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An Existence Theorem for an Integro-Differential Equation of Neutral Type

Twierdzenie o istnieniu rozwiązań równania całkowo-różniczkowego typu neutralnego

Теорема о существовании решений интегрально-диференциального уравнения нейтрального типа

I. Introduction

In this paper we shall consider the problem of existence of the solution for the neutral integro-differential equation of the form

$$(0)\qquad \begin{aligned} x'(t) &= f\Big(t, \int_0^{\infty} x(t-s)\,d_sG(t,s), \int_0^{t} x'(t-s)\,d_sK(t,s)\Big) \quad \text{for } t\in \langle 0, a\rangle \\ x(t) &= \xi(t) \quad \text{for } t\in(-\infty, 0\rangle. \end{aligned}$$

To solve the problem we shall use the method based on the notion of "measure of noncompactness" and the fixed point theorem of Darbo [3]. To do that we shall need the exact formula for measure of noncompactness in the spaces C, C^1 of continuous and differentiable functions on a compact interval. Those formulas have been proved by Goebel [4]. Besides those formulas the paper [4] contains some remarks which show how to check the assumptions of Darbo's theorem for concrete transformations in the spaces C and C^1. It appears that it is enough to check that the mappings change the modulus of continuity of the argument function in a "regular way" (see Definition 1). In our case the transformations we shall have to study will be some linear integral operations. So the main stress will be put on finding the conditions under which they change modulus of continuity as required in Definition 1.

II. On some integral operators

Let $C_{(-\infty,a\rangle}$ be the Banach space of bounded and continuous functions $\varphi(t)$ defined on an interval $(-\infty, a\rangle$ with the norm $\|\varphi\|_{(-\infty,a\rangle} = \sup\{|\varphi(t)|: t\in(-\infty, a\rangle\}$. Let $G(t, s)$ be a real function defined on $\langle 0, a\rangle\times\langle 0, \infty)$.

We shall deal with the mapping $\mathscr{G}$ in the following form

$$(1) \qquad (\mathscr{G}\varphi)(t) = \int_0^\infty \varphi(t-s)\,d_sG(t, s) \quad \text{for } t\in\langle 0, a\rangle.$$

In [1] Bielecki gave some conditions on the kernel $G(t, s)$ which are sufficient for $\mathscr{G}$ to map the class $C_{(-\infty,a\rangle}$ into a class $C_{\langle 0,a\rangle}$. Those assumptions are:

1° $G(t, 0) = 0$ for $t\in\langle 0, a\rangle$.

2° The function $G(t, s)$ is of bounded variation with respect to s for any fixed value $t\in\langle 0, a\rangle$ and satisfies the inequality

$$\overset{\infty}{\underset{s=0}{\Upsilon}} G(t, s) \leqslant V_1 = \text{const}.$$

3° For an arbitrary $\varepsilon > 0$ exists a number $K > 0$ that

$$\overset{\infty}{\underset{s=K}{\Upsilon}} G(t, s) < \varepsilon.$$

4° For an arbitrary fixed number $K' > 0$ and $\bar{t}\in\langle 0, a\rangle$

$$\lim_{t\to\bar{t}} \int_0^{K'} |G(t, s) - G(\bar{t}, s)|\,ds = 0.$$

Theorem 1. *Under the assumptions* 1^0-4^0 *the mapping* $\mathscr{G}$ *maps any function* $\varphi\in C_{(-\infty,a\rangle}$ *into a function of the class* $C_{\langle 0,a\rangle}$ *and this mapping is continuous.*

Proof. The proof that $\mathscr{G}\colon C_{(-\infty,a\rangle}\to C_{\langle 0,a\rangle}$ has been given in [2]. We shall prove the continuity. Because the mapping is linear, we shall only show that there exists a constant M and

$$\|\mathscr{G}\varphi\|_{\langle 0,a\rangle} \leqslant M\,\|\varphi\|_{(-\infty,a\rangle}.$$

In fact

$$|(\mathscr{G}\varphi)(t)| = \left|\int_0^\infty \varphi(t-s)\,d_sG(t, s)\right| \leqslant \|\varphi\|_{(-\infty,a\rangle} \overset{\infty}{\underset{s=0}{\Upsilon}} G(t, s) \leqslant V_1\cdot\|\varphi\|_{(-\infty,a\rangle}.$$

This ends the proof of Theorem 1.

Let us go through a new problem. Let us consider a function $K(t, s)$ defined for $t\in\langle 0, a\rangle$ and $s\in\langle 0, a\rangle$ with real value. We assume that the function $K(t,s)$ has bounded variation with respect to s for any fixed $t \in \langle 0, a\rangle$ and satisfies inequality

(2) $$\overset{a}{\underset{s=0}{\Upsilon}} K(t,s) \leqslant V_2 = \text{const}.$$

We deal with a mapping $\mathscr{K}$ of the form

(3) $$(\mathscr{K}\varphi)(t) = \int_0^t \varphi(t-s)\, d_s K(t,s) \quad \text{for } t \in \langle 0, a\rangle \quad \text{and} \quad \varphi \in C_{\langle 0,a\rangle}.$$

We shall use a concept of the modulus of continuity $\omega(x, h)$ for a function $x \in C_{\langle 0,a\rangle}$ as

$$\omega(x, h) = \sup[|x(t) - x(\bar{t})| : t, \bar{t} \in \langle 0, a\rangle, \quad |t-\bar{t}| < h].$$

Definition 1. We say, that transformation $\mathscr{K}$ changes the modulus of continuity in the regular way, if there exist a constant L and functions $a(h)$ and $\beta(h)$ such that $0 \leqslant a(h) \to 0$ and $0 \leqslant \beta(h) \to 0$ for $h \to 0$ and for every $\varphi \in C_{\langle 0,a\rangle}$

(4) $$\omega\big((\mathscr{K}\varphi), h\big) \leqslant L\omega\big(\varphi, a(h)\big) + \beta(h).$$

Now let us raise a question; what conditions on $K(t,s)$ are sufficient for the mapping $\mathscr{K}$ to change the modulus of continuity in the regular way.

The following theorem holds.

Theorem 2. *If the kernel $K(t,s)$ of the transformation $\mathscr{K}$ is given by*

(5) $$K(t,s) = K_1(t,s) + K_2(t,s),$$

where $K_1(t,s)$ is a continuous function of s for every t and satisfies:

1° *for every* $t \in \langle 0, a\rangle$ $\overset{a}{\underset{s=0}{\Upsilon}} K_1(t,s) \leqslant V_3 = \text{const}$,

2° $\lim\limits_{(t-\bar{t})\to 0} \overset{a}{\underset{s=0}{\Upsilon}} \big(K_1(t,s) - K_1(\bar{t},s)\big) = 0$ *for* $t, \bar{t} \in \langle 0, a\rangle$ *and* $t \geqslant \bar{t}$,

3° $\lim\limits_{(t-\bar{t})\to 0} \overset{t}{\underset{s=\bar{t}}{\Upsilon}} K_1(t,s) = 0$ *for* $t, \bar{t} \in \langle 0, a\rangle$ *and* $t \geqslant \bar{t}$,

and $K_2(t,s)$ is of the form

(6) $$K_2(t,s) = \sum_{i=1}^{\infty} a_i H\big(v_i(t) - s\big),$$

where 4° *$H(u)$ is the Heaviside's function*

$$H(u) = \begin{cases} 0, & u < 0 \\ 1, & u \geqslant 0, \end{cases}$$

5° *The functions* $r_i(t)$, $i = 1, 2, \ldots$ *satisfies* $0 \leqslant v_i(t) \leqslant t$ *and all are equicontinuous i.e. there exists a function* $\Omega(h)$ *defined on the interval* $\langle 0, a\rangle$ *such that for every* i $\omega(v_i, h) \leqslant \Omega(h)$ *and* $0 \leqslant \Omega(h) \to 0$, *if* $h \to 0$,

6° *The coefficients* a_i *form an absolutly convergent series* $\sum_{i=1}^{\infty} |a_i| < \infty$, *then the transformation* $\mathscr{K}$ *maps* $C_{\langle 0,a\rangle}$ *into itself, is continuous and changes the modulus of continuity in the regular way.*

Proof. It is easy to verify that the transformation $\mathscr{K}$ maps $C_{\langle 0,a\rangle}$ into $C_{\langle 0,a\rangle}$ and is continuous. We prove only the last part of the thesis.

Let $t, \bar{t} \in \langle 0, a\rangle$ and $t > \bar{t}$. By (5), (6) we have

$$|(\mathscr{K}\varphi)(t) - (\mathscr{K}\varphi)(\bar{t})| = \left|\int_0^t \varphi(t-s)\,d_sK(t,s) - \int_0^{\bar{t}} \varphi(\bar{t}-s)\,d_sK(\bar{t},s)\right|$$

$$\leqslant \left|\int_0^t \varphi(t-s)\,d_sK_1(t,s) - \int_0^{\bar{t}} \varphi(\bar{t}-s)\,d_sK_1(\bar{t},s)\right| +$$

$$+\left|\int_0^t \varphi(t-s)\,d_s\sum_{i=1}^{\infty} a_iH\big(v_i(t)-s\big) - \int_0^{\bar{t}} \varphi(\bar{t}-s)\,d_s\sum_{i=1}^{\infty} a_iH\big(v_i(\bar{t})-s\big)\right|.$$

Now we estimate both parts separately.

$$s_1 = \left|\int_0^t \varphi(t-s)\,d_sK_1(t,s) - \int_0^{\bar{t}} \varphi(\bar{t}-s)\,d_sK_1(\bar{t},s)\right|$$

$$\leqslant \left|\int_0^{\bar{t}} \varphi(t-s)\,d_sK_1(t,s) + \int_{\bar{t}}^t \varphi(t-s)\,d_sK_1(t,s) - \int_0^{\bar{t}} \varphi(\bar{t}-s)\,d_sK_1(\bar{t},s)\right|$$

$$\leqslant \left|\int_0^{\bar{t}} \varphi(t-s)\,d_sK_1(t,s) - \int_0^{\bar{t}} \varphi(\bar{t}-s)\,d_sK_1(t,s)\right| +$$

$$+\left|\int_0^{\bar{t}} \varphi(\bar{t}-s)\,d_sK_1(t,s) - \int_0^{\bar{t}} \varphi(\bar{t}-s)\,d_sK_1(\bar{t},s)\right| + \left|\int_{\bar{t}}^t \varphi(t-s)\,d_sK_1(t,s)\right|$$

$$\leqslant \left|\int_0^{\bar{t}} \big(\varphi(t-s) - \varphi(\bar{t}-s)\big)\,d_sK_1(t,s)\right| +$$

$$+\left|\int_0^{\bar{t}} \varphi(\bar{t}-s)\,d_s\big(K_1(t,s) - K_1(\bar{t},s)\big)\right| + \left|\int_{\bar{t}}^t \varphi(t-s)\,d_sK_1(t,s)\right|$$

$$\leqslant \sup_{s\in\langle 0,\bar{t}\rangle} \left|\varphi(t-s) - \varphi(\bar{t}-s)\right| \cdot \Upsilon_{s=0}^{a} K_1(t,s) +$$

$$+\sup_{s\in\langle 0,\bar{t}\rangle} |\varphi(\bar{t}-s)| \cdot \Upsilon_{s=0}^{a} \big(K_1(t,s) - K_1(\bar{t},s)\big) + \sup_{s\in\langle \bar{t},t\rangle} |\varphi(t-s)|\, \Upsilon_{s=\bar{t}}^{t} K_1(t,s).$$

Thus in view of 1°, 2°, 3° we obtain

$$s_1 \leqslant \omega(\varphi, |t-\bar{t}|)\cdot V_3 + \|\varphi\|_{\langle 0,a\rangle}\cdot\gamma(|t-\bar{t}|),$$

where

$$\gamma(|t-\bar{t}|) = \mathop{\Upsilon}_{s=0}^{a}\big(K_1(t, s) - K_1(\bar{t}, s)\big) + \mathop{\Upsilon}_{s=\bar{t}}^{t} K_1(t, s). \tag{7}$$

However, by the assumptions 4°, 5°, 6° we get

$$s_2 = \Big|\int_0^t \varphi(t-s)\,d_s \sum_{i=1}^{\infty} a_i H\big(v_i(t)-s\big) - \int_0^{\bar{t}} \varphi(\bar{t}-s)\,d_s \sum_{i=1}^{\infty} a_i H\big(v_i(\bar{t})-s\big)\Big|$$

$$= \Big|\sum_{i=1}^{\infty} a_i\Big(\int_0^t \varphi(t-s)\,d_s H\big(v_i(t)-s\big) - \int_0^{\bar{t}} \varphi(\bar{t}-s)\,d_s H\big(v_i(\bar{t})-s\big)\Big)\Big|$$

$$\leqslant \sum_{i=1}^{\infty} |a_i|\big|\varphi\big(t-v_i(t)\big)-\varphi\big(\bar{t}-v_i(\bar{t})\big)\big| \leqslant \sum_{i=1}^{\infty} |a_i|\cdot\omega\big(\varphi, |t-\bar{t}|+\omega(v_i, |t-\bar{t}|)\big)$$

$$\leqslant \Big(\sum_{i=1}^{\infty} |a_i|\Big)\cdot\omega\big(\varphi, |t-\bar{t}|+\Omega(|t-\bar{t}|)\big).$$

Therefore the modulus of continuity for the transformation $\mathscr{K}$ can be estimated in the following way:

$$\omega\big((\mathscr{K}\varphi), h\big) \leqslant V_3\cdot\omega(\varphi, h) + \sum_{i=1}^{\infty} |a_i|\cdot\omega\big(\varphi, h+\Omega(h)\big) + \|\varphi\|_{\langle 0,a\rangle}\cdot\gamma(h)$$

for $|t-\bar{t}| < h$.

Then there exist $\alpha(h) = h+\Omega(h)$ and $\beta(h) = \gamma(h)$ and $L = V_3 + \sum_{i=1}^{\infty} |a_i|$ such that conditions of Definition 1 are satisfied and

$$\omega\big((\mathscr{K}\varphi), h\big) \leqslant \Big(V_3 + \sum_{i=1}^{\infty} |a_i|\Big)\cdot\omega\big(\varphi, h+\Omega(h)\big) + \|\varphi\|_{\langle 0,a\rangle}\cdot\gamma(h) \tag{8}$$

for every $\varphi\in C_{\langle 0,a\rangle}$.

This ends the proof of Theorem 2.

III. Remarks about "measures of noncompactness"

Let (M, ϱ) be a metric space M with a metric ϱ. By $\mathfrak{M}$ we denote a class of all nonempty closed and bounded subsets of the space M.

Definition 2. If $X\in\mathfrak{M}$, then $\mu(X)$ is the greatest lower bound of such numbers r that X can be covered by a finite number of balls of radius r. We call $\mu(X)$ the "measure of noncompactness" of the set X.

Now let us consider a space $C_{\langle 0,a\rangle}$ of all real functions $x(t)$ continuous on an interval $\langle 0, a\rangle$ and the space $C^1_{\langle 0,a\rangle}$ consisted of the differentiable functions in continuous way. The norm in the space $C^1_{\langle 0,a\rangle}$ we introduce

$$\|x\|_{C^1} = \|x\|_C + \|Dx\|_C,$$

where $\|x\|_C = \sup[|x(t)|: t \in \langle 0, a\rangle]$ and $D = \frac{d}{dt}$ is differential operation.

Let $\mathfrak{M}_C$ and $\mathfrak{M}_{C^1}$ be classes of all nonempty bounded subsets of the spaces C and C^1 respectively. We quote the basic theorems concerning methods of estimate of the function μ in the spaces $C_{\langle 0,a\rangle}$ and $C^1_{\langle 0,a\rangle}$ (cf [4]).

Theorem 3. *For arbitrary set $X \in \mathfrak{M}_C$ we have*

$$\mu_C(X) = \frac{1}{2}\lim_{h\to 0}\omega(X, h) = \frac{1}{2}\lim_{h\to 0}\{\sup[\omega(x, h): x \in X]\}.$$

Theorem 4. *For arbitrary set $X \in \mathfrak{M}_{C^1}$*

$$\mu_{C^1}(X) = \mu_C(DX), \text{ where } DX = \left[\frac{d}{dt}x(t): x \in X\right].$$

Moreover we use

Theorem 5 (Darbo [3]). *Let $(B, \|\ \|)$ be a Banach space and let E be a bounded closed and convex subset of the space B. We assume that a transformation T maps the set E into itself and is continuous. If $\mu(TX) \leqslant k\mu(X)$ for all closed subsets X of E, where $0 \leqslant k < 1$ and μ means the "measure of noncompactness" for sets in $(B, \|\ \|)$, then the transformation T has a fixed point in the set E.*

According to the above remarks we can see that in order to check the inequality in Darbo's theorem in C^1 space, it is enough to verify that

$$\lim_{h\to 0}\omega\big(D(TX), h\big) \leqslant k \lim_{h\to 0}\omega(DX, h)$$

for all bounded sets $X \in \mathfrak{M}_{C^1}$. Notice also that this inequality holds if we can find two functions $\alpha_1(h)$, $\beta_1(h)$ such that for any $x \in C^1$

$$\omega\big(D(Tx), h\big) \leqslant \omega\big(Dx, \alpha_1(h)\big) + \beta_1(h)$$

and

$$\lim_{h\to 0}\alpha_1(h) = \lim_{h\to 0}\beta_1(h) = 0.$$

Similarly to the Definition 1 we could say that the mapping $DT: C^1 \to C$ changes the modulus of continuity of the derivative in the regular way.

IV. An existence theorem.

Let $f(t, x, y)$ be a real continuous function defined on $D = \langle 0, a\rangle \times \times R \times R$. In the set D we define a metric

$$\varrho\big((t, x, y), (\bar{t}, \bar{x}, \bar{y})\big) = \max[|t-\bar{t}|, |x-\bar{x}|, |y-\bar{y}|].$$

Let $G(t, s)$ and $K(t, s)$ be functions such as in Theorem 1 and Theorem 2. We shall deal with the differential equation

$$(9)\qquad x'(t) = f\Big(t, \int_0^\infty x(t-s)\, d_s G(t, s), \int_0^t x'(t-s)\, d_s K(t, s)\Big) \text{ for } t \in \langle 0, a\rangle,$$

$$(10)\qquad x(t) = \xi(t) \quad \text{for} \quad t \in (-\infty, 0\rangle.$$

Theorem 6. *Assume that*

1° *the function $f(t, x, y)$ is bounded $|f(t, x, y)| \leqslant M$ and satisfies a Lipschitz condition with respect to a variable y*

$$|f(t, x, y) - f(t, x, \bar{y})| \leqslant k|y-\bar{y}|,$$

where $0 \leqslant k < \frac{1}{L}$, $L = V_3 + \sum_{i=1}^{\infty} |a_i|$,

2° *the function $G(t, s)$ satisfies the assumptions of Theorem 1,*

3° *the function $K(t, s)$ satisfies the inequality (2) and the assumptions of Theorem 2,*

4° *$\xi(t)$ is an initial, bounded and continuous function defined on an interval $(-\infty, 0\rangle$.*

By those assumptions a differential equation of the from (9) has at least one solution $x(t)$ defined and continuous for $t \in (-\infty, a\rangle$, belonging to the class C^1 on an interval $\langle 0, a\rangle$ and satisfying the initial condition (10).

Proof. Let us consider the space $C^1_{\langle 0,a\rangle}$ and a integral operation $F\colon C^1_{\langle 0,a\rangle} \to C^1_{\langle 0,a\rangle}$ given by

$$(11)\qquad (F\varphi)(t) = \xi(0) + \int_0^t f\Big(\tau, \int_0^\infty \varphi(\tau-s)\, d_s G(\tau, s), \int_0^\tau \varphi'(\tau-s)\, d_s K(\tau, s)\Big)\, d\tau \quad \text{for } t \in \langle 0, a\rangle$$

and let

$$(12)\qquad (F\varphi)(t) = \xi(t) \quad \text{for } t \in (-\infty, 0\rangle.$$

By notations and assumptions 2°, 3°, 4° and by above formula it is easy to verify that F is the continuous transformation.

The norms in spaces $C_{\langle 0,a\rangle}$ and $C^1_{\langle 0,a\rangle}$ we introduce as usual

$$\|(F\varphi)\|_C = \sup[|(F\varphi)(t)| : t \in \langle 0, a\rangle],$$
$$\|(F\varphi)\|_{C^1} = \|(F\varphi)\|_C + \|(F\varphi)'\|_C.$$

Now we estimate

$$|(F\varphi)(t)| \leqslant |\xi(0)| + \int_0^t \left| f\left(\tau, \int_0^\infty \varphi(\tau-s)\,d_sG(\tau,s), \int_0^\tau \varphi'(\tau-s)\,d_sK(\tau-s)\right)\right| d\tau$$

and thus

$$\|(F\varphi)\|_C \leqslant |\xi(0)| + aM.$$

Because

$$|(F\varphi)'(t)| = \left| f\left(t, \int_0^\infty \varphi(t-s)\,d_sG(t,s), \int_0^t \varphi'(t-s)\,d_sK(t,s)\right)\right|,$$

then

$$\|(F\varphi)'\|_C \leqslant M.$$

Hence

$$\|(F\varphi)\|_{C^1} \leqslant |\xi(0)| + aM + M = |\xi(0)| + (a+1)\cdot M = r.$$

It means that transformation F maps the set $C^1_{\langle 0,a\rangle}$ in a ball $B(\theta, r)$. In particular this ball maps into itself.

Now we denote the modulus of continuity for $(F\varphi)'$ for $\varphi \in B(\theta, r)$. Then for $t > \bar{t}$, where $t, \bar{t} \in \langle 0, a\rangle$, we have

$$(13)\qquad |(F\varphi)'(t) - (F\varphi)'(\bar{t})| \leqslant \left| f\left(t, \int_0^\infty \varphi(t-s)\,d_sG(t,s), \int_0^t \varphi'(t-s)\,d_sK(t,s)\right) - \right.$$

$$-f\left(t, \int_0^\infty \varphi(t-s)\,d_sG(t,s), \int_0^{\bar{t}} \varphi'(\bar{t}-s)\,d_sK(\bar{t},s)\right)\Big| +$$

$$+\left| f\left(t, \int_0^\infty \varphi(t-s)\,d_sG(t,s), \int_0^{\bar{t}} \varphi'(\bar{t}-s)\,d_sK(\bar{t},s)\right) - \right.$$

$$\left. -f\left(\bar{t}, \int_0^\infty \varphi(\bar{t}-s)\,d_sG(\bar{t},s), \int_0^{\bar{t}} \varphi'(\bar{t}-s)\,d_sK(\bar{t},s)\right)\right|$$

$$\leqslant k\left| \int_0^t \varphi'(t-s)\,d_sK(t,s) - \int_0^{\bar{t}} \varphi'(\bar{t}-s)\,d_sK(\bar{t},s)\right|$$

$$+\omega\left(f, \max_{t,\bar{t}\in\langle 0,a\rangle}\left(|t-\bar{t}|, \left|\int_0^\infty \varphi(t-s)\,d_sG(t,s) - \int_0^\infty \varphi(\bar{t}-s)\,d_sG(\bar{t},s)\right|\right)\right).$$

We apply Theorem 2 to the first part of above sum. Let

$$(\mathscr{K}\varphi')(t) = \int_0^t \varphi'(t-s)\,d_sK(t,s).$$

By the assumptions of Theorem 2 and the formula (8) the modulus of continuity for $\mathscr{K}$ can be estimated as follows:

$$\omega\big((\mathscr{K}\varphi'), h\big) \leqslant \Big(V_3 + \sum_{i=1}^{\infty} |a_i|\Big)\cdot\omega\big(\varphi', h + \Omega(h)\big) + \|\varphi'\|_{\langle 0,a\rangle}\gamma(h). \tag{14}$$

In view of Theorem 1 we can also estimate the second part. It is easy to verify that a domain of the function f is a compact set, so f is the uniformly continuous function. Since the modulus of continuity for the function f satisfies

$$\lim_{|t-\bar{t}|\to 0} \omega\big(f, \eta(|t-\bar{t}|)\big) = 0, \tag{15}$$

where $\eta(|t-\bar{t}|) = \max_{t,\bar{t}\in\langle 0,a\rangle} [|t-\bar{t}|, |(\mathscr{G}\varphi)(t)-(\mathscr{G}\varphi)(\bar{t})|]$ (see the formula (1)).

Returning to the formula (13) in view of the inequalities (14) and (15) we obtain for $|t-\bar{t}| < h$

$$\omega\big((F\varphi)', h\big) \leqslant k\Big[\Big(V_3 + \sum_{i=1}^{\infty} |a_i|\Big)\omega\big(\varphi', h + \Omega(h)\big) + \|\varphi'\|_{\langle 0,a\rangle}\cdot\gamma(h)\Big] + \omega\big(f, \eta(h)\big).$$

For arbitrary set $X \subset B(\theta, r)$ we have

$$\omega\big((FX)', h\big) \leqslant k\Big[\Big(V_3 + \sum_{i=1}^{\infty} |a_i|\Big)\cdot\omega\big(X', h + \Omega(h)\big) + r\cdot\gamma(h)\Big] + \omega\big(f, \eta(h)\big)$$

and by (15), (7) and the assumptions 2°, 3° of Theorem 2

$$\lim_{h\to 0} \omega\big((FX)', h\big) \leqslant k\Big(V_3 + \sum_{i=1}^{\infty} |a_i|\Big)\cdot\lim_{h\to 0}\omega\big(X', h + \Omega(h)\big).$$

Since, by Theorem 4 we obtain

$$\begin{aligned}\mu_{C^1}(FX) &= \mu_C\big((FX)'\big) = \tfrac{1}{2}\lim_{h\to 0}\omega\big((FX)', h\big)\\ &\leqslant \tfrac{1}{2}k\Big(V_3 + \sum_{i=1}^{\infty} |a_i|\Big)\lim_{h\to 0}\omega\big(X', h + \Omega(h)\big)\\ &= k\Big(V_3 + \sum_{i=1}^{\infty} |a_i|\Big)\mu_C(X') = k\Big(V_3 + \sum_{i=1}^{\infty} |a_i|\Big)\cdot\mu_{C^1}(X).\end{aligned}$$

By the assumption 1° $0 \leqslant k < \dfrac{1}{V_3 + \sum_{i=1}^{\infty} |a_i|}$. Then all assumptions of Darbo's Theorem for F are satisfied. Therefore the transformation F has a fixed point, which is a solution of our differential equation (9) and (10).

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STRESZCZENIE

W pracy tej rozważamy problem istnienia rozwiązań dla równania całkowo-różniczkowego typu neutralnego postaci (0). Do rozwiązania tego problemu stosujemy metodę opartą na pojęciu ,,miary niezwartości" i twierdzenie Darbo o punkcie stałym.

РЕЗЮМЕ

В работе рассматривается проблема существования решения интегрально-дифференциального уравнения нейтрального типа вида [0]. Для решения этой проблемы применяется метод, опирающийся на понятии „меры некомпактности" и теорема Дарбо о неподвижной точке.