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Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

ZDZISŁAW RYCHLIK

The Convergence of Rosen's Series for the Sums of a Random Number of Independent Random Variables

O zbieżności szeregów Rosena dla sum niezależnych zmiennych losowych z losową liczbą składników

О сходимости рядов Розена для сумм случайного числа независимых случайных величин

1. Introduction

In the present paper we shall give an extension of B. Rosen's theorems [6] to the sums of a random number of independent nonidentically distributed random variables. Some generalizations of his results may by found in [1], [2], [3] and [4]. The results given in this paper are extensions or generalizations of results of the above-mentioned papers.

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with corresponding characteristic functions $\{\varphi_n(t), n \geqslant 1\}$ and let $S_n = \sum_{k=1}^{n} X_k$.

In the following by N we shall denote a positive integer-valued random variable which has the distribution function dependent on a parameter $\lambda (\lambda > 0)$ i.e. $P[N = n] = p_n$ $(n = 1, 2, \ldots)$, where the p_n are functions of λ such that for all λ, $p_n \geqslant 0$ and $\sum_{n=1}^{\infty} p_n = 1$. We assume that the random variables $N, X_1, X_2, \ldots$, are independent, and $\alpha = EN$, $\gamma^2 = \sigma^2 N$ exist for all λ.

Under the above-mentioned conditions and notations the distribution function $F_\lambda(x)$ and the characteristic function $\varphi_\lambda(t)$ of the random variaple

$$S_N = X_1 + X_2 + \ldots + X_N$$

depend on the parameter λ and

$$F_\lambda(x) = \sum_{n=1}^{\infty} p_n P[S_n < x],$$

$$\varphi_\lambda(t) = \sum_{n=1}^{\infty} p_n \prod_{k=1}^{n} \varphi_k(t).$$

In what follows absolute, in general different, positive constants will be denoted by C. Further on, let I_λ be an interval on the x-axis and let $\mu(I_\lambda)$ be its length.

2. Upper bounds for the probabilities $P[S_N \in I_\lambda]$

In this Section we give upper bounds for the probabilities $P[S_N \in I_\lambda]$ for some different types of interval families.

Definition 1. A sequence $[X_n, n \geq 1]$ of independent random variables is said to satisfy the condition (**A**), if there exist some constants $\delta_0 > 0$, n_0 and a function $g(n)$ such that for every $n \geq n_0$

$$\int_{|t| \leq \delta_0} \prod_{k=1}^{n} |\varphi_k(t)|\, dt \leq C_0 / g(n), \tag{1}$$

where C_0 is a constant not depending on n, and $g(n) \to \infty$ as $n \to \infty$.

One can observe that a sequence of independent random variables normally distributed with standard deviation σ_k such that $\sum_{k=1}^{n} \sigma_k^2 \to \infty$, when $n \to \infty$, satisfies the condition (**A**) with $g(n) = (\sum_{k=1}^{n} \sigma_k^2)^{1/2}$. By Lemma 1 of [6], we see that any sequence of independent nondegenerate identically distributed random variables satisfies the condition (**A**) with the function $g(n) = n^{1/2}$. The same fact concerns the random variables considered by L. H. Koopmans [3] and by C. C. Heyde [2]. Another example of random variables which satisfy (1) can be found in [5].

The following Theorem is an extension of Theorem 1 [6].

Theorem 1. *Let $[X_n, n \geq 1]$ be a sequence of independent random variables satisfying the condition* (**A**). *If $a \to \infty$ as $\lambda \to \infty$, then*

(a) *if $\mu(I_\lambda) \leq [g(a/2)]^{2p}$, $0 < p < 1/2$, then*

$$P[S_N \in I_\lambda] \leq C[g(a/2)]^{2p-1}[1 + \gamma^2 g(a/2)/a^2],$$

(b) *if $\mu(I_\lambda) \leq \varepsilon g(a/2)$, $\varepsilon > 0$, then*

$$P[S_N \in I_\lambda] \leq C\varepsilon\{1 + \gamma^2 g(a/2)/a^2 + \eta(\varepsilon, \lambda)\},$$

where for every fixed $\varepsilon > 0$ $\eta(\varepsilon, \lambda) \to \infty$ *as* $\lambda \to \infty$,

(c) *if* $\mu(I_\lambda) \leqslant \text{const.}$, *then*

$$P[S_N \in I_\lambda] \leqslant C\{\gamma^2/a^2 + 1/g(a/2)\},$$

(d) $\max_x P[S_N = x] \leqslant C\{\gamma^2/a^2 + 1/g(a/2)\}$.

C is a constant independent of λ *and* I_λ.

Proof. Let $f_\lambda(t)$ and $h_\lambda(x)$ be the functions such that

$$\int_{-\infty}^{\infty} |f_\lambda(t)|\,dt < \infty, \quad |f_\lambda(t)| \leqslant 1, \tag{2}$$

$$h_\lambda(x) = \int_{-\infty}^{\infty} e^{itx} f_\lambda(t)\,dt \geqslant 0. \tag{3}$$

If $F_\lambda(x)$ is the distribution function of the random variable S_N, then

$$\int_{-\infty}^{\infty} h_\lambda(x)\,dF_\lambda(x) = \int_{-\infty}^{\infty} f_\lambda(t)\varphi_\lambda(t)\,dt,$$

where $\varphi_\lambda(t)$ is the characteristic function of S_N. But

$$\int_{-\infty}^{\infty} h_\lambda(x)\,dF_\lambda(x) \geqslant \min_{x\in I_\lambda} h_\lambda(x) \int_{I_\lambda} dF_\lambda(x),$$

hence, by the simple calculations

$$P[S_N \in I_\lambda] \leqslant \{\min_{x\in I_\lambda} h_\lambda(x)\}^{-1} \Big\{ \int_{|t|\leqslant\delta_0} |\varphi_\lambda(t)|\,dt + \int_{|t|>\delta_0} |f_\lambda(t)|\,dt \Big\}.$$

On the other hand, because of $P[N \leqslant a/2] \leqslant 4\gamma^2/a^2$, we have

$$|\varphi_\lambda(t)| \leqslant \gamma^2/a^2 + \sum_{n\geqslant a/2} p_n \prod_{k=1}^{n} |\varphi_k(t)|.$$

Thus

$$P[S_N \in I_\lambda] \leqslant \{\min_{x\in I_\lambda} h_\lambda(x)\}^{-1} \Big\{ 8\delta_0\gamma^2/a^2 + \int_{|t|>\delta_0} |f_\lambda(t)|\,dt + \sum_{n\geqslant a/2} p_n \int_{|t|>\delta_0} \prod_{k=1}^{n} |\varphi_n(t)|\,dt \Big\}$$

holds.

Now let us choose λ_0 so that for all $\lambda > \lambda_0$ $a/2 \geqslant n_0$. Then by our assumptions

$$\sum_{n\geqslant a/2} p_n \int_{|t|\leqslant\delta_0} \prod_{k=1}^{n} |\varphi_k(t)|\,dt \leqslant C/g(a/2)$$

holds for every $\lambda > \lambda_0$. Hence

(4) $$P[S_N \in I_\lambda] \leqslant C\{\min_{x \in I_\lambda} h_\lambda(x)\}^{-1}\Big\{\gamma^2/a^2 + 1/g(a/2) + \int_{|t|>\delta_0} |f_\lambda(t)|\,dt\Big\}.$$

To prove (a) we choose

$$h_\lambda(x) = \sqrt{2\pi}\exp\{-(x-\mu_\lambda)^2/2[g(a/2)]^{4p}\}/[g(a/2)]^{2p}$$

and

$$f_\lambda(t) = \exp\{-\tfrac{1}{2}t^2[g(a/2)]^{4p} - i\mu_\lambda t\},$$

where μ_λ is the midpoint of I_λ. It is easy to verify that $f_\lambda(t)$ and $h_\lambda(x)$ are functions satisfying the conditions (2) and (3). Furthermore, we have

$$\min_{x \in I_\lambda} h_\lambda(x) \geqslant \sqrt{2\pi}\exp(-1/8)/[g(a/2)]^{2p}$$

and

$$\int_{|t|>\delta_0} |f_\lambda(t)|\,dt \leqslant C_1/g(a/2),$$

where C_1 is a constant independent of λ and I_λ. Thus, the last two inequalities and (4) prove (a).

In case (b) we choose

$$h_\lambda(x) = \sqrt{2\pi}\exp\{-(x-\mu_\lambda)^2/2\varepsilon^2[g(a/2)]^2\}/\varepsilon g(a/2),$$

$$f_\lambda(t) = \exp\{-\tfrac{1}{2}t^2\varepsilon^2[g(a/2)]^2 - i\mu_\lambda t\}.$$

Obviously these functions satisfy (2) and (3). But we have

$$\min_{x \in I_\lambda} h_\lambda(x) \geqslant \exp(-1/8)\sqrt{2\pi}/\varepsilon g(a/2).$$

Thus (4) gives

$$P[S_N \in I_\lambda] \leqslant C\varepsilon\{1 + \gamma^2 g(a/2)/a^2 + \eta(\varepsilon, \lambda)\},$$

where

$$\eta(\varepsilon, \lambda) = g(a/2)\int_{|t|>\delta_0} \exp\{-\tfrac{1}{2}t^2\varepsilon^2[g(a/2)]^2\}\,dt.$$

It is easy to see that for every fixed $\varepsilon > 0$

$$\lim_{\lambda\to\infty}\eta(\varepsilon, \lambda) = \lim_{\lambda\to\infty}\frac{1}{\varepsilon}\int_{|z|>\delta_0\varepsilon g(a/2)} \exp(-z^2/2)\,dz = 0,$$

which proves (b).

In order to prove (c), let us put

$$h_\lambda(x) = \delta\left(\frac{\sin\frac{1}{2}\delta(x-\mu_\lambda)}{\frac{1}{2}\delta(x-\mu_\lambda)}\right)^2,$$

$$f_\lambda(t) = \begin{cases}(1-|t/\delta|)\exp(i\mu_\lambda t) & \text{for } |t| \leqslant \delta,\\ 0 & \text{for } |t| > \delta,\end{cases}$$

where δ is chosen so that $\delta \leqslant \delta_0$, and $M \leqslant 2\pi/\delta$ which assures that

$$\min_{x \in I_\lambda} h_\lambda(x) \geqslant \varrho > 0 \quad \text{and} \quad \int_{|t|>\delta_0} |f_\lambda(t)|\,dt = 0.$$

Therefore (4) gives (c).

The statement (d) immediately follows from (c).

Corollary 1. *Let $[X_n, n \geqslant 1]$ be a sequence of independent random variables satisfying the condition* (**A**) *with $g(n) = \sqrt{n}$. If $a \to \infty$, $\gamma = 0(a^{3/4})$ as $\lambda \to \infty$, then*

(a_1) *if* $\mu(I_\lambda) \leqslant a^p$, $0 < p < 1/2$, *then*

$$P[S_N \in I_\lambda] \leqslant C a^{p-1/2},$$

(b_1) *if* $\mu(I_\lambda) \leqslant \varepsilon a^{1/2}$, $\varepsilon > 0$, *then*

$$P[S_N \in I_\lambda] \leqslant C\varepsilon\{1 + \eta(\varepsilon, \lambda)\},$$

where for every fixed $\varepsilon > 0$, $\eta(\varepsilon, \lambda) \to 0$ *as* $\lambda \to \infty$,

(c_1) *if* $\mu(I_\lambda) \leqslant M = \text{const.}$, *then* $P[S_N \in I_\lambda] \leqslant C a^{-1/2}$,

(d_1) $\max_X P[S_N = x] \leqslant C a^{-1/2}$.

C is a constant independent of λ and I_λ.

B. Rosen in [6] has proved that if $F(x)$ is a distribution function and $\varphi(t)$ its characteristic function, then

$$(5)\quad \tfrac{1}{2}[F(x+0) + F(x-0)] = \frac{1}{2} + \frac{1}{2\pi i}\int_0^\delta t^{-1}\{e^{itx}\varphi(-t) - e^{-itx}\varphi(t)\}\,dt + \\ + \frac{1}{\pi}\int_{-\infty}^{\infty} dF(y)\int_\delta^\infty [\sin(x-y)t/t]\,dt$$

provided $\int_{-\infty}^{\infty} (1+|x|)\,dF(x) < \infty$.

Thus, using the equality (5) to the distribution function $F_\lambda(x)$ with characteristic function $\varphi_\lambda(t)$, we obtain

$$(6)\qquad [F(x+0) + F(x-0)]/2 = 1/2 + \frac{1}{2\pi i}\int_0^\delta t^{-1}\{e^{itx}\varphi_\lambda(-t) - \\ - e^{-itx}\varphi_\lambda(t)\}\,dt + R_\lambda(x, \delta),$$

where δ is a positive number and

$$R_\lambda(x, \delta) = \frac{1}{\pi}\int_{-\infty}^{\infty} dF_\lambda(y)\int_\delta^\infty [\sin(x-y)t/t]\,dt.$$

Lemma. *Let $[X_n, n \geqslant 1]$ be a sequence of independent random variables satisfying the condition* (**A**). *If $a \to \infty$ as $\lambda \to \infty$, then for every ε, $0 < \varepsilon < 1/2$, a constant C exists, independent of x and λ, such that*

$$(7)\qquad |R_\lambda(x, \delta)| \leqslant C[g(a/2]^{2\varepsilon-1}\{1+\gamma^2 g(a/2)/a^2\}.$$

Proof. By definition $R_\lambda(x, \delta)$, we have

$$\pi|R_\lambda(x,\delta)| \leqslant \int_{-\infty}^{\infty} dF_\lambda(y)\left|\int_\delta^\infty [\sin(x-y)t/t]\,dt\right|$$

$$= \int_{|x-y|\leqslant[g(a/2)]^{2\varepsilon}} \left|\int_\delta^\infty [\sin(x-y)t/t]\,dt\right| dF_\lambda(y) +$$

$$+ \sum_{j\leqslant[g(a/2)]^2} \int_{Bj} \left|\int_\delta^\infty [\sin(x-y)t/t]\,dt\right| dF_\lambda(y) +$$

$$+ \int_{|x-y|>[g(a/2)]^{2\varepsilon+2}} \left|\int_\delta^\infty [\sin(x-y)t/t]\,dt\right| dF_\lambda(y) = I_1 + I_2 + I_3$$

where $B_j = \{y\colon j[g(a/2])^{2\varepsilon} < |x-y| \leqslant (j+1)[g(a/2)]^{2\varepsilon}\}$.

Now using Theorem 1 (a) and the fact that $\left|\int_\delta^\infty (\sin ut/t)\,dt\right| \leqslant C_1$ ($C_1 = \text{const.}$), we get

$$I_1 \leqslant C[g(a/2)^{2\varepsilon-1}[1+\gamma^2 g(a/2)/a^2].$$

Further on, form Theorem 1 (a) and by the bounded

$$\left|\int_\delta^\infty (\sin ut/t)\,dt\right| \leqslant C_2/\delta|u| \quad (C_2 = \text{const.}),$$

we have

$$I_2 \leqslant C' \sum_{j\leqslant[g(a/2)]^2} \int_{B_j} |x-y|\,dF_\lambda(y) \leqslant C''[1/g(a/2)+\gamma^2/a^2] \sum_{j\leqslant[g(a/2)]^1} 1/j \leqslant$$

$$\leqslant C[g(a/2)]^{2\varepsilon-1}[1+\gamma^2 g(a/2)/a^2],$$

what with the following inequality

$$I_3 \leqslant C \int_{|x-y|\geqslant[g(a/2)]^{2\varepsilon+2}} \frac{1}{|x-y|}\,dF_\lambda(y) \leqslant C[g(a/2)]^{-2\varepsilon-2}$$

ends the proof of the Lemma.

3. The Convergence of Rosen's Series for the Sums of a Random Number of Independent Random Variables

We shall now assume that the parameter λ belongs to the set of positive integers. Thus we shall consider the sequences $\{N_n, n \geqslant 1\}$ of a positive integer-valued random variables independent of X_n, $n = 1, 2, \ldots$.

Let us put $p_k(n) = P[N_n = k]$, $EN_n = a_n$, $\gamma_n^2 = \sigma^2 N_n$, and let $F_k(x)$ be the distribution function of X_k $k = 1, 2, \ldots$.

The following three theorems constitute some generalizations or extensions of the results given in [7], [6], [2], [3] and [4].

Theorem 2. *Let $|x|^{2+r}$ be uniformly integrable with respect to F_k, $k = 1, 2, \ldots$, for some r, $0 < r \leqslant 1$, and let $EX_k = 0$, $\sigma^2 X_k = \sigma_k^2 \geqslant \sigma_0^2 = \text{const} > 0$.*

If $\{N_n, n \geqslant 1\}$ is a sequence of a positive integer-valued random variables independent of X_n, $n = 1, 2, \ldots$ such that

$$\sum_{n=1}^{\infty} a^{-1-(r-s)/2} < \infty \text{ and } \gamma_n = O(a^{3/4}), \text{ where } 0 \leqslant s < r,$$

then

$$\sum_{n=1}^{\infty} a_n^{-1+s/2} |P[S_{N_n} < 0] - 1/2| < \infty. \tag{7}$$

Proof. Let $\varphi_{N_n}(t)$ denote the characteristic function of the random variable S_{N_n}. Putting $x = 0$ in (6), we get

$$P[S_{N_n} < 0] - 1/2 = \frac{1}{2}[F_{N_n}(0+) + F_{N_n}(0-)] - P[S_{N_n} = 0]/2 - 1/2$$

$$= \frac{1}{2\pi i}\int_0^{\delta} t^{-1}[\varphi_{N_n}(-t) - \varphi_{N_n}(t)]\,dt - \frac{1}{2}P[S_{N_n} = 0] + R_{N_n}(0, \delta),$$

where δ is a positive number, $F_{N_n}(x) = P[S_{N_n} < x]$, and

$$R_{N_n}(0, \delta) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dF_{N_n}(y) \int_{\delta}^{\infty} (\sin yt/t)\,dt.$$

Hence, we get

$$\sum_{n=1}^{\infty} a_n^{-1+s/2}\left|P[S_{N_n} < 0] - \frac{1}{2}\right| \leqslant \frac{1}{2\pi}\sum_{n=1}^{\infty} a_n^{-1+s/2}\left|\int_0^{\delta} t^{-1}[\varphi_{N_n}(-t) - \right. \tag{8}$$

$$\left. - \varphi_{N_n}(t)]\,dt\right| + \frac{1}{2}\sum_{n=1}^{\infty} a_n^{-1+s/2} P(S_{N_n} = 0] + \sum_{n=1}^{\infty} a_n^{-1+s/2} |R_{N_n}(0, \delta)|.$$

From Lemma 4 [3] it follows that there exist the positive constants $\delta_1 > 0$ and C, $0 < C < \infty$, such that for $|t| \leqslant \delta_1$

$$(9) \qquad |\varphi_k(t)| \leqslant 1 - Ct^2, \quad \text{uniformly in } k.$$

Thus the sequence $\{X_n, n \geqslant 1\}$ satisfies the condition (A) with the function $g(n) = \sqrt{n}(\delta_0 = \delta_1, n_0 = 1)$. Hence by Corollary 1 ($d_1$) $P[S_{N_n} = 0] \leqslant Ca_n^{-1/2}$ holds, and therefore

$$(10) \qquad \sum_{n=1}^{\infty} a_n^{-1+s/2} P[S_{N_n} = 0] \leqslant C \sum_{n=1}^{\infty} a_n^{-1-(1-s)/2} < \infty.$$

On the other hand, putting $x = 0$ and $\varepsilon < (1-r)/2$ in the Lemma, we get

$$(11) \qquad \sum_{n=1}^{\infty} a_n^{-1+s/2} |R_{N_n}(0, \delta)| \leqslant C \sum_{n=1}^{\infty} a^{-1-(r-s)/2} < \infty.$$

Now let us observe that

$$(12) \qquad \Big|\int_0^\delta t^{-1}[\varphi_{N_n}(-t) - \varphi_{N_n}^{(t)}]dt\Big| \leqslant \sum_{k \leqslant a_{n/2}} p_k(n) \int_0^\delta |t^{-1}| \prod_{j=1}^{k} |\varphi_j(t)| \times$$
$$\times \Big|\sin\Big(\sum_{j=1}^{k} \arg\varphi_j(t)\Big)\Big| dt + \sum_{k \geqslant a_{n/2}} p_k(n) \int_0^\delta |t^{-1}| \prod_{j=1}^{k} |\varphi_j(t)| \Big|\sin\Big(\sum_{j=1}^{k} \arg\varphi_j(t)\Big)\Big| dt.$$

It follows by Lemmas 2 and 5 [3] that there exists δ_2 such that for every $|t| \leqslant \delta_2$

$$\Big|\sin\Big(\sum_{j=1}^{k} \arg\varphi_j(t)\Big)\Big| \leqslant C_1 \sum_{j=1}^{k} |I_j(t)| \leqslant C_2 k |t|^{2+r}, \quad k = 1, 2, \ldots,$$

where $I(t)$ is the imaginary part of $\varphi_j(t)$, $j = 1, 2, \ldots$, and C_1, C_2 are positive constants independent of t and k.

We choose δ in (6) to be $\delta = \min(\delta_1, \delta_2)$, where δ_1 is as in (9). Then, we get

$$\Big|\int_0^\delta t^{-1}[\varphi_{N_n}(-t) - \varphi_{N_n}(t)]dt\Big| \leqslant C \sum_{k \leqslant a_{n/2}} kp_k(n) \int_0^\delta |t|^{1+r} \exp(-Ckt^2)dt +$$
$$+ C \sum_{k \geqslant a_{n/2}} kp_k(n) \int_0^\delta |t|^{1+r} \exp(-Ckt^2)dt.$$

Taking into account

$$(13) \qquad \int_0^\delta |t|^{1+r} \exp(-Ckt^2)dt \leqslant Ck^{-1-r/2},$$

we obtain

$$(14)\qquad \sum_{k\leqslant a_{n/2}} kp_k(n)\int_0^\delta |t|^{1+r}\exp(-Ckt^2)\,dt \leqslant CP[N_N\leqslant a_n/2]\leqslant Ca_n^{-1/2},$$

and

$$(15)\qquad \sum_{k\geqslant a_{n/2}} kp_k(n)\int_0^\delta |t|^{1+r}\exp(-Ckt^2)\,dt \leqslant Ca_n^{-r/2}.$$

Thus, because of (14) and (15), we have

$$\frac{1}{2\pi}\sum_{n=1}^{\infty} a_n^{-1+s/2}\left|\int_0^\delta t^{-1}[\varphi_{N_n}(-t)-\varphi_{N_n}(t)]\,dt\right|$$

$$\leqslant C\sum_{n=1}^{\infty} a_n^{-1-(1-s)/2}+C\sum_{n=1}^{\infty} a_n^{-1-(r-s)/2}<\infty.$$

The last inequality, (10) and (11) prove (7).

Theorem 3. *Let $\{X_n, n\geqslant 1\}$ be a sequence of independent random variables such that $EX_k=0$, $\sigma^2 X_k=\sigma_k^2\geqslant\sigma_0^2>0$, $k=1,2,\ldots$ and $|x|^{2+r}$ is uniformly integrable with respect to F_k, $k=1,2,\ldots$, for some r, $0<r\leqslant 1$.*

If $\{N_n, n\geqslant 1\}$ is a sequence of a positive integer-valued random variables independent of X_n, $n=1,2,\ldots$, such that

$$\gamma_n = 0(a_n^{3/4}) \quad \text{and} \quad \sum_{n=1}^{\infty} a_n^{-1-(r-s)/2}<\infty,\quad 0\leqslant s<r,$$

then for every p, $0\leqslant p<(1-s)/2$ and every x, $-\infty<x<\infty$,

$$\sum_{n=1}^{\infty} a_n^{-1+s/2}|P[S_{N_n}<a_n^p x]-1/2|<\infty.$$

Proof. Let us observe that

$$[S_{N_n}<a_n^p x]\subset\begin{cases}[S_{N_n}<x] & \text{if}\quad x<0,\\ [S_{N_n}<x]\cup[|S_{N_n}|<a_n^p x] & \text{if}\quad x\geqslant 0.\end{cases}$$

We see that $a_n\to\infty$ as $n\to\infty$. Hence for any ϱ, $p<\varrho<(1-r)/2$, there exists $n_0=n_0(\varrho, x)$ such that for $n\geqslant n_0$

$$[|S_{N_n}|<a_n^p x]\subset[|S_{N_n}|<a_n^\varrho/2].$$

Furthermore, it is easy to see, by the proof of Theorem 2, that the sequence $\{X_n, n\geqslant 1\}$ satisfies the condition (A) with the function $g(n)=\sqrt{n}$. Thus by Corollary 1 (a_1) we get

$$P[|S_{N_n}|<a_n^\varrho/2]\leqslant Ca_n^{\varrho-1/2}.$$

Hence for every x

$$\sum_{n=1}^{\infty} a_n^{-1+s/2} |P[S_{N_n} < a_n^{\varrho} x] - 1/2|$$

$$\leqslant \sum_{n=1}^{\infty} a_n^{-1+s/2} |P[S_{N_n} < 0] - 1/2| + C \sum_{n=1}^{\infty} a_n^{-1+s/2} P[|S_{N_n}| < a_n^{\varrho}/2].$$

By Theorem 2 the first series on the right hand side of the last inequality converges. On the other hand we have

$$\sum_{n=1}^{\infty} a_n^{-1+s/2} P[|S_{N_n}| < a_n^{\varrho}/2] \leqslant C \sum_{n=1}^{\infty} a_n^{\varrho+(s-3)/2} < \infty,$$

since $\varrho < (1-r)/2$. Thus Theorem 3 is proved.

Definition 2. If there exist a nondegenerate random variable with the characteristic function $\varphi(t)$ and constants C_1, $\delta' > 0$ and $\eta > 0$ such that

$$\max_k |\varphi_k(t)| \leqslant C1 - C_1 t^2) \quad \text{for} \quad |t| \leqslant \delta',$$

$$\max_k |I_k(t)| \leqslant |I(t) \quad \text{for} \quad |t| \leqslant \eta,$$

we shall say that the sequence $\{X_n, n \geqslant 1\}$ satisfies the condition (**B**).

Here, and in what follows $I(t)$ denotes imaginary part of $\varphi(t)$.

It is easy to see that the random variables considered in [2] satisfy the condition (B).

Theorem 4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables satisfying the condition* (**B**) *with a random variable X such that*

$$\int_{|x|>z} x^2 dP[X < x] = 0(z^{-r}), \quad 0 < r < 1.$$

If $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables independent of X_n, $n = 1, 2, \ldots$ such that $\gamma_n = 0(a_n^{3/4})$ and $\sum_{n=1}^{\infty} a_n^{-1-r/2} < \infty$, then

$$\sum_{n=1}^{\infty} a_n^{-1} |P[S_{N_n} < 0] - 1/2| < \infty.$$

Proof. In the same way as in the proof of Theorem 2 one can obtain the following inequality

$$(16) \quad \sum_{n=1}^{\infty} a_n^{-1} |P[S_{N_n} < 0] - 1/2| \leqslant \frac{1}{2} \sum_{n=1}^{\infty} P[S_{N_n} = 0] +$$

$$+ \sum_{n=1}^{\infty} a_n^{-1} |R_{N_n}(0, \delta)| + \frac{1}{2\pi} \sum_{n=1}^{\infty} a_n^{-1} \left| \int_0^{\delta} t^{-1} [\varphi_{N_n}(-t) - \varphi_{N_n}(t)] dt \right|,$$

where $R_{N_n}(0, \delta)$ is as in the above.

It can shown that if $\{X_n, n \geqslant 1\}$ satisfies the condition (**B**) then $\{X_n, n \geqslant 1\}$ satisfies (**A**) with $g(n) = \sqrt{n}$, by Corollary 1 (d_1). we get

$$\sum_{n=1}^{\infty} a_n^{-1} P[S_{N_n} = 0] \leqslant C \sum_{n=1}^{\infty} a_n^{-3/2} < \infty. \tag{17}$$

Moreover, it follows from the Lemma with ε chosen less than $(1-r)/2$ that

$$\sum_{n=1}^{\infty} a_n^{-1} |R_{N_n}(0, \delta)| \leqslant C \sum_{n=1}^{\infty} a_n^{-1-r/2} < \infty. \tag{18}$$

For the first series on the right hand side of (16) we have

$$\sum_{n=1}^{\infty} a_n^{-1} \Big| \int_0^{\delta} t^{-1} [\varphi_{N_n}(-t) - \varphi_{N_n}(t)] dt \Big| \tag{19}$$

$$\leqslant \sum_{n=1}^{\infty} a_n^{-1} \sum_{k \leqslant a_{n/2}} p_k(n) \int_0^{\delta} |t^{-1}| \prod_{j=1}^{k} |\varphi_j \cdot (t)| \sum_{j=1}^{k} |\arg \varphi_j \cdot (t)| dt +$$

$$+ \sum_{n=1}^{\infty} a_n^{-1} \sum_{k \geqslant a_{n/2}} p_k(n) \int_0^{\delta} |t^{-1}| \prod_{j=1}^{k} |\varphi_j \ (t)| \sum_{j=1}^{k} |\arg \varphi_j(t)| dt,$$

where δ is a positive constant to be determined later.

Now we can write

$$\varphi_j(t) = R_j(t) + iI_j(t),$$

where $R_j(t)$ and $I_j(t)$ are real functions, bounded on any finite interval. Thus, we have

$$\arg \varphi_j(t) = \operatorname{arctg}\{I_j(t)/R_j(t)\}.$$

But $R_j(t) = \{\varphi_j(t) + \varphi_j(-t)\}/2$ is itself a characteristic function and therefore it is continuous about $R_j(0) = 1$ in a neighbourhood of the origin. Therefore for every $\varepsilon > 0$ there exists $\delta_j > 0$ such that $|R_j(t) - 1| < \varepsilon$ in $|t| \leqslant \delta_j$. Choose $\delta'' = \min_j \delta_j$ (clearly $\delta'' > 0$). Then, uniformly in k for $|t| \leqslant \delta''$ we must have

$$|\arg \varphi_j(t)| \leqslant C |I_j(t)| \leqslant C |I(t)|,$$

where $I(t)$ is as in the condition (B). But by Lemma 2 [4] $|I(t)| = 0(|t|^{2+r})$, so

$$|\arg \varphi_j(t)| = 0(|t|^{2+r}), \; j = 1, 2, \ldots. \tag{20}$$

We choose δ in (19) to be $\delta = \min(\eta, \delta', \delta'')$. On the basis of (20) and (13) we get

$$(21)\qquad \sum_{k\leqslant a_{n/2}} p_k(n)\int_0^\delta |t^{-1}|\prod_{j=1}^{k}|\varphi_j(t)|\sum_{j=1}^{k}|\arg\varphi_j(t)|\,dt$$

$$\leqslant \sum_{k\leqslant a_{n/2}} kp_k(n)\int_0^\delta |t|^{1+r}\exp(-C_0kt^2)\,dt \leqslant Ca_n^{-1/2}$$

and

$$(22)\qquad \sum_{k\geqslant a_{n/2}} p_k(n)\int_0^\delta |t^{-1}|\prod_{j=1}^{k}|\varphi_j(t)|\sum_{j=1}^{k}|\arg\varphi_j(t)|\,dt$$

$$\leqslant C\sum_{k\geqslant a_{n/2}} k^{-r/2}p_k(n) \leqslant Ca_n^{-r/2}.$$

Thus from (19), (21) and (22) we obtain

$$\sum_{n=1}^{\infty} a_n^{-1}\left|\int_0^\delta t^{-1}[\varphi_{N_n}(-t)-\varphi_{N_n}(t)]\,dt\right|$$

$$\leqslant C\sum_{n=1}^{\infty} a_n^{-3/2} + C\sum_{n=1}^{\infty} a_n^{-1-r/2} < \infty,$$

and what with (16), (17) and (18) ends the proof.

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STRESZCZENIE

W pracy podano rozszerzenia twierdzeń Rosena [6] na przypadek sum niezależnych zmiennych losowych z losową liczbą składników. Otrzymane twierdzenia rozszerzają bądź uogólniają wyniki podane w pracach [1], [2], [3] i [4].

РЕЗЮМЕ

В работе получено расширения теорем Розена [6] на случай сумм случайного числа независимых случайных величин. Получены теоремы являются обобщениями либо расширениями задач исследованных в [1], [2], [3] и [4].