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Gamma-Starlike Functions

Funkcje gama-gwieździste

Гамма-звездообразные функции

1. In recent papers [3,4,5] several authors have investigated regular function $f(z)$, defined in the unit disc D, with the property that the real part of an arithmetic mean of the quantities $(zf'(z)/f(z))$ and $(1+zf''(z)/f'(z))$ is positive, i.e.

$$\operatorname{Re}\left[(1-a)z\frac{f'(z)}{f(z)}+a\left(1+z\frac{f''(z)}{f'(z)}\right)\right]>0$$

for $z\in D$ and for some fixed real a. Functions satisfying this condition are said to belong to the class of alpha-convex (or alpha-starlike) functions $\mathcal{M}_a$, and they have been shown to be starlike. In this paper we consider regular functions $f(z)$, defined in D, with the property that the real part of a geometric mean of the quantities $zf'(z)/f(z)$ and $1+zf''(z)/f'(z)$ is positive. We will show that these functions are starlike and will call such functions gamma-starlike to suggest the use of the geometric mean in their definition.

Definition 1. Let $f(z) = z+\sum\limits_{a}^{\infty} a_n z^n$ be regular in the unit disc D, with $f(z)$, $f'(z)$ and $[1+zf''(z)/f'(z)]\neq 0$ in $0<|z|<1$. Suppose γ is real and

$$\operatorname{Re}\left[\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right]>0, \tag{1}$$

for $z\in D$, where the powers appearing in (1) are meant as principal values.

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[1] This work was carried out while the second author was an IREX Scholar in Poland.

Then we say that $f(z)$ is a gamma-starlike function and we denote the class of such functions by $\mathscr{L}^{\gamma}$.

Remarks. (i) Condition (1) is equivalent to the following condition:

$$\left|(1-\gamma)\arg\frac{zf'(z)}{f(z)}+\gamma\arg\left(1+\frac{f''(z)}{f'(z)}\right)\right|<\frac{\pi}{2} \tag{2}$$

(ii) If $\gamma = 0$, $\mathscr{L}_0 \equiv S^*$, the class of starlike functions, while if $\gamma = 1$, $\mathscr{L}_1 \equiv C$, the class of convex functions.

2. We now show that if $f(z)$ is a gamma-starlike function then $f(z)$ is starlike and univalent.

Theorem 1. *$\mathscr{L}_\gamma \subset S^*$, for all real γ.*

Proof. If $f(z) \in \mathscr{L}_\gamma$ and we set

$$\frac{1+w(z)}{1-w(z)} = \frac{zf'(z)}{f(z)}. \tag{3}$$

for $z \in D$, then $w(0) = 0$, $w(z) \neq \pm 1$ and $w(z)$ is a meromorphic function. We will show that $|w(z)| < 1$, and this will imply that

$$\mathrm{Re}[zf'(z)/f(z)] > 0.$$

Let $w(z) = R(z)e^{i\varphi(z)}$, $R \geqslant 0$ for $z \in D$, and suppose that z_0 is a point of D such that

$$\max_{|z|\leqslant|z_0|}|w(z)| = |w(z_0)| = 1. \tag{4}$$

Then $\frac{\partial}{\partial \theta} R(z_0) = 0$, and since

$$\frac{zw'(z)}{w(z)} = \frac{\partial \Phi(z)}{\partial\theta} - i\frac{1}{R}\frac{\partial R(z)}{\partial \theta},$$

we must have $z_0w'(z_0)/w(z_0) = \partial\Phi(z_0)/\partial\theta$, and hence $z_0w'(z_0)/w(z_0)$ must be a real number. A simple geometric argument can show even more. If we assume $\partial\Phi(z_0)/\partial\theta < 0$ then $w(z)$ would be locally univalent at z_0 and this would lead to a contradiction of (4). Thus we see that $\partial\Phi(z_0)/\partial\theta$ must be non-negative and so we can set

$$\frac{z_0w'(z_0)}{w(z_0)} = B, \tag{5}$$

where $B \geqslant 0$.

Since $|w(z_0)| = 1$ and $w(z_0) \neq \pm 1$, we must have

$$\frac{1+w(z_0)}{1-w(z_0)} = Ai, \tag{6}$$

where A is real and $A \neq 0$.

From (1) and (3) we have

$$\operatorname{Re} I(\gamma, f(z)) =$$
$$= \operatorname{Re}\left[\left(\frac{1+w(z)}{1-w(z)}\right)^{1-\gamma}\left(\frac{1+w(z)}{1-w(z)} + \frac{zw'(z)}{w(z)}\left(\frac{w(z)}{1+w(z)} + \frac{w(z)}{1-w(z)}\right)\right)^{\gamma}\right]$$

where

$$I(\gamma, f(z)) \equiv \left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + z\frac{f''(z)}{f'(z)}\right)^{\gamma},$$

and thus at $z = z_0$, by using (5) and (6) we obtain

$$\operatorname{Re} I(\gamma, f(z_0)) = \operatorname{Re}\left[(Ai)^{1-\gamma}\left(Ai + \frac{B}{a}\left(A + \frac{1}{A}\right)i\right)^{\gamma}\right].$$

If we let $C = A + B(A+1/A)/2$, then since $B \geqslant 0$ and $A \neq 0$ we have $AC > 0$ and obtain

$$\operatorname{Re} I(\gamma, f(z_0)) = \operatorname{Re}[(Ai)^{1-\gamma}(Ci)^{\gamma}] = \operatorname{Re}(|A|^{1-\gamma}|C|^{\gamma} i) = 0.$$

This is a contradiction of (1) and so we must have $|w(z)| < 1$ for $z \in D$ and thus $f(z) \in S^*$.

Note that Theorem 1 shows that if $f \in \mathscr{L}_\gamma$, then $f \in \mathscr{L}_0 = S^*$. We can show more than this.

Theorem 2. *If* $0 \leqslant \delta \leqslant \gamma$ (*or* $\gamma \leqslant \delta \leqslant 0$) *then* $\mathscr{L}_\gamma \subset \mathscr{L}_\delta$.

Proof. The case $\delta = 0$ has been handled in Theorem 1, so we only need to consider the case $0 < \delta/\gamma < 1$.

If $f \in \mathscr{L}_\gamma$, then there is a function $P_1(z) \in \mathscr{P} = \{P(z) | P(0) = 1,\ P(z)$ is regular in D and $\operatorname{Re} P(z) > 0\}$ satisfying

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma} \equiv P_1(z).$$

By Theorem 1 we have $f(z) \in S^*$ and hence there exists $P_2(z) \in \mathscr{P}$ such that

$$\frac{zf'(z)}{f(z)} \equiv P_2(z). \tag{8}$$

If we raise sides of (7) to the δ/γ power we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)^{\delta/\gamma-\delta}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\delta} \equiv P_1(z)^{\delta/\gamma}, \tag{9}$$

and if we raise both sides of (8) to the $(1-\delta/\gamma)$ power we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\delta/\gamma} \equiv P_2(z)^{1-\delta/\gamma}. \tag{10}$$

Multiplying equation (9) by equation (10) we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\delta}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\delta} \equiv P_1(z)^{\delta/\gamma}P_2(z)^{1-\delta/\gamma} \equiv P_3(z). \tag{11}$$

Since $P_1(z)\in\mathscr{P}$ and $P_2(z)\in\mathscr{P}$, we have $P_3(0)=1$ and $|\arg P_3(z)|\leqslant\frac{\delta}{\gamma}|\arg P_1(z)|+\left(1-\frac{\delta}{\gamma}\right)|\arg P_2(z)|<\pi/2$, i.e. Re $P_3(z)>0$ and $P_3(z)\in\mathscr{P}$, Consequently from (10) we have $f(z)\in\mathscr{L}_\gamma$.

Note that the last theorem shows that if $f(z)\in\mathscr{L}$ and $\gamma\geqslant 1$, then $f(z)$ is a convex function.

3. It is possible to obtain bounds on the coefficients of gamma-starlike functions by using certain "standard" methods.

Theorem 3. *If $f\in\mathscr{L}_\gamma$, $f(z)=z+a_2z^2+n$, and if μ is a complex constant, then*

$$|a_2(1+\gamma)|\leqslant 2, \tag{12}$$

$$|a_3(4+8\gamma)+a_2^2(\gamma^2-7\gamma-2)|\leqslant 4, \tag{13}$$

$$(1+\gamma)^2|1+2\gamma||a_3-\mu a_2|\leqslant \mathrm{Max}\left[(1+\gamma)^2,\ |4\mu(1+2\gamma)-3(3\gamma+1)|\right] \tag{14}$$

all hold.

Remarks. (i) $\gamma=0$ (14) reduces to

$$|a_3-\mu a_2^2|\leqslant \mathrm{Max}[1,\ |4\mu-3|],$$

which is a result of Keogh and Merkes [2].

(ii) For $\gamma=1$ (14) reduces to

$$|a_3-\mu a_2^2|\leqslant \mathrm{Max.}\left[\frac{1}{3},\ |\mu-1|\right].$$

(iii) If $f(z)\in\mathscr{L}_\gamma$, $\gamma\geqslant 0$, then

$$|a_2|\leqslant 2/(1+\gamma) \tag{15}$$

and

$$(16)\qquad |a_3| \leqslant \begin{cases} \dfrac{3(3\gamma+1)}{(1+2\gamma)(1+\gamma)^2}, & 0 \leqslant \gamma \leqslant \dfrac{7+\sqrt{57}}{2}, \\ \dfrac{1}{1+2\gamma}, & \dfrac{7+\sqrt{57}}{2} < \gamma. \end{cases}$$

hold.

Inequalities (12) – (16) may **not** be sharp. They would be sharp if it were possible to prove that the differential equation

$$(17)\qquad \left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+z\frac{f''(z)}{f'(z)}\right)^{\gamma} = \frac{1+z}{1-z},$$

with initial condition $f(0) = 0$, $f'(0) = 1$, has a solution that is a regular function in the unit disc. The authors have not been able to prove this for arbitrary γ, but suspect that a solution exists for $\gamma \geqslant 0$.

The class $\mathcal{M}_a$ satisfies Theorems 1, 2 and inequality (13) and it is also true that $\mathcal{M}_0 = \mathcal{L}_0 = S^*$ and $\mathcal{M}_1 = \mathcal{L}_1 = C$. However, in general $\mathcal{M}_p \neq \mathcal{L}_\beta$; This can be seen by considering $\beta = 1/2$. By using infinite series, it is possible to show that (17) does have a regular solution for $\gamma = \beta = 1/2$, and for this solution $a_3 = 5/3$. For functions in $\mathcal{M}_{1/2}$ we must have $|a_3| \leqslant 29/18$ [3], and thus $\mathcal{M}_{1/2} \neq \mathcal{L}_{1/2}$.

We conclude by indicating a refinement in the class of gamma-starlike functions.

Definition 2. Let $f(z) = z + \sum_{2}^{\infty} a_n z^n$ be regular in the unit disc D with $f(z), f'(z), 1+zf''(z)/f'(z) \neq 0$ in $0 < |z| < 1$, and suppose γ is a real constant, $0 \leqslant \gamma < 1$. If

$$(18)\qquad \operatorname{Re}\left[\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+z\frac{f''(z)}{f'(z)}\right)^{\gamma}\right] > a.$$

for $z \in D$, then we say that $f(z)$ is a gamma-starlike function of order a, and we denote the class of such functions by $\mathcal{L}_\gamma(a)$. If (18) is replaced by

$$\left|(1-\gamma)\arg\left(\frac{zf(z)}{f(z)}\right) + \gamma\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}a$$

for $z \in D$, then we say that $f(z)$ is a strongly gamma-starlike function of order a, and we denote the class of such functions by $\mathcal{L}^*_\gamma(a)$.

Note that $\mathcal{L}^*(a)$ and $\mathcal{L}^*_1(a)$ are respectively the classes of strongly-starlike and strongly –convex functions of order a introduced by Brannan and Kirwan [1].

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STRESZCZENIE

Niech $f(z) = z+\sum_{2}^{\infty} a_n z^n$ będzie funkcją holomorficzną w kole jednostkowym $D = \{z: |z| < 1\}$ i taką, że $f(z) \neq 0$, $f'(z) \neq 0$ oraz

$$1+zf''(z)/f'(z) \neq 0 \quad \text{dla} \quad 0 < |z| < 1.$$

Jeśli zachodzi nierówność

$$\mathrm{Re}\{[zf'(z)/f(z)]^{1-\gamma}\ [zf''(z)/f'(z)+1]\} > 0 \ \text{dla}\ z \in D$$

dla dowolnego, ustalonego rzeczywistego γ to mówimy, że $f(z)$ jest funkcją gamma-gwiaździstą.

W pracy tej autorzy dowodzą, że funkcje gamma-gwiaździste są jednolistne i gwiaździste. Podane są też pewne oszacowania współczynników dla rozważanych funkcji.

РЕЗЮМЕ

Пусть $f(z) = z+\sum_{2}^{\infty} a_n z^n$ будет голоморфной функцией в единичном круге $D = \{z: |z| < 1\}$ подчиненной условиям $f(z) \neq 0$, $f'(z) \neq 0$ и $1+zf''(z)/f'(z) \neq 0$ для $0 < |z| < 1$.

Если исполнится неравенство для вещественого фиксированного γ

$$\mathrm{Re}\{[zf'(z)/f(z)]^{1-\gamma}\ [zf''(z)/f'(z)+1]) < 0 \quad \text{для}\ z \in D$$

то тогда мы говорим, что $f(z)$ — это гамма-звездообразная функция.

Авторы доказывают, что гамма-звездообразные являются однолистными и звездообразными функциями. Даются также некоторые оценки коэффициентов рассматриваемых функций.