B|^2r^{2n}} \leqslant \operatorname{im} p(z) \leqslant \frac{\frac{1}{2i}(A\bar{B}-\bar{A}B)r^{2n}+|A+B|r^n}{1-|B|^2r^{2n}},$$

$$\operatorname{arctg}\frac{\operatorname{im}(A\bar{B})r^{2n}}{1+r^{2n}\operatorname{re}(A\bar{B})} - \arcsin\frac{|A+B|r^n}{|1+A\bar{B}r^{2n}|} \leqslant \arg p(z)$$

$$\leqslant \operatorname{arctg}\frac{\operatorname{im}(A\bar{B})r^{2n}}{1+r^{2n}\operatorname{re}(A\bar{B})} + \arcsin\frac{|A+B|r^n}{|1+A\bar{B}r^{2n}|}.$$

Now $H(K_1)$ is a convex domain; according to the coefficient theorem for a function which is subordinate to a convex function (see Golusin [3], p. 326) we have

Theorem 2.4. *If* $p(z) = 1+c_nz^n+c_{n+1}z^{n+1}+\ldots \in P_n(A, B)$, *then*

(2.6) $$|c_k| \leqslant |A+B|, \ k = n, n+1, \ldots$$

Equality in (2.6) *holds for the functions* $p(z) = H(\eta z^n)$, $|\eta| = 1$.

Proof. The function

$$H(z) = 1+(A+B)z+\ldots$$

is a convex function and the inequality (2.5) is a consequence of Theorem 5 ([3], p. 326).

Theorem 2.5. *If* $p(z) = 1+c_nz^n+c_{n+1}z^{n+1}+\ldots \in P_n(A, B)$ *and* $|B| < 1$, *then*

(2.7) $$\sum_{k=n}^{N}|c_k|^2 \leqslant |A+B|^2\frac{1-|B|^{2N}}{1-|B|^2}, \ N = n, n+1, n+2, \ldots.$$

In particular

(2.8) $$\sum_{k=n}^{\infty}|c_k|^2 \leqslant \frac{|A+B|^2}{1-|B|^2}.$$

The estimation (2.7) *is sharp for* $n = 1$ *only, and the estimation* (2.8) *is sharp for every natural* n. *The extremal function has a form* $p(z) = H(\eta z^n)$, $|\eta| = 1$.

Proof. The function H has a form:

$$H(z) = 1+\sum_{k=1}^{\infty} a_k z^k, \text{ where } a_k = (A+B)B^{k-1}.$$

Then $\sum_{k=1}^{n} |a_k|^2 = |A+B|^2 \frac{1-|B|^{2N}}{1-|B|^2}$ and the inequality (2.7) is a consequence of the Theorem 215 (Littlewood [10], p. 168). The inequality (2.8) can be obtained from (2.7) letting $N\to\infty$.

3. Now we give some distortion theorems.

Theorem 3.1. *If* $p \in P_n(A, B)$, *then*

$$(3.1)\qquad |p'(z)| \leqslant \begin{cases} \dfrac{n|A+B||z|^{n-1}}{(1-|B||z|^n)^2}, & \text{for } |z| \leqslant \sqrt[n]{|B|}, \\ \dfrac{n|A+B||z|^{n-1}}{(1-|B|^2)(1-|z|^{2n})}, & \text{for } \sqrt[n]{|B|} < |z|. \end{cases}$$

For $|z| \leqslant \sqrt[n]{|B|}$ *estimate* (3.1) *is sharp.*

Theorem 3.2. *If* $p \in P_n(A, B)$ *and* $A+B \neq 0$, *then*

$$(3.2)\qquad |p'(z)| \leqslant \begin{cases} \dfrac{n|z|^{n-1}}{|A+B|}\,|A+Bp(z)|^2, & \text{for } |z| \leqslant \varrho_0, \\ \dfrac{|z|^{n-2}[4|z|^2+n^2(1-|z|^2)^2]}{4|A+B|(1-|z|^2)}\,|A+Bp(z)|^2, & \text{for } \varrho_0 < |z|, \end{cases}$$

where $\varrho_0 = (\sqrt{1+n^2}-1)n^{-1}$.
If $A+B = 0$, *then* $p'(z) = 0$. *The estimation* (3.2) *is sharp.*

Proof of Theorem 3.1. In view of the definition of $P_n(A, B)$ we have

$$(3.3)\qquad p(z) = \frac{1+A\omega(z)}{1-B\omega(z)}, \text{ where } |\omega(z)| \leqslant |z|^n, \ (z \in K_1).$$

Hence

$$p'(z) = \frac{(A+B)\omega'(z)}{[1-B\omega(z)]^2}.$$

Using some generalization of Schwarz's Lemma (see [3], p. 290), we obtain the following estimation:

$$(3.4)\qquad |p'(z)| \leqslant \frac{n|A+B||z|^{n-1}}{1-|z|^{2n}} \cdot \frac{1-|\omega(z)|^2}{1-|B||\omega(z)|}.$$

Now, the function $\Phi(x) = \dfrac{1-x^2}{(1-|B|x)^2}$, $0 \leqslant x < 1$, increases in the interval $[0, |B|]$ and decreases in the interval $[|B|, 1)$. Thus for $|z| \leqslant \sqrt[n]{|B|}$, the expression $\Phi(|\omega(z)|)$ attains its maximum with respect to ω at the point $|\omega(z)| = |z|^n$. This proves the inequality (3.1) for $|z| \leqslant \sqrt[n]{|B|}$. If $\sqrt[n]{|B|} \leqslant |z|$, then the expression $\Phi(|\omega(z)|)$ attains its maximum at the point $|\omega(z)| = |B|$. From this we have the second part of (3.1). The function $p(z) = H(\eta z^n)$ with $\eta = \dfrac{|Bz_0^n|}{Bz_0^n}$, gives equality in (3.1) for $z = z_0$, $|z_0| \leqslant \sqrt[n]{|B|}$.

Proof of Theorem 3.2. From the identity (3.3) we calculate $\omega(z) = (p(z)-1)(A+Bp(z))^{-1}$. Hence

$$\omega'(z) = \frac{(A+B)p'(z)}{\big(A+Bp(z)\big)^2}. \tag{3.5}$$

From (3.5) we have

$$|p'(z)| = \frac{|\omega'(z)|}{|A+B|}|A+Bp(z)|^2. \tag{3.6}$$

Using the estimates of $|\omega'(z)|$ (see [12]), together with (3.6) we obtain the inequality (3.2). The equality takes place in (3.2) for the function $p(z) = H\big(\omega(z)\big)$ where $\omega(z)$ is the function realizing maximum of $|\omega'(z)|$. (see [12]).

4. Let us denote by $R_n(A, B)$, $n \in N$, $|A| \leqslant 1$, $|B| \leqslant 1$, the class of functions

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots, \quad z \in K_1, \tag{4.1}$$

such that

$$f'(z) \in P_n(A, B). \tag{4.2}$$

If $n = 1$, $A = B = 1$, then this class is the class of univalent functions with bounded rotation. In general case $R_n(A, B)$ is a subclass of univalent functions.

Remark 1. Because of the condition (4.2) we can obtain the estimations of $|f'(z)|$, $\mathrm{re} f'(z)$, $\mathrm{im} f'(z)$ immediately from the Theorem 2.3, putting $p(z) = f'(z)$.

Remark 2. From the estimations of $|f'(z)|$, where $f \in R_n(A, B)$ we obtain the following estimates

$$\int_0^{|z|} \frac{|1+A\bar{B}r^{2n}|-|A+B|r^n}{1-|B|^2r^{2n}}\,dr \leqslant |f(z)| \leqslant \int_0^{|z|} \frac{|1+A\bar{B}r^{2n}|+|A+B|r^n}{1-|B|^2r^{2n}}\,dr.$$

Theorem 4.1. *If* $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots \in R_n(A, B)$, *then*

$$|a_k| \leqslant \frac{|A+B|}{k}, \quad k = n+1, n+2, \ldots. \tag{4.3}$$

Furthermore, if $|B| < 1$, *then*

$$\sum_{k=n+1}^{N+1} k^2|a_k|^2 \leqslant |A+B|^2 \frac{1-|B|^{2N}}{1-|B|^2}, \quad N = n, n+1, n+2, \ldots, \tag{4.4}$$

$$\sum_{k=n+1}^{\infty} k^2|a_k|^2 \leqslant \frac{|A+B|^2}{1-|B|^2}. \tag{4.5}$$

The estimates (4.3) *and* (4.5) *are sharp. The estimate* (4.4) *is sharp for* $n = 1$ *only.*

Proof. If $f \in R_n(A, B)$ then $f'(z) = p(z) \in P_n(A, B)$. Hence $ka_k = c_{k-1}$ and (4.3), (4.4), (4.5) follows from (2.6), (2.7) and (2.8) resp.

5. Let $\sum_n^*(A, B)$ denote the class of meromorphic functions

$$f(z) = \frac{1}{z} + b_{n-1}z^{n-1} + b_nz^n + \ldots, \quad 0 < |z| < 1, \tag{5.1}$$

such that

$$-\frac{zf'(z)}{f(z)} = p(z) \in P_n(A, B). \tag{5.2}$$

The class $\sum_1^*(1, 1) = \sum^*$ is well known class of starlike meromorphic functions. From the definition of the class $\sum_n^*(A, B)$ it follows immediately that the region of variability of the functional $-zf'(z)/f(z)$ (z being fixed) is the disc $|w-s| \leqslant R$, where s and R have been given by (2.5). Now, we shall find the estimations of coefficients in the class $\sum_n^*(A, B)$.

Theorem 5.1. *If* $f(z)$ *belongs to the class* $\sum_n^*(A, B)$ *and has the form* (5.1), *then*

$$|b_m| \leqslant \frac{|A+B|}{m+1}, \quad m = n-1, n, n+1, \ldots. \tag{5.3}$$

The extremal functions are the solutions of the equation

$$-\frac{zf'(z)}{f(z)} = H(\eta z^n),\ |\eta| = 1, \tag{5.4}$$

where the function H is defined by the formula (2.1).

Proof. From the definition of the class $\sum_n^*(A, B)$ we have

$$-\frac{zf'(z)}{f(z)} = \frac{1+A\omega(z)}{1-B\omega(z)}, \tag{5.5}$$

where $\omega(z)$ satisfies the condition (2.3). Now, the condition (5.5) after simple calculations takes a form

$$[Bzf'(z)-Af(z)]\omega(z) = f(z)+zf'(z). \tag{5.6}$$

Using (2.3) and (5.1) we obtain the equality

$$\Big[-(A+B)+\sum_{k=n-1}^{\infty}(kB-A)b_k z^{k+1}\Big]\Big(\sum_{k=n}^{\infty}a_k z^k\Big) = \sum_{k=n-1}^{\infty}(k+1)b_k z^{k+1}. \tag{5.7}$$

Comparing coefficients of both sides (5.7) we have

$$(m+1)b_m = -(A+B)a_{m+1},\ m = n-1, n, n+1, \ldots, 2n-2. \tag{5.8}$$

Because $|a_k| \leqslant 1$ for all k, then from (5.8) the estimate (5.3) holds for $m = n-1, n, n+1, \ldots, 2n-2$.

For every $m \geqslant 2n-1$, the equality (5.7) may be written in the following form

(5.9)

$$\Big[-(A+B)+\sum_{k=n-1}^{m-n}(kB-A)b_k z^{k+1}\Big]\omega(z) = \sum_{k=n-1}^{m}(k+1)b_k z^{k+1}+\sum_{k=m+2}^{\infty}c_k z^k,$$

where

$$\sum_{k=m+2}^{\infty}c_k z^k = \sum_{k=m+1}^{\infty}(k+1)b_k z^{k+1} - \omega(z)\sum_{k=m-n+1}^{\infty}(kB-A)b_k z^{k+1}.$$

Now, we shall use an analogous reasoning as in Clunie paper [2]. Integrating the squares of modulus the both sides of equality (5.9) along the circle $|z| = r$, $0 < r < 1$, and using the inequality $|\omega(z)| \leqslant 1$, we obtain (for $r \to 1$) the inequality

$$|A+B|^2+\sum_{k=n-1}^{m-n}|kB-A|^2|b_k|^2 \geqslant \sum_{k=n-1}^{m}(k+1)^2|b_k|^2+\sum_{k=m+2}^{\infty}|c_k|^2. \tag{5.10}$$

Ignoring the last sum in (5.10) we obtain after some calculations

$$(5.11)\quad (m+1)^2|b_m|^2$$
$$\leqslant |A+B|^2 - \sum_{k=n-1}^{m-n} \big((k+1)^2 - |kB-A|^2\big)|b_k|^2 - \sum_{k=m-n+1}^{m-1} (k+1)^2|b_k|^2.$$

In view of inequalities $|A| \leqslant 1$, $|B| \leqslant 1$ we have $|kB-A| \leqslant k+1$. Therefore both sums in (5.11) are nonnegative and we can drop them. Then

$$(5.12)\qquad (m+1)^2|b_m|^2 \leqslant |A+B|^2$$

holds. This implies the inequality (5.3) for $m = 2n-1, 2n, 2n+1, \ldots$. The proof is complete.

It is easy to see that the function f which satisfies the condition (5.4) has the form

$$f(z) = \frac{1}{z} + \frac{A+B}{m+1}\eta z^m + \ldots .$$

Thus $|b_m| = \dfrac{|A+B|}{m+1}$.

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STRESZCZENIE

W tej pracy zbadano niektóre funkcjonały (rzeczywiste i zespolone) określone na klasie funkcji

$$P_n(A, B) = \left\{p:\ p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots \dashv \frac{1+Az}{1-Bz},\ |z| < 1\right\},$$

gdzie n jest dowolną ustaloną liczbą naturalną, zaś A, B są to ustalone liczby zespolone, takie że $|A| < 1$, $|B| < 1$.

Następnie przedmiotem rozważań jest klasa funkcji f regularnych w kole $|z| < 1$, odpowiednio unormowanych i takich że $f' \in P_n(A, B)$.

W ostatnim paragrafie wykazano następujące twierdzenie:

Jeśli funkcja meromorficzna f ma rozwinięcie

$$f(z) = \frac{1}{z} + b_{n-1} z^{n-1} + b_n z^n + \ldots,\ 0 < |z| < 1,$$

oraz

$$-\frac{zf'(z)}{f(z)} \in P_n(A, B),$$

to

$$|b_m| \le \frac{|A+B|}{m+1},\ m = n-1, n, n+1, \ldots .$$

Przez odpowiedni dobór parametrów n, A, B wyniki niniejszej pracy pozwalają uzyskać znane twierdzenia w teorii funkcji analitycznych.

РЕЗЮМЕ

В работе изучаются некоторые функционалы (действительные и комплексные), определенные в классе функций

$$P_n(A, B) = \left\{p:\ p(z) = 1 + c_n z^n + \ldots \dashv \frac{1+Az}{1-Bz},\ |z| < 1\right\}$$

где n — произвольное, фиксированное натуральное число, A, B — фиксированные комплексные числа, $|A| < 1$, $|B| < 1$.

Следующим предметом рассуждений является класс функций f, регулярных в круге $|z| < 1$ и соответственно нормированных, таких, что $f' \in P_n(A, B)$.

Доказывается также следующая теорема. *Если мероморфная функция имеет развертку вида*

$$f(z) = \frac{1}{z} + b_{n-1} z^{n-1} + b_n z^n + \ldots,\ 0 < |z| < 1$$

и

$$-\frac{zf'(z)}{f(z)} \in P_n(A, B)$$

то

$$|b_m| \leqslant \frac{|A+B|}{m+1}, \quad m = n-1, n, n+1, \ldots$$

Соответственным подбором параметров n, A, B результаты настоящей работы дают возможность получения известных в теории аналитических функций теорем.