ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. XXVIII, 1 SECTIO A 1974

Department of Mathematics, Brigham Young University, Provo, Utah 84602, USA
Department of Mathematics, Marquette University, Milwaukee, Wisconsin 53233, USA

DOUGLAS M. CAMPBELL, MICHAEL R. ZIEGLER

The Argument of the Derivative of Linear-Invariant Families of Finite Order and the Radius of Close-to-Convexity

Argument pochodnej i promień prawie wypukłości liniowo-niezmienniczej rodziny funkcji skończonego rzędu

Аргумент производной и радиус почти выпуклости линейно-инвариантного семейства функций конечного порядка

I. Preliminary Remarks

We begin by stating some of the basic definitions and results in the theory of linear invariant families of locally univalent analytic functions.

Let L be the set of Möbius transformations of D onto D where $D = \{z: |z| < 1\}$.

Pommerenke [13] has defined a family of functions of the form $f(z) = z + \ldots$, analytic and locally univalent ($f'(z) \neq 0$) in D to be a **linear invariant family** if and only if for each $\Phi(z)$ in L and every f in M the function

$$\Lambda_\Phi[f(z)] = \frac{f[\Phi(z)] - f[\Phi(0)]}{f'[\Phi(0)]\Phi'(0)} = z + \ldots \tag{1.1}$$

is also in M.

If M is a linear invariant family, then the **order** of M is defined in [13] as

$$a = \sup\{|f''(0)/2|: f \in M\}. \tag{1.2}$$

The order of a linear invariant family is always greater than or equal to one. Let $\mathfrak{U}_a$ denote the union of all linear invariant families of order at most a. Then the (**universal**) family $\mathfrak{U}_a$ is itself linear invariant. If $f(z) = z + \ldots$ is analytic and locally univalent in D, then we may consider the linear invariant family $M(f)$ generated by $f(z)$; namely,

$$M(f) = \{\Lambda_\Phi[f(z)]: \Phi(z) \in L\}.$$

The order of $f(z)$ is the order of the linear-invariant family which it generates. As an aid in computing the order of $f(z)$, denoted order f, we have [13, p. 115]

$$\text{(1.3)}\qquad \text{order } f = \sup_{z\in D}\left|-\bar{z}+(1-|z|^2)f''(z)/2f'(z)\right| = \sup\{|g''(0)/2|:\ g\in M(f)\}.$$

Linear invariant families exist in great profusion in classical geometric function theory. The set S of normalized analytic univalent functions is a linear invariant family of order 2. Normalized convex univalent functions, close-to-convex functions of order β, and functions with boundary rotation bounded by $k\pi$ (denoted by V_k) are linear invariant families of order 1, $\beta+1$, and $k/2$, respectively. On the other hand, the starlike univalent functions are not linear-invariant.

The family $\mathfrak{U}_1$ is exceptional in that it is precisely the set of all normalized convex univalent functions while for each $a>1$, $\mathfrak{U}_a$ contains the function

$$\text{(1.4)}\qquad f_{i\gamma}(z) = \frac{1}{2i\gamma}\left[\left(\frac{1+z}{1-z}\right)^{i\gamma}-1\right],\ \gamma=(a^2-1)^{1/2}$$

whose order is a and which has infinite valence [13, p. 128]. Despite this gross discontinuity between the possible valence of functions in $\mathfrak{U}_a$, $a>1$, and $\mathfrak{U}_1$, many properties are purely a function of the order of the linear invariant family rather than any intrinsic geometry of the family. For example, the radius of convexity of a linear invariant family of order a is always $a-(a^2-1)^{1/2}$ [13, p. 133].

II. Linear Invariance and the Functions $G(r)$ and $\gamma(t)$.

In order to gain some control over the behavior of $\arg f'(z)$ for $f\in M$, we introduce the following:

Definition. *Let M be a family of normalized functions which are analytic and locally univalent in D. Then for $0\leqslant r<1$, let*

$$\text{(2.1)}\qquad G(r, M) = G(r) = \sup_{f\in M}\max_{|z|=r} \arg f'(z),$$

where the argument varies continuously from the initial value of $\arg f'(0)=0$.

Lemma 2.1. *For any linear invariant family M*

$$G(r) = -\inf_{f\in M}\min_{|z|=r}\arg f'(z).$$

Proof. Let $f(z)$ be in M, z and ζ in D and

$$f(z,\zeta)=\frac{f\left(\dfrac{z+\zeta}{1+\bar{\zeta}z}\right)-f(\zeta)}{f'(\zeta)(1-|\zeta|^2)}.$$

Since M is a linear invariant family, $f(z,\zeta)$ also belongs to M. If $z^* = (z+\zeta)/(1+\bar{\zeta}z)$ then a brief calculation shows that

$$(2.2)\qquad (1-|z|^2)f'(z,\zeta)=\frac{1-|z^*|^2}{1-|\zeta|^2}\cdot\frac{1+\zeta\bar{z}}{1+\bar{\zeta}z}\cdot\frac{f'(z^*)}{f'(\zeta)}.$$

In particular, when $z=-\zeta$ we have

$$(2.3)\qquad (1-|z|^2)f'(-\zeta,\zeta)=[(1-|\zeta|^2)f'(\zeta)]^{-1},$$

from which the lemma follows.

Since $\max\{\arg f'(z)\colon |z|=r,\ 0\leqslant r<1\}$ is a monotone increasing function of r, $G(r)$ is also monotone increasing. In general, the supremum of monotone increasing piecewise analytic continuous functions need not be continuous. Nevertheless, $G(r)$ is in fact continuous.

Theorem 2.2. *Let M be a linear invariant family of finite order. Let $M^\wedge$ denote the closure of M in the topology of uniform convergence on compacta. Let $M(*)=\{f(sz)/s\colon f\in M \text{ and } 0<s\leqslant 1\}$. Then*

$$(2.4)\qquad G(r,M)=G(r,M(*))=G(r,M^\wedge).$$

Furthermore, $G(r)$ is a monotone increasing continuous function of r satisfying

$$2\arcsin r\leqslant G(r),\qquad 0\leqslant r<1.$$

Proof. Since $G(r,M)\leqslant G(r,M^\wedge)$ and $G(r,M)\leqslant G(r,M(*))$, to establish (2.4) it will suffice to show that $G(r,M^\wedge)\leqslant G(r,M)$ and $G(r,M(*))\leqslant G(r,M)$. $M^\wedge$ is a compact linear invariant family. Hence there is an $f(z)$ in $M^\wedge$ such that $G(r,M^\wedge)=\arg f'(r)$. Since $M^\wedge$ is the closure of M, there is a sequence of functions f_n from M which converges to f locally uniformly. Thus $G(r,M)\geqslant\lim\arg f_n'(r)(n\to\infty)=G(r,M^\wedge)$. To obtain the second inequality we chose a sequence of functions f_n from $M(*)$ and a sequence of points z_n in D, $|z_n|=r$, such that $\arg f_n'(z_n)\to G(r,M(*))$. Since $f_n'(z_n)=g_n'(s_nz_n)$, $g_n\in M$, $0<s_n\leqslant 1$, we have

$$\arg f_n'(z_n)\leqslant\max_{|z|=r}\arg g_n'(z)\leqslant G(r,M).$$

Taking the limit as $n\to\infty$ yields $G(r,M(*))\leqslant G(r,M)$ and completes the proof of (2.4).

Since $G(r)$ is monotone increasing, in order to establish the continuity of $G(r)$ it suffices to show $G(r^-) \geqslant G(r^+)$ for all r in $(0, 1)$. We may assume M is compact by (2.4). Choose f_n in M and $r_n \to r$ such that $\arg f_n'(r_n) \to G(r^+)$. By compactness there is an f in M such that $\arg f'(r) = G(r^+)$. The continuity of $\arg f'(r)$ implies

$$G(r^-) \geqslant \arg f'(r) = G(r^+),$$

which concludes the proof of continuity.

If $f(z)$ is any function in M, then $g(z) = 2f(z/2)$ is in $M(*)$ and satisfies

$$(1-z)g''(z)/g'(z) = (1-z)f''(z/2)/[2f'(z/2)].$$

Consequently, $\lim (1-z)g''(z)/g'(z) = 0$ as $z \to 1$ and, by theorem 3.14 in [14], the function $z/(1+z)$ is in $M(*)\hat{}$. Since $G(r, M(*)) = G(r, M(*)\hat{})$ and $\max \arg z/(1+z) = 2 \arcsin r$ ($|z| = r$), we have

$$G(r) = G(r, M(*)) \geqslant 2 \arcsin r.$$

Corollary 2.3. *If M is a linear invariant family then*

$$\sup_{f \in M} \sup_{z \in D} \arg f'(z) \geqslant \pi.$$

Proof. If M is of finite order then this is immediate from Theorem 2.2. If M is of infinite order then Theorem 2.10 shows that $\sup \sup \arg f'(z)$ is actually ∞ which is certainly greater than π.

Corollary 2.4. *If M is any linear invariant family of convex univalent functions, then $G(r) = 2 \arcsin r$.*

Proof. This is immediate from Theorem 2.2 and the fact that $|\arg f'(z)| \leqslant 2 \arcsin r$ for any convex univalent function.

Kirwan [9] defines a family M to be **rotationally invariant** if whenever f is in M then $f(tz)/t$, $0 < |t| \leqslant 1$, t complex, is also in M. The convex functions, close-to-convex functions, V_k, S, and $\mathfrak{U}_a$ [1, Theorem 5] are examples of linear invariant families which are also rotationally invariant.

Theorem 2.5. *If M is a compact rotationally linear invariant family of finite order, then M contains the function $z/(1+z)$.*

Remark. This places an immediate constraint on distortion results for families of this type.

Proof. If M is compact rotationally linear invariant, then $M = M(*) = M(*)\hat{}$ and the last part of the proof of Theorem 2.2 shows that $z/(1+z)$ must be in M.

For several well-known linear invariant families $G(r, M)$ can be determined explicitly. For the convex functions $G(r) = 2 \arcsin r$, for close-

-to-convex functions $G(r) = 4\arcsin r$, for functions in V_k $G(r) = k\arcsin r$, for functions β-close-to- V_k $G(r) = (k+2\beta)\arcsin r$ [2], and for functions in S $G(r) = 4\arcsin r$ if $0 \leq r \leq 1/\sqrt{2}$ while $G(r) = \pi + \log[r^2/(1-r^2)]$ if $1/\sqrt{2} \leq r < 1$ [6, p. 115]. Theorem 2.2 indicates that we cannot determine the linear invariant family M if we know $G(r)$. However the following results show that $G(r)$ does uniquely define the order of M.

Theorem 2.6. *Let M be a linear invariant family of order a, let $t \in (0, \infty)$, $r = \tanh t$, and define*

$$(2.5) \qquad \gamma(t) \equiv G(\tanh t)/2t = \sup_{f \in M} \max_{|z| = \tanh t} (1/2t)\arg f'(z).$$

Then

1) $\gamma(t) = -\inf_{f \in M} \min_{|z| = \tanh t} (1/2t)\arg f'(z)$.

2) $(t_1+t_2)\gamma(t_1+t_2) \leq t_1\gamma(t_1) + t_2\gamma(t_2)$.

e) $\lim_{t\to\infty} \gamma(t) = \gamma(\infty)$ *exists.*

4) $0 \leq \gamma(\infty) \leq \gamma(t) \leq a$.

5) $0 \leq \gamma(\infty) \leq (a^2-1)^{1/2}$

6) $\gamma(t)$ *is continuous in* $(0, \infty)$ *and* $\lim_{t\to 0}\gamma(t) = a$.

7) *Let a and γ be real numbers with $a \geq 1$ and γ in $[0, (a^2-1)^{1/2}]$. Then there is a linear invariant family of order a with $\gamma(\infty) = \gamma$.*

Proof. 1) follows directly from lemma 2.1. Let t_k $(k = 1, 2)$ be given in $(0, \infty)$, $r_k = \tanh t_k$ and $z_k = r_k e^{i\theta}$. If $r = \tanh(t_1+t_2)$ and $z = re^{i\theta}$, then $(z_1+z_2)/(1+z_1\bar{z}_2) = z$. Using z, z_1, and z_2 in (2.2) yields

$$(1-|z_1|^2)f'(z_1, z_2) = \frac{1-|z|^2}{1-|z_2|^2}\cdot\frac{f'(z)}{f'(z_2)}$$

which implies

$$\arg f'(z) = \arg f'(z_1, z_2) + \arg f'(z_2) \leq 2t_1\gamma(t_1) + 2t_2\gamma(t_2).$$

Hence

$$2(t_1+t_2)\gamma(t_1+t_2) \leq 2t_1\gamma(t_1) + 2t_2\gamma(t_2)$$

which proves the second assertion.

The third claim follows immediately from (2) and a problem in Polya and Szego [12, Vol. 1, p. 17]. Furthermore, (2) implies $\gamma(nt) \leq \gamma(t)$ for any integer n, thus $\gamma(\infty) \leq \gamma(t)$ for all t in $(0, \infty)$. Since $r = \tanh t$ is equivalent to $t = (1/2)\log[(1+r)/(1-r)]$, the estimates [13, p. 126]

$$|\log(1-|z|^2)f'(z)| \leq a\log[(1+r)/(1-r)]$$

and

$$|\arg f'(z)| \leq (a^2-1)^{1/2}\log[(1+r)/(1-r)] + 2\arcsin r$$

immediately yield $\gamma(t) \leqslant a$ and $\gamma(\infty) \leqslant (a^2-1)^{1/2}$, which completes the proof of (4) and (5).

The first part of (6) follows from Theorem 2.2. From (4) we have $\gamma(t) \leqslant a$ for all t; thus the remainder of (6) follows upon showing that as $t \to 0^+$, $\liminf \gamma(t) \geqslant a$. As in Theorem 2.2 we may assume that M is compact and choose an f in M such that $f''(0)/2 = a_2 = a$. Thus for z sufficiently small,

$$\arg f'(z) = \arg\big(1+2a_2 z + O(z^2)\big)$$

and

$$\max_{|z|=r} \arg f'(z) = \arcsin[2a_2 r + O(r^2)].$$

Consequently

$$\gamma(t) = \sup_{f \in M} \max_{|z|=\tanh t} \frac{\arg f'(z)}{2t}$$

$$\geqslant \frac{\arcsin[2a_2 r + O(r^2)]}{\log[(1+r)/(1-r)]}$$

and

$$\liminf_{t\to 0^+} \gamma(t) \geqslant \lim_{r \to 0^+} \frac{\arcsin[2a_2 r + O(r^2)]}{\log[(1+r)/(1-r)]} = a_2 = a.$$

Since (7) is trivial for $a = 1$, we may suppose that $a > 1$ and choose any number γ in $[a, (a^2-1)^{1/2}]$. Let

$$c = a(a^2-1-\gamma^2)(a^2-1)^{-1/2} + i\gamma,$$

and consider

$$f_c(z) = \frac{1}{2c}\left[\left(\frac{1+z}{1-z}\right)^c - 1\right]. \tag{2.6}$$

Then the order of $f_c(z)$ is [4, Theorem 2.1]

$$\frac{1}{\sqrt{2}}\{|c|^2+1+[(1-|c|^2)^2+4\gamma^2]^{1/2}\}^{1/2},$$

and a computation shows that this reduces to a. Thus to prove (7) it suffices to show that $\gamma(\infty) = \gamma$ for the linear invariant family M generated by $f_c(z)$. For any $\Phi(z)$ in L we have

$$\ln \Lambda'_\Phi[f_c(z)] = \ln\big[f'_c(\Phi(z))\Phi'(z)/f'_c(\Phi(0))\Phi'(0)\big].$$

Letting $c = a + i\gamma$,

$$\arg \Lambda'_\Phi[f_c(z)] = \gamma \ln\left|\frac{(1+\Phi(z))(1-\Phi(0))}{(1-\Phi(z))(1+\Phi(0))}\right| + (a+1)\arg\left(\frac{1+\Phi(z)}{1+\Phi(0)}\right) + $$
$$+ (a-1)\arg\left(\frac{1-\Phi(z)}{1-\Phi(0)}\right) - 2\arg(1+\zeta z).$$

If $z = re^{i\theta}$, then

$$|\{[1+\Phi(z)][1-\Phi(0)]\}/\{[(1-\Phi(z)][1+\Phi(0)]\}| \leqslant (1+r)/(1-r)$$

and thus

$$\text{(2.7)} \quad \arg \Lambda'_\Phi[f_c(z)] \leqslant \gamma \ln[(1+r)/(1-r)] + |a+1|\pi + |1-a|\pi + \pi$$
$$\leqslant \gamma \ln[(1+r)/(1-r)] + (2a+3)\pi .$$

On the other hand

$$\text{(2.8)} \qquad \arg f'_c(r) = \gamma \ln(1+r)/(1-r),$$

hence (2.7) and (2.8) yield

$$\gamma \leqslant G(r, M)/\ln[(1+r)/(1-r)] = \gamma(t) \leqslant \gamma + \frac{(2a+3)\pi}{t}$$

which shows that $\gamma(\infty) = \gamma$ and completes the proof of the theorem.

Corollary 2.7. *Let M be a linear invariant family of order a. Then $G'(0^+)$ always exists and satisfies $G'(0^+) = 2a$.*

Proof. We have

$$G'(0^+) = \lim_{r\to 0^+} \frac{G(r)}{r} = \lim_{r\to 0^+} \frac{\ln[(1+r)/(1-r)]}{r} \frac{G(r)}{\ln[(1+r)/(1-r)]}$$
$$= 2 \lim_{r\to 0^+} \gamma(t) = 2a.$$

Pommerenke's best estimates [13] on $\arg f'(z)$ for $f(z)$ in $\mathfrak{U}_a$ are

$$|\arg f'(z)| \leqslant 2\int_0^r \frac{(a^2-x^2)^{1/2}}{1-x^2}\,dx \leqslant (a^2-1)^{1/2}\log\frac{1+r}{1-r} + 2\arcsin r$$

while, for any z in D, there is an $f(z)$ in $\mathfrak{U}_a$ with

$$\text{(2.9)} \qquad |\arg f'(z)| \geqslant (a^2-1)^{1/2}\log[(1+r)/(1-r)].$$

One might therefore conjecture that for $\mathfrak{U}_a$, $G(r)$ is either

$(a^2-1)^{1/2}.\log[(1+r)/(1-r)]$ or $(a^2-1)^{1/2}\log[(1+r)/(1-r)] + 2\arcsin r$.

Neither conjecture is true for any $a > 1$ since in the first case $G'(0) = 2(a^2-1)^{1/2} \neq 2a$, while in the second $G'(0) = 2[(a^2-1)^{1/2}+1] \neq 2a$. This suggests that it should be possible to improve (2.9) and it is.

Theorem 2.8. *For each a in $(1, \infty)$ and for each z satisfying $0 < |z| < 1/a$, there is an $f(z)$ in $\mathfrak{U}_a$ with $\arg f'(z) > (a^2-1)^{1/2}\log[(1+r)/(1-r)]$.*

Proof. Since $\mathfrak{U}_a$ is rotationally invariant we may assume $z = r$, $0 < r < 1/a$. Let

$$f_r(z) = \int_0^z (1+we^{i\lambda})^{a-1}(1-we^{-i\lambda})^{-a-1}dw$$

where $\lambda = \arccos r$. The function f_r is in V_{2a} since it is generated by the measure with weight $a-1$ at $\theta = \lambda$ and weight $a+1$ at $\theta = -\lambda$. Furthermore, $\arg f_r'(r) = 2a\arcsin r$. Since $V_{2a} \subset \mathfrak{U}_a$, it now suffices to show that

$$2a\arcsin r > (a^2-1)^{1/2}\log[(1+r)(1-r)]$$

for $0 < r < 1/a$. An elementary calculation shows that

$$h(r) = 2a\arcsin r - (a^2-1)^{1/2}\log[(1+r)/(1-r)]$$

is a strictly increasing function of r, $r \in (0, 1/a)$, and, since $h(0) = 0$, this completes the proof.

A careful examination of Pommerenke's proof that

$$|\arg f'(z)| \leqslant 2\int_0^r (a^2-x^2)^{1/2}(1-x^2)^{-1}dx,$$

$|z| = r$, $f \in \mathfrak{U}_a$, leads one to consider

$$f(z) = \int_0^z \exp\left[2i\int_0^w (a^2-x^2)^{1/2}(1-x^2)^{-1}dx\right]dw$$

as a possible extremal function for the maximum of the argument of the derivative. Indeed, in this case $\arg f'(r) = 2\int_0^r (a^2-x^2)^{1/2}(1-x^2)^{-1}dx$ which would certainly make it extremal. Unfortunately, $f(z)$ is not in $\mathfrak{U}_a$. This is difficult to verify directly from the definition of the order of $f(z)$; however, if we note that $(1-z)f''(z)/f'(z) \to i(a^2-1)^{1/2}$ as $z \to 1$ in any angle, then f has as a limit function [14, Theorem 3.14]

$$f_c(z) = (1/2c)\{[(1+z)/(1-z)]^c - 1\},$$

$c = -1+i(a^2-1)^{1/2}$. Furthermore, the order of $f_c(z)$ [4, Theorem 2.1] is

$$\beta = [a^2+1+(a^4+2a^2-3)^{1/2}]^{1/2}/\sqrt{2}.$$

A computation shows that $\beta > a$ for all $a > 1$. If M is the linear invariant family generated by $f(z)$, then $f_c(z)$ is in $M^\wedge$ and, since order M = order $M^\wedge$, it follows that order $f(z) \geqslant$ order $f_c(z) = \beta > a$. Consequently, $f(z)$ is not in $\mathfrak{U}_a$.

One fruitful method of investigation of $\mathfrak{U}_a$ has been to place various normed linear space structures on $\mathbf{X} = \bigcup_{a \geqslant 1} \mathfrak{U}_a$ [3]. Following Hornich, we define an addition and a multiplication on the set of normalized locally univalent analytic functions in D as follows:

$$[f+g](z) = \int_0^z f'(\omega) g'(\omega)\, d\omega \qquad (f, g \in \mathbf{X}),$$

$$[af](z) = \int_0^z [f'(\omega)]^a\, d\omega \qquad (f \in \mathbf{X},\ a \text{ real})$$

where square brackets denote the algebraic operations on $\mathbf{X}$.

Theorem 2.9. *If f is in $\mathbf{X}$ and a is real, then*

(2.10) $$|a|\gamma_f(\infty) = \gamma_{[af]}(\infty),$$

where $\gamma_g(\infty)$ denotes the value of $\gamma(\infty)$ for the linear invariant family M_g which g generates.

Proof. We actually show that

(2.11) $$\big||a|\gamma_f(t) - \gamma_{[af]}(t)\big| \leqslant \pi|a-1|/2t,$$

from which (2.10) is obvious. For any $\Phi(z)$ in L and any r in $[0, 1]$ a computation shows

$$|a|\,|\arg \Lambda'_\Phi(f(z))| = |\arg \Lambda'_\Phi([af](z)) + (a-1)\arg\{\Phi(z)/\Phi'(0)\}|$$
$$\leqslant |\arg \Lambda'_\Phi([af](z))| + |a-1|\pi,$$

where we have used the fact that $\Phi(z)$ in L implies $\Phi(z)$ is of the form $\Phi(z) = e^{i\theta}(z+\zeta)/(1+\bar{\zeta}z)$ and therefore $\Phi'(z)/\Phi'(0) = (1+\bar{\zeta}z)^{-2}$, hence $|\arg\{\Phi'(z)/\Phi'(0)\}| \leqslant \pi$.

Therefore

$$|a|\,|\arg \Lambda'_\Phi(f(z))| \leqslant G(r, M_{[af]}) + |a-1|\pi$$

and consequently

$$|a|G(r, M_f) \leqslant G(r, M_{[af]}) + |a-1|\pi.$$

Upon reversing the roles of f and $[af]$, we obtain

$$\big||a|G(r, M_f) - G(r, M_{[af]})\big| \leqslant |a-1|\pi$$

from which (2.11) follows directly.

It is perhaps appropriate to remark at this stage of development that a function $\beta(t)$, similar to $\gamma(t)$, was introduced by Pommerenke for the study of the distortion of $|f'(z)|$ in linear invariant families. To his conclusions [13, Theorem 2.2] one can add the facts that $\beta(t)$ is continuous, $\lim_{t\to 0^+}\beta(t) = a$ and for each β in $[1, a]$ there is a linear invariant family M with $\beta(\infty) = \beta$. There are several differences in the behavior of $\beta(t)$ and $\gamma(t)$. Although $\beta(t)$ may be constant, $\gamma(t)$ cannot be constant. This is obvious since $\gamma(\infty) < \gamma(0)$. The function $\gamma_{[af]}(\infty)$ is homogeneous in $|a|$ while $\beta_{[af]}(\infty)$ is not. Finally, Pommerenke was able to characterize compact linear invariant families of order a for which $\beta(\infty) = a$; these were those families containing the function $\left(\frac{1}{2a}\right)[((1+z)/(1-z))^a - 1]$. In would be of interest to obtain a comparable proposition concerning the function $\gamma(t)$.

We have previously mentioned that $G(r) \equiv k \arcsin r$ for the linear invariant family V_k. We have also seen (Corollary 2.4) that $G(r, M) \equiv 2 \arcsin r$ for any linear invariant subfamily M of V_2 (the convex univalent functions). Nevertheless, the boundary rotation of a linear invariant family M of V_k, $k > 2$, is not enough to determine $G(r, M)$ for either small or large values of r. For example, let $f(z)$ be a close-to-convex univalent function with boundary rotation 100π (such a function is easily constructed). Let M be the linear invariant family generated by $f(z)$. Then $G(r, M)$ does not behave as $100 \arcsin r$ for either small or large values of r. In fact since M consists of close-to-convex functions, we know that $G(r, M) \leqslant 4 \arcsin r$, $0 \leqslant r < 1$. This shows that $G(r)$ need not depend simply on the boundary rotation of a linear invariant family.

We know (Corollary 2.7) that $G'(0^+)$ is intimately related to the order of the linear invariant family generating $G(r)$. If

$$G(1) \equiv \lim_{r\to 1} G(r) = \sup_{f\in M} \sup_{|z|<1} \arg f'(z)$$

is finite, then $G(1)$ is also related to the order of the linear invariant family generating $G(r)$. We can obtain a relationship between $G(1)$ and the order of M by utilizing the class $K(\beta)$ of generalized close-to-convex functions of order β examined by A. W. Goodman [7]. A function f in $\mathbf{X}$ is in $K(\beta)$, $\beta \geqslant 0$, if for each r in $[0, 1)$ and each pair θ_1 and θ_2, $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$, we have

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}[1 + re^{i\theta} f''(re^{i\theta})/f'(re^{i\theta})]\,d\theta \geqslant -\beta\pi; \tag{2.13}$$

equivalently if there is a nonzero complex number C and a normalized convex univalent function $\Phi(z)$ such that for z in D

$$|\arg\{cf'(z)/\Phi'(z)\}| \leqslant \beta\pi/2.$$

Theorem 2.10. *Let M be a linear invariant satisfying*

$$\sup_{f\in M}\ \sup_{|z|<1} \arg f'(z) = \beta\pi < \infty.$$

Then M is contained in the linear invariant family $K(\beta)$. Furthermore, $|\arg f'(z)| \leqslant 2(\beta+1)\arcsin r$. Finally, if $a =$ order M, then a is finite and satisfies $1 \leqslant a \leqslant \beta+1$.

Proof. We show that M is in $K(\beta)$ but is not in $K(\beta-2)$ (when $\beta > 2$). Let $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$, $0 \leqslant \theta_1 < \theta_2 < 2\pi$. Then

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}[1+re^{i\varphi}f''(re^{i\varphi})/f'(re^{i\varphi})]\,d\varphi = \arg[z_2 f'(z_2)/(z_1 f'(z_1))].$$

Since M is rotationally invariant the minimum value of the above integral over M depends only on r and $\theta_2-\theta_1$. We therefore set $\theta = \theta_2-\theta_1$ and

$$\Delta(r,\theta) = \inf\{\arg[z_2 f'(z_2)/z_1 f'(z_1)]\colon f\in M\}.$$

Since

$$\arg\left[\frac{z_2 f'(z_2)}{z_1 f'(z_1)}\right] = \arg\left[\frac{z_2}{z_1}\left(\frac{1-|z_1|^2}{1-\bar z_1 z_2}\right)^2\right] + \arg f'(\zeta_0, z_1),$$

where we define ζ_0 by $\zeta_0 = (z_2-z_1)/(1-\bar z_1 z_2)$, we see that

$$\Delta(r,\theta) = 2\operatorname{arccot}\left[\left(\frac{1-r^2}{1+r^2}\right)\cot(\tfrac{1}{2}\theta)\right] + \inf_{f\in M} \arg f'(\zeta_0, z_1). \tag{2.4}$$

Because of the linear invariance of M,

$$\inf[\arg f'(\zeta_0, z_1) : f\in M] = \inf[\arg f'(\zeta_0) : f\in M]$$

and hence by the hypothesis we have

$$(2-\beta)\pi \geqslant \inf_{|z|<1} \Delta(r,\theta) \geqslant -\beta\pi.$$

Thus M is a subset of $K(\beta)$ but, for $\beta > 2$, M is not a subset of $K(\beta-2)$. The remainder of the theorem now follows from well-known results for $K(\beta)$. Namely, order $K(\beta) = \beta+1$ and

$$|\arg f'(z)| \leqslant 2(\beta+1)\arcsin|z|,\ f \text{ in } K(\beta).$$

Much better results are presented in Theorem 3.5, but under considerably stronger hypothesis.

Concluding this section we remark that Theorem 2.6 may be used to show certain families are not linear invariant. For example, as one type of generalization of V_k, Pinchuk [11] let V_k^λ denote the class of functions in $\mathbf{X}$ which satisfy

$$\sup_{0\leqslant r<1} \int_0^{2\pi} |\operatorname{Re}[e^{i\lambda}(1+re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta}))]|\,d\theta = k\pi\cos\lambda,$$

$k \geqslant 2$, $|\lambda| < \pi/2$. One can show that

$$a = \sup\{a_2 \colon f \in V_k^\lambda\} = k|1 + e^{-2i\lambda}|/4$$

while $\gamma(\infty, V_k^\lambda) \geqslant (k+2)|\sin 2\lambda|/4$. It is easy to see that the inequality $\gamma(\infty) \leqslant (a^2-1)^{1/2}$ is not valid for various values of λ and k and hence, by Theorem 2.6, for these values of λ and k, V_k^λ cannot be a linear invariant family. We conjecture that V_k^λ is linear invariant if and only if $\lambda = 0$.

III. $G(r)$ and the Radius of Close-to-Convexity

The radius of close-to-convexity of a family is useful in that it provides an upper bound for the radius of starlikeness and a lower bound for the radius of univalence. In this section we place additional restrictions on $G(r)$ and are thereby able to determine the precise radius of close-to-convexity of various linear invariant families. All known examples of $G(r)$ satisfy the conditions we assume and we are thus able to obtain previously known results as corollaries.

Theorem 3.1. *Let M be a linear invariant family of order a, $a > 1$. If $G'(r, M)$ exists for each r in $[0, 1)$ and if for each fixed ϱ in $(2a/(1+a^2), 1)$ the equation*

$$G'(r) = 2(\varrho^2 - r^2)^{-1/2} \tag{3.1}$$

has a unique root r_ϱ within $[0, 1)$, then the radius of close-to-convexity of M is $\varrho_0[1+(1-\varrho_0^2)^{1/2}]^{-1}$ where

$$\varrho_0 = \sup\{\varrho \in (2a/(1+a^2), 1) \colon 2\arcsin(r_\varrho/\varrho) - G(r_\varrho) \geqslant -\pi\}. \tag{3.2}$$

Proof. Let $z_2 = e^{i\theta} z_1$, $|z_1| = r$, $\theta \in (0, 2\pi)$ and set

$$\Delta(r, \theta) = \inf_{f \in M} \arg[z_2 f'(z_2)/z_1 f'(z_1)].$$

The radius of close-to-convexity of M is the supremum of all r in $[0, 1)$ such that $\Delta(r, \theta) \geqslant -\pi$ for all $\theta \in (0, 2\pi)$, [8]. We may assume M is compact since the radii of close-to-convexity of M and $M^\wedge$ are the same. As in the proof of Theorem 2.10, we have

$$\begin{aligned} \Delta(r, \theta) &= 2\,\mathrm{arccot}\left[\left(\frac{1-r^2}{1+r^2}\right)\cot\frac{\theta}{2}\right] + \inf\{\arg f'(\zeta_0) \colon f \in M\} \\ &= 2\,\mathrm{arccot}\left[\left(\frac{1-r^2}{1+r^2}\right)\cot\frac{\theta}{2}\right] - G(|\zeta_0|), \end{aligned} \tag{3.3}$$

where $|\zeta_0| = r[2(1-\cos\theta)/(1-2r^2\cos\theta + r^4)]^{1/2}$. The compactness of M implies that for any z_1 and z_2 there is an f in M for which $\arg[z_2 f'(z_2)/z_1 f'(z_1)]$

$=\Delta(r,\theta)$. We wish to find the minimum of $\Delta(r,\theta)$ for fixed r. Upon differentiating $\Delta(r,\theta)$ with respect to θ we obtain

$$\frac{\partial}{\partial\theta}\Delta(r,\theta)=\frac{1-r^4}{1-2r^2\cos\theta+r^4}\times \tag{3.4}$$

$$\left[1-G'(|\zeta_0|)\left(\frac{1-r^2}{1+r^2}\right)\cdot\frac{\sin\theta}{[2(1-\cos\theta)]^{1/2}}\cdot\frac{r}{(1-2r^2\cos\theta+r^4)^{1/2}}\right].$$

Since we are only interested in a minimum, and because $G'(|\zeta_0|)\geqslant 0$, we need consider $\theta\in(0,\pi)$. Letting $\varrho=2r/(1+r^2)$ and noting that

$$\frac{\sin\theta}{[2(1-\cos\theta)]^{1/2}}=\frac{1}{\varrho}\left[\frac{\varrho^2-|\zeta_0|^2}{1-|\zeta_0|^2}\right]^{1/2},$$

the inner factor of (3.4) can be written as

$$h(|\zeta_0|)=1-\frac{1}{2}G'(|\zeta_0|)[\varrho^2-|\zeta_0|^2]^{1/2}. \tag{3.5}$$

Since $f(z)$ is starlike, and hence close-to-convex for all r in $[0,1/a]$, [13, p. 134], we need only consider r in $(1/a,1)$ or, equivalently, ϱ in $\big(2a/(1+a^2),1\big)$. As θ varies from 0 to π, $|\zeta_0|$ varies from 0 to ϱ. Our hypothesis guarantees for each ϱ in $\big(2a/(1+a^2),1\big)$ that $h(|\zeta_0|)$ has exactly one root, denoted by r_ϱ, within the interval $[0,\varrho]$. (Note that we do not require the continuity of G' nor do we postulate anything about the number of or lack of roots of h in $(\varrho,1)$). It is easy to verify that this value r_ϱ yields a minimum for $\Delta(r,\theta)$. From

$$r_\varrho=r\left[\frac{2(1-\cos\theta_\varrho)}{1-2r^2\cos\theta_\varrho+r^4}\right]^{1/2},$$

it is easy to deduce by a half angle formula that

$$\cos\frac{1}{2}\theta_\varrho=\frac{1}{\varrho}\left[\frac{\varrho^2-r_\varrho^2}{1-r_\varrho^2}\right]^{1/2}.$$

Consequently,

$$\cot\frac{1}{2}\theta_\varrho=\frac{1}{r_\varrho}\left[\frac{\varrho^2-r_\varrho^2}{1-\varrho^2}\right]^{1/2}.$$

Thus,

$$\Delta(r)=\inf_\theta\Delta(r,\theta)=2\operatorname{arccot}\left[\frac{(1-r_\varrho^2)}{(1+r_\varrho^2)}\cot\frac{1}{2}\theta_\varrho\right]-G(r_\varrho) \tag{3.6}$$

$$=2\arcsin(r_\varrho/\varrho)-G(r_\varrho),$$

and, by compactness, there is a function f in M for which $\arg[z_2 f'(z_2)/z_1 f'(z_1)] = \Delta(r)$. Since $\Delta(r)$ is a decreasing functions of r and $\varrho = 2r/(1+r^2)$ is equivalent to $r = \varrho[1+(1-\varrho^2)^{1/2}]^{-1}$, the radius of close-to-convexity of M is $\varrho_0[1+(1-\varrho_0^2)^{1/2}]^{-1}$ where ϱ_0 is the supremum of those ϱ's in $(2a/(1+a^2), 1)$ for which

$$2\arcsin(r_\varrho/\varrho) - G(r_\varrho) \geqslant -\pi.$$

This completes the proof of the theorem.

Corollary 3.2. *For each ϱ in $(4/5, 1)$, let r_ϱ denote the unique solution in $(0, \varrho)$ of*

$$r^6 - 2r^4 + 2r^2 - \varrho^2 = 0.$$

Then the radius of close-to-convexity of S is $\varrho_0[1+(1-\varrho_0^2)^{1/2}]^{-1}$, where ϱ_0 is the unique solution of

$$2\arcsin(r_\varrho/\varrho) - \log[r_\varrho^2/(1-r_\varrho^2)] = 0.$$

Proof. S is a linear invariant family of order 2 and $G(r)$ is $4\arcsin r$ if $0 \leqslant r \leqslant 1/\sqrt{2}$ and $\pi + \log[r^2/(1-r^2)]$ if $1/\sqrt{2} \leqslant r < 1$ [6, p. 115]. If $\varrho \in (4/5, 1)$, then there is a solution to

$$G'(r) = 2(\varrho^2 - r^2)^{-1/2} \tag{3.7}$$

only if $r \geqslant 1/\sqrt{2}$ and these solutions are roots of

$$p(r) = r^6 - 2r^4 + 2r^2 - \varrho^2 = 0.$$

Since $p(0) = -\varrho^2$ and $p(\varrho) = \varrho^2(\varrho^2-1)^2$, there is at least one root in $(0, \varrho)$. However, $q(t) \equiv p(\sqrt{t}) = t^3 - 2t^2 + 2t - \varrho^2$ is monotone increasing and therefore there is only one root of (3.7) in $(0, \varrho)$ and it is denoted by r_ϱ. The remainder of the corollary now follows from Theorem 3.1 upon recalling that $\Delta(r) \to -\infty$ as $r \to 1$ for S.

Corollary 3.2 was first proved by Krzyż [10] who expressed his solution in a slightly different form. It is easy to see Krzyż's result agrees with ours. We need only show that if r_ϱ is the root of $r^6 - 2r^4 + 2r^2 - \varrho^2 = 0$, then

$$x = \frac{1}{r_\varrho}\left[\frac{\varrho^2 - r_\varrho^2}{1-\varrho^2}\right]^{1/2}$$

is a root of

$$X^3 - AX^2 + A^2X - A = 0$$

where $A = (1+r^2)/(1-r^2)$. If we notice that we can rewrite x as $A(1-r_\varrho^2)$ (by using 3.7) and if we use the fact that $A^2 = (1-\varrho^2)^{-1}$, then it is a routine verification that x is indeed a root of $X^3 - AX^2 + A^2X - A = 0$.

Corollary 3.3. *If M is a linear invariant family for which $G(r) = 2a\arcsin r$, then the radius of close-to-convexity is* 1 *if* $1 \leqslant a \leqslant 2$ *and* $\varrho_0[1+(1-\varrho_0^2)^{1/2}]^{-1}$ *where ϱ_0 is the unique solution of*

$$2\arcsin\left(\frac{a^2\varrho^2-1}{a^2\varrho^2-\varrho^2}\right)^{1/2} - 2\arcsin\left(\frac{a^2\varrho^2-1}{a^2-1}\right)^{1/2} = -\pi$$

if $a > 2$.

Proof. Since for $G(r) = 2a\arcsin r$ the solution of

$$G'(r) = 2(\varrho^2 - r^2)^{1/2}$$

is

$$r_\varrho = [(a^2\varrho^2-1)/(a^2-1)]^{1/2},$$

the radius of close-to-convexity is $\varrho_0[1+(1-\varrho_0^2)^{1/2}]^{-1}$ where ϱ_0 is the supremum of those ϱ's in $(2a/(1+a^2), 1)$ for which

$$h(\varrho) = 2\arcsin\left(\frac{a^2\varrho^2-1}{a^2\varrho^2-\varrho^2}\right)^{1/2} - 2a\arcsin\left(\frac{a^2\varrho^2-1}{a^2-1}\right)^{1/2} \geqslant -\pi.$$

However, $\lim_{\varrho\to 1} h(\varrho) = \pi - \pi a$ and $h(\varrho)$ is monotone decreasing in ϱ for ϱ in $(2a/(1+a^2), 1)$. Thus for any a, $1 \leqslant a \leqslant 2$, $h(1) \geqslant -\pi$ and the radius of close-to-convexity is 1.

Corollary 3.3 yields the radius of close-to-convexity for functions in V_k, functions β-close-to-convex, and functions β-close-to-V_k [2], since $G(r)$ for the above classes is $k\arcsin r$, $(2\beta+2)\arcsin r$, and $(2\beta+k)\arcsin r$, respectively. The radius of close-to-convexity of V_k had been determined previously by Coonce and Ziegler [5].

Corollary 3.4. *The radius of close-to-convexity for $\mathfrak{U}_a$, r_a, is greater than or equal to* $\varrho_0[1+(1-\varrho_0^2)^{1/2}]^{-1}$ *where ϱ_0 is the unique solution in* $(2a/(1+a^2), 1)$ *of*

$$(3.8)\qquad 2\arcsin(\omega/\varrho) - 2\int_0^{\omega}(a^2-x^2)^{1/2}(1-x^2)^{-1}dx = -\pi$$

where $\omega \equiv (\varrho^2a^2-1)(\varrho^2+a^2-2)]^{1/2}$. *Furthermore,* $\liminf_{a\to\infty} ar_a \geqslant \beta$, *where β is the unique root of the equation*

$$(3.9)\qquad \operatorname{arccot}[4\beta^2-1)^{-1/2}] - [(4\beta^2-1)]^{1/2} = -\pi/2.$$

in the interval $[\pi/4, \pi/2]$. *(β is approximately* 1.4858).

Proof. If $G(r, M) \leqslant F(r)$ and $F(r)$ is a function satisfying the conditions of Theorem 3.1, then an examination of the proof of Theorem 3.1 shows that the radius of close-to-convexity for M is at least $\varrho_0[1+(1-\varrho_0^2)^{1/2}]^{-1}$ where

$$\varrho_0 = \sup\{\varrho \in (2a/(1+a^2), 1): 2\arcsin(r_\varrho/\varrho) - F(r_\varrho) \geqslant -\pi\}.$$

In the case $M = \mathfrak{U}_a$ we may let $F(r) = 2\int_0^r (a^2-x^2)^{1/2}(1-x^2)^{-1}dx$ and note that r_ϱ is $[\varrho^2a^2-1)/(\varrho^2+a^2-2)]^{1/2}$ for ϱ in $(2a/(1+a^2), 1)$. The first conclusion follows as before. For large a the inequality $1/a \leqslant r_a \leqslant \pi/(2a)$ follows from known results on the radii of starlikeness and univalence of $\mathfrak{U}_a$ [13, p. 135]. In order to prove $\liminf_{a\to\infty} ar_a \geqslant \beta$, it is sufficient to show

$$\lim_{a\to\infty} a\varrho_0[1+(1-\varrho_0^2)^{1/2}]^{-1} = \beta.$$

If a_n is any sequence such that

$$\liminf_{n\to\infty} a_n\varrho_{0_n}[1+(1-\varrho_{0_n}^2)^{1/2}]^{-1} = \beta,$$

then it follows from (3.8) that

$$2\operatorname{arc\,cot}(4\beta^2-1)^{-1/2} - 2(4\beta^2-1)^{1/2} = -\pi$$

which is (3.9). Differentation shows that the left hand side of (3.9) is a monotone decreasing function with a unique root in $[\pi/4, \pi/2]$ and thus $\lim a_n\varrho_{0_n}[1+(1-\varrho_{0_n}^2)^{1/2}]^{-1}$ must exist.

Theorem 3.5. *Let M be a linear invariant family of order a. Suppose $G(r)$ is bounded on $[0, 1)$ and the unique solution r_ϱ in $(0, \varrho)$ of $G'(r) = 2(\varrho^2-r^2)^{-1/2}$ for ϱ in $(2a/(1+a^2), 1)$ tends to 1 as $\varrho\to 1$. Then M is contained in the close-to-convex functions of order $\beta = G(1)\pi^{-1}-1$. Consequently, $G(r) \leqslant 2G(1)\pi^{-1}\arcsin r$, and $a \leqslant G(1)\pi^{-1}$.*

Proof. We actually establish that M is contained in the close-to-convex functions of order $\beta = G(1)\pi^{-1}-1$ but not in the close-to-convex functions of any lower order. Thus the result is best possible. To establish this stronger claim it will suffice to prove that

$$\inf_{r\in[0,1)}\ \inf_{\theta\in(0,2\pi)} \Delta(r, \theta) = -(G(1)/\pi-1)\pi = \pi - G(1).$$

Using (3.3) of Theorem 3.1 we see that

$$\inf_{\theta\in(0,2\pi)} \Delta(r, \theta) = 2\arcsin(r_\varrho/\varrho) - G(r_\varrho).$$

Since $\varrho = 2r/(1+r^2)$, $r_\varrho \to 1$ as $\varrho \to 1$, G is continuous, and $\inf\{\Delta(r, \theta): :\theta \in (0, 2\pi)\} = 2 \arccos r_\varrho/\varrho - G(r_\varrho)$ is a decreasing function of r, we obtain

$$\inf_{r\in[0,1)} \inf_{\theta\in(0,2\pi)} \Delta(r, \theta) = 2 \arcsin 1 - G(1) = \pi - G(1),$$

which completes the proof of the first part of the theorem. The latter claims follow from previously cited facts about $K(\beta)$.

If the root r_ϱ Theorem 3.5 tends to $R < 1$, then the salient conclusion is that

$$\alpha = \frac{1}{\pi}\left(G(R) - 2 \arcsin R\right) + 1. \tag{3.10}$$

This condition can be used to show that no $G(r)$ can be of the form $G(r) = 2ar$. If this were the case, then $r_\varrho = (a^2\varrho^2 - 1)^{1/2}/a \to (a^2-1)^{1/2}/a = R$ as $\varrho \to 1$ and (3.10) becomes $\pi a \leqslant 2(a^2-1)^{1/2} - 2 \arcsin[(a^2-1)^{1/2}/a] + \pi$. However, a differentiation shows this last inequality is false for all $a > 1$. This shows that no linear invariant family can have $G(r) = 2ar$. It would be of interest to establish other positive and negative results on the possible forms of $G(r)$ for various linear invariant families.

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STRESZCZENIE

Praca dotyczy liniowo niezmienniczej rodziny M rzędu a, wprowadzonej przez Pommerenke. W szczególności suma mnogościowa $\mathfrak{U}_a$ wszystkich liniowo niezmienniczych rodzin rzędu co najwyżej a jest również rodziną liniowo niezmienniczą. Autorzy otrzymują oszacowanie $\arg f'(z)$ w klasie $\mathfrak{U}_a$ oraz znajdują promień prawie wypukłości dla rodzin liniowo niezmienniczych.

РЕЗЮМЕ

Работа посвящена линейно-инвариантному семейству M порядка a, введенному Померенке. В частности, сумма $\mathfrak{U}_a$ всех линейно-инвариантных семейств порядка не большего чем a также является линейно-инвариантным семейством. Авторы получают оценку $\arg f'(z)$ в классе $\mathfrak{U}_a$ а также находят радиус почти выпуклости для линейно-инвариантных семейств.