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Estimation of Variance Components in Unbalanced Mixed Models

Estymacja komponentów wariancyjnych w modelach mieszanych nieortogonalnych

Оценивание компонент дисперсии в несбалансированных смешанных моделях

1. Introduction In this paper we want to consider experiments in which the observed random variable is described as a sum of population mean, effects of the levels of treatments and the effect of experimental error. If inferences are going to be drawn about just the levels of the treatment that appear in experiment, the effects of such a treatment are considered as fixed. When the levels of the treatment occuring in the experiment are considered as a random sample from an infinite population of levels, then the effects of such a treatment are considered random. These random effects are regarded as random variables having zero means. Variances of these random variables are called the variance components.

In many cases, one sees the need for a model in which some effects are fixed, others random. Such a model will be called a mixed one. The population mean is always considered as fixed and the effect of error as random.

For balanced data the most frequently used technique of estimation of variance components is the analysis of variance method, which consists in equating the observed mean squares to their expected values, and solving the resulting equations.

In a balanced case for a mixed model when random effects are assumed independent and normal, the analysis of variance method leads to unbiased estimators with minimum variance [1]. The analysis of variance method cannot be used for unbalanced data. Henderson [3] proposed a method of estimation for an unbalanced mixed model, and that method was then developed by Oktaba [8] and Searle [10]. For unbalanced mixed models the unbiased estimators with minimum variance are unknown. There exist methods that give unbiased estimators. In mixed unbalanced models

Henderson's method requires the inversion of matrix. In practice there appear situations in which this method requires the inversion of matrix of the rank about 500.

Koch [4,5] developed for random models a method which is computationally simpler than that of Henderson. Niedokos [7] proposed a method of estimation for mixed unbalanced models which is similar to that given by Koch and does not require the inversion of matrix. Paper [7] deals with unbalanced two-way cross classification, and a model with the combination of two-way cross classification and nested classification. In this paper these two models are dealt with again without restrictions that are imposed in [7] on the fixed effects, and some estimators have been given a simpler form. Furthermore, the author developed methods of estimation for others three unbalanced mixed models consisting of a combination of nested and crossed classifications. The numerical example illustrating the use of estimators given in [7] is presented in [9] by Oktaba and Wesołowska.

2. Notation Letters A, B and C denote the treatments that occur in experiment; $A \times B$ denotes two-way cross classification; $A_i B_j$ denotes the ij-th subclass of the classification $A \times B$; $A \times B \times C$ denotes three-way cross classification of the treatments A, B and C ; $A_i B_j C_k$ denotes the ijk-th subclass of the classification $A \times B \times C$. Symbol $C(A)$ denotes the two--stage nested classification with the levels of C nested in the levels of A. The k-th level of C nested in the i-th level of A will be denoted by $C^i_{k(i)}$. Symbol $C(A \times B)$ will denote the combination of the nested and cross classification in which the levels of C are nested in the subclasses of the two-way cross classification $A \times B$. The k-th level of C nested in the ij-th subclass of the classification $A \times B$ will be denoted by $C^{ij}_{k(ij)}$. In distinct situations k will be used instead of $k(i)$ and $k(ij)$.

3. Two-way cross classification $A \times B$ Let us consider a model of two--way cross classification $A \times B$ with fixed effects of the treatment A and the random ones of the treatment B.

We shall asume the model

$$(3.1) \qquad y_{ijk} = \mu + a_i + b_j + ab_{ij} + e_{ijk}, \quad i = 1, 2, \ldots, a; j = 1, 2, \ldots, b; k = 1, 2, \ldots, n_{ij} \geqslant 2,$$

where y_{ijk} are the observations, μ is a population mean, a_i is the fixed effect of A_i, b_j is the random effect of B_j, ab_{ij} is the random effect of the subclass $A_i B_j$, and e_{ijk} is the random effect of error, associated with a given observation.

Let the random effects satisfy the following assumptions.

a. The $\{b_j\}$ are random variables that are independent and have means 0 and constant variances σ_b^2.

b. The $\{ab_{ij}\}$ are random variables that have means 0 and constant variances $(a-1)a^{-1}\sigma_{ab}^2$, and covariances

$$\text{(3.2)} \qquad \operatorname{Cov}(ab_{ij}, ab_{rs}) = \begin{cases} -\dfrac{1}{a}\sigma_{ab}^2, & \text{if } i \neq r, j = s \\ 0, & \text{otherwise.} \end{cases}$$

c. The $\{e_{ijk}\}$ are random variables that are independent and have 0 means and constant variances σ_e^2.

d. The random vectors $\{b_j\}$, $\{ab_{ij}\}$ and $\{e_{ijk}\}$ are independent of one another.

An assumption similar to (3.2b) is used by Gaylor and Hartwell [2], Niedokos [6], and Searle and Fawcett [11]. It means that interaction effects which are associated with the same level of the random treatment B and different levels of the fixed treatment A are correlated with constant correlation coefficient for all pairs of indices (i, r) with $i \neq r$. From the assumptions (3.2), it follows that

$$\operatorname{Var}(y_{ijk}) = \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2 + \sigma_e^2.$$

The variances σ_b^2, $\dfrac{a-1}{a}\sigma_{ab}^2$ and σ_e^2 are the variance components that are to be estimated. We want to find the estimators that are linear functions of the products of the following type: $y_{ijk}y_{rst}$ and $y_{ij.}y_{rs.}$, where $y_{ij.}$ denotes the arithmetical mean of all observations from the subclass A_iB_j.

Using (3.2), we obtain the following expectations:

$$\text{(3.3)} \qquad \begin{cases} \text{a. } E(y_{ijk}y_{ijt}) = (\mu+a_i)^2 + \sigma_b^2 + \dfrac{a-1}{a}\sigma_{ab}^2, \text{ if } k \neq t. \\[2ex] \text{b. } E(y_{ij.}\,y_{rs.}) = (\mu+a_i)^2 + \sigma_b^2 + \dfrac{a-1}{a}\sigma_{ab}^2 + \dfrac{1}{n_{ij}}\sigma_e^2, \text{ if } i = r,\ j = s; \\[2ex] \qquad (\mu+a_i)^2, \quad \text{if } i = r,\ j \neq s; \\[2ex] \qquad (\mu+a_i)(\mu+a_r) + \sigma_b^2 - \dfrac{1}{a}\sigma_{ab}^2, \text{ if } i \neq r,\ j = s; \\[2ex] \qquad (\mu+a_i)(\mu+a_r), \text{ if } i \neq r, j \neq s. \end{cases}$$

To find the estimators of variance components, we consider the following normalized sums

$$S_1 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2,$$

$$S_2 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}(n_{ij}^2-n_{ij})^{-1}\sum_{k\neq l} y_{ijk}y_{ijl},$$

(3.4) $$S_3 = \frac{1}{a(b^2-b)}\sum_{i=1}^{a}\sum_{j\neq s} y_{ij.}\,y_{is.}\,,$$

$$S_4 = \frac{1}{(a^2-a)b}\sum_{j=1}^{b}\sum_{i\neq r} y_{ij.}\,y_{rj.}\,,$$

$$S_5 = \frac{1}{(a^2-a)(b^2-b)}\sum_{i\neq r}\sum_{j\neq s} y_{ij.}\,y_{rs.}\,.$$

By using (3.3), we can obtain the expectations of the S's. Detailed derivations are given only for S_2 and S_4. We have

$$E\Big(\sum_{k\neq l} y_{ijk}y_{ijl}\Big) = \sum_{k\neq l}\Big[(\mu+a_i)^2+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2\Big]$$

$$=(n_{ij}^2-n_{ij})\Big(\mu^2+a_i^2+2\mu a_i+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2\Big).$$

Hence

$$E(S_2) = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\Big(\mu^2+a_i^2+2\mu a_i+\sigma_b^5+\frac{a-1}{a}\sigma_{ab}^2\Big)$$

$$=\mu^2+\frac{1}{a}\sum_{i=1}^{a}(a_i^2+2\mu a_i)+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2.$$

To obtain $E(S_4)$, we first find

$$E\Big(\sum_{i\neq r} y_{ij.}\,y_{rj.}\Big) = \sum_{i\neq r}\Big[(\mu+a_i)(\mu+a_r)+\sigma_b^2-\frac{1}{a}\sigma_{ab}^2\Big]$$

$$=(a^2-a)\Big(\mu^2+\sigma_b^2-\frac{1}{a}\sigma_{ab}^2\Big)+\sum_{i\neq r}[a_i a_r+\mu(a_i+a_r)].$$

Finally, we get

$$E(S_4) = \frac{1}{b}\sum_{j=1}^{b}\left\{\mu^2+\sigma_b^2-\frac{1}{a}\sigma_{ab}^2+\frac{1}{a^2-a}\sum_{i\neq r}[\alpha_i\alpha_r+\mu(\alpha_i+\alpha_r)]\right\}$$

$$= \mu^2+\frac{1}{a^2-a}\sum_{i\neq r}[\alpha_i\alpha_r+\mu(\alpha_i+\alpha_r)]+\sigma_b^2-\frac{1}{a}\sigma_{ab}^2.$$

By methods exactly analogous to the above, we get

$$E(S_1) = \mu^2+\frac{1}{a}\sum_{i=1}^{a}(\alpha_i^2+2\mu\alpha_i)+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2+h_n^{-1}\sigma_e^2,$$

where

$$h_n = ab\left(\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}}\right)^{-1},$$

$$E(S_2) = \mu^2+\frac{1}{a}\sum_{i=1}^{a}(\alpha_i^2+2\mu\alpha_i)+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2,$$

(3.5) $$E(S_3) = \mu^2+\frac{1}{a}\sum_{i=1}^{a}(\alpha_i^2+2\mu\alpha_i),$$

$$E(S_4) = \mu^2+\frac{1}{a^2-a}\sum_{i\neq r}[\alpha_i\alpha_r+\mu(\alpha_i+\alpha_r)]+\sigma_b^2-\frac{1}{a}\sigma_{ab}^2,$$

$$E(S_5) = \mu^2+\frac{1}{a^2-a}\sum_{i\neq r}[\alpha_i\alpha_r+\mu(\alpha_i+\alpha_r)].$$

If S_w are equated to their expectations the unbiased estimators of σ^2's are obtained by solving the resulting equations.

The estimators are as follows:

(3.6) $$\hat{\sigma}_e^2 = h_n(S_1-S_2),\quad \frac{a-1}{a}\hat{\sigma}_{ab}^2 = \frac{a-1}{a}(S_2-S_3-S_4+S_5),$$

$$\hat{\sigma}_b^2 = \frac{a-1}{a}(S_4-S_5)+\frac{1}{a}(S_2-S_3).$$

These estimators are unbiased. If we impose the restriction $\Sigma\alpha_i = 0$, then

(3.7) $$\sum_{i\neq r}(\alpha_i+\alpha_r) = 0 \text{ and } \sum_{i\neq r}\alpha_i\alpha_r = -\sum_{i=1}^{a}\alpha_i^2.$$

It may be shown that if (3.7) is true, then we can get the following unbiased estimators of the functions of parameters

(3.8) $$\hat{\mu}^2 = \frac{a-1}{a} S_5 + \frac{1}{a} S_3 \text{ and } \widehat{\frac{1}{a-1}\sum_{i=1}^{a} a_i^2} = S_3 - S_5.$$

Using the identity

(3.9) $$\sum_{i \neq r} z_i z_r = \Big(\sum_{i=1}^{a} z_i\Big)^2 - \sum_{i=1}^{a} z_i^2,$$

we shall rewrite the formulas (3.4) in a form more adaptable to computation. We have

$$S_2 = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} (n_{ij}^2 - n_{ij})^{-1} \Big[\Big(\sum_{k=1}^{n_{ij}} y_{ijk}\Big)^2 - \sum_{k=1}^{n_{ij}} y_{ijk}^2\Big],$$

$$S_3 = \frac{1}{a(b^2-b)} \sum_{i=1}^{a} \Big[\Big(\sum_{j=1}^{b} y_{ij.}\Big)^2 - \sum_{j=1}^{b} y_{ij.}^2\Big],$$

$$S_4 = \frac{1}{(a^2-a)b} \sum_{j=1}^{b} \Big[\Big(\sum_{i=1}^{a} y_{ij.}\Big)^2 - \sum_{i=1}^{a} y_{ij.}^2\Big],$$

$$S_5 = \frac{1}{(a^2-a)(b^2-b)} \Big[\Big(\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}\Big)^2 - \sum_{i=1}^{a}\Big(\sum_{j=1}^{b} y_{ij.}\Big)^2 - \sum_{j=1}^{b}\Big(\sum_{i=1}^{a} y_{ij.}\Big)^2 + \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2\Big].$$

If an experiment is balanced, then formulas (3.6) give the estimators which coincide with those obtained by the analysis of variance method. It can be shown by replacing in formulas (3.4) n_{ij} by n and $y_{ij.}$ by $n^{-1}\sum_{k=1}^{n} y_{ijk}$.

Suppose that some value β is added to each observation y_{ijk}, then the S_w in (3.4) will be transformed as follows

(3.10) $$S'_w = S_w + \frac{2\beta}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij.} + \beta^2, \quad w = 1, 2, 3, 4, 5,$$

where S'_w represents the transformed value of S_w. It is easy to see that (3.10) represents a translation of S_w and differences between S_w are invariant under such a transformation. These arguments show that the estimators (3.6) are unchanged if we subject the observations to the translation: $y_{ijk} \to y_{ijk} + \beta$.

4. Combination of cross and nested classifications $C(\mathbf{A} \times \mathbf{B})$. We shall consider a model in which the fixed treatment A and the random treatment B form the cross classification $A \times B$ and the levels of the random treatment C are nested in the subclasses of the classification $A \times B$. The mathematical model can be written

$$y_{ijkl} = \mu + a_i + b_j + ab_{ij} + c(ab)_{k(ij)} + e_{ijk(ij)l(ijk)}, \tag{4.1}$$

$$i = 1, 2, \ldots, a; \; j = 1, 2, \ldots, b; \; k(ij) = 1, 2, \ldots, n_{ij} \geqslant 2;$$
$$l(ijk) = 1, 2, \ldots, p_{ijk} \geqslant 2,$$

where y_{ijkl} are the observations, μ is a population mean, a_i is the fixed effect of A_i, b_j is the random effect of B_j, ab_{ij} is the random effect of the subclass A_iB_j, $c(ab)_{k(ij)}$ is the random effect of the subclass $C^{ij}_{k(ij)}$, and $e_{ijk(ij)l(ijk)}$ is the random effect of error associated with a given observation. Suppose that the random effects satisfy the following assumptions:

a. The $\{b_j\}$ are random variables that are idependent and have 0 means and constant variances σ_b^2.

b. The $\{ab_{ij}\}$ are random variables that have 0 means, constant variances $(a-1)a^{-1}\sigma_{ab}^2$, and covariances

$$\operatorname{Cov}(ab_{ij}, ab_{rs}) = \begin{cases} -\dfrac{1}{a}\sigma_{ab}^2, & \text{if } i \neq r, j = s \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

c. The $\{c(ab)_{k(ij)}\}$ are random variables that are independent and have 0 meas and constant variances σ_c^2.

d. The $\{e_{ijk}\}$ are random variables that are independent and have 0 means and constant variances σ_e^2.

e. The random vectors $\{b_j\}$, $\{ab_{ij}\}$, $\{c(ab_{k(ij)}\}$ and $\{e_{ijkl}\}$ are independent of one another.

From these assumptions, it follows that

$$\operatorname{Var}(y_{ijkl}) = \sigma_b^2 + \frac{a-1}{\sigma}\sigma_{ab}^2 + \sigma_c^2 + \sigma_e^2.$$

Using (4.2), we can write

a. $E(y_{ijkl}y_{ijku}) = (\mu + a_i)^2 + \sigma_b^2 + \dfrac{a-1}{a}\sigma_{ab}^2 + a_c^2$, if $l \neq u$.

b. $E(y_{ijk(ij).}\, y_{ijt(ij).})$

$$= (\mu + a_i)^2 + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2 + p_{ijk}^{-1}\sigma_c^2, \text{ if } k(ij) = t(ij), \tag{4.3}$$

$$(\mu + a_i)^2 + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2, \text{ if } k(ij) \neq t(ij).$$

c. $E(y_{ij..}y_{rs..}) = (\mu + a_i)^2$, if $i = r, j \neq s$,

$$(\mu + a_i)(\mu + a_r) + \sigma_b^2 - \frac{1}{a}\sigma_{ab}^2, \text{ if } i \neq r, j = s,$$

$$(\mu + a_i)(\mu + a_r), \text{ if } i \neq r, j \neq s,$$

where $y_{ijkl(ij).}$ and $y_{ij..}$ denote the arithmetical means of all observations in the subclasses $C_{k(ij)}^{ij}$ and A_iB_j respectively.

To obtain the estimators of the variance components, we form the following normalized sums:

(4.4)
$$S_1 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}\sum_{k=1}^{n_{ij}} y_{ijk\cdot}^2,$$

$$S_2 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}\sum_{k=1}^{n_{ij}} (p_{ijk}^2 - p_{ijk})^{-1}\sum_{l\neq u} y_{ijkl}y_{ijku},$$

$$S_3 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} (n_{ij}^2 - n_{ij})^{-1}\sum_{k\neq l} y_{ijk\cdot}y_{ijl\cdot},$$

$$S_4 = \frac{1}{a(b^2-b)}\sum_{i=1}^{a}\sum_{j\neq s} y_{ij..}y_{is..},$$

$$S_5 = \frac{1}{(a^2-a)b}\sum_{j=1}^{b}\sum_{i\neq r} y_{ij..}y_{rj..},$$

$$S_6 = \frac{1}{(a^2-a)(b^2-b)}\sum_{i\neq r}\sum_{j\neq s} y_{ij..}y_{rs..}.$$

Using (4.3), we can obtain the expectations of S_w. We have

$$E(S_1) = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}\sum_{k=1}^{n_{ij}}\left[(\mu + a_i)^2 + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2 + p_{ijk}^{-1}\sigma_c^2\right]$$

$$= \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\left[(\mu + a_i)^2 + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2\right] + \sigma_e^2\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}\sum_{k=1}^{n_{ij}} p_{ijk}^{-1}$$

$$= \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(a_i^2 + 2\mu a_i) + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2 + h_p\sigma_e^2,$$

where

$$h_p = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}\sum_{k=1}^{n_{ij}} p_{ijk}^{-1}. \tag{4.5}$$

In a similar manner, we can find the remaining expectations. They are as follows

$$\begin{aligned}
E(S_1) &= \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(\alpha_i^2+2\mu\alpha_i) + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2 + h_p\sigma_e^2,\\
E(S_2) &= \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(\alpha_i^2+2\mu\alpha_i) + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2,\\
E(S_3) &= \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(\alpha_i^2+2\mu\alpha_i) + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2,\\
E(S_4) &= \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(\alpha_i^2+2\mu\alpha_i),\\
E(S_5) &= \mu^2 + \frac{1}{a^2-a}\sum_{i\neq r}[\alpha_i\alpha_r + \mu(\alpha_i+\alpha_r)] + \sigma_b^2 - \frac{1}{a}\sigma_{ab}^2,\\
E(S_6) &= \mu^2 + \frac{1}{a^2-a}\sum_{i\neq r}[\alpha_i\alpha_r + \mu(\alpha_i+\alpha_r)].
\end{aligned} \tag{4.6}$$

After equating these expectations to their observed values and solving the resulting equations, we obtain the estimators of the variance components. They are as follows:

$$\begin{aligned}
\hat{\sigma}_e^2 &= h_p^{-1}(S_1-S_2),\quad \hat{\sigma}_c^2 = S_2-S_3,\\
\frac{a-1}{a}\hat{\sigma}_{ab}^2 &= \frac{a-1}{a}(S_3-S_4-S_5+S_6),\\
\hat{\sigma}_b^2 &= \frac{a-1}{a}(S_5-S_6)+\frac{1}{a}(S_3-S_4).
\end{aligned} \tag{4.7}$$

These estimators are unbiased. In a balanced case, i.e. when n_{ij} and p_{ijk} are constant, then the estimators (4.7) coincide with those obtained by the analysis of variance method. Imposing the restriction (3.7), we can obtain the unbiased estimators

$$\hat{\mu}^2 = \frac{a-1}{a}S_6 + \frac{1}{a}S_4 \quad\text{and}\quad \widehat{\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i^2} = S_4 - S_6.$$

By using the identity (3.9), we can obtain the computing formulas for S_w. We have

$$S_2 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}\sum_{k=1}^{n_{ij}}(p_{ijk}^2-p_{ijk})^{-1}\Big[\Big(\sum_{l=1}^{p_{ijk}} y_{ijkl}\Big)^2-\sum_{l=1}^{p_{ijk}} y_{ijkl}^2\Big],$$

$$S_3 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}(n_{ij}^2-n_{ij})^{-1}\Big[\Big(\sum_{k=1}^{n_{ij}} y_{ijk.}\Big)^2-\sum_{k=1}^{n_{ij}} y_{ijk.}^2\Big],$$

(4.8) $$S_4 = \frac{1}{a(b^2-b)}\sum_{i=1}^{a}\Big[\Big(\sum_{j=1}^{b} y_{ij..}\Big)^2-\sum_{j=1}^{b} y_{ij..}^2\Big],$$

$$S_5 = \frac{1}{(a^2-a)b}\sum_{j=1}^{b}\Big[\Big(\sum_{i=1}^{a} y_{ij..}\Big)^2-\sum_{i=1}^{a} y_{ij..}^2\Big],$$

$$S_6 = \frac{1}{(a^2-a)(b^2-b)}\Big[\Big(\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}\Big)^2-\sum_{i=1}^{a}\Big(\sum_{j=1}^{b} y_{ij..}\Big)^2 -\sum_{j=1}^{b}\Big(\sum_{i=1}^{a} y_{ij..}\Big)^2+\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}^2\Big].$$

Let us subject the observations to the translation: $y_{ijkl}\to y_{ijkl}+\beta$. Then the S_w will be transformed as follows

$$S'_w = S_w+\frac{2\beta}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}\sum_{k=1}^{n_{ij}} y_{ijk.}+\beta^2,\ w=1,2,3;$$

$$S'_w = S_w+\frac{2\beta}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}+\beta^2,\ w=4,5,6.$$

It is easy to see that replacing of S_w by S'_w in (4.7) and (4.8) does not change the estimators of σ_e^2, σ_c^2, σ_b^2 and $(a-1)a^{-1}\Sigma a_i^2$, but the estimators of σ_{ab}^2 and μ^2 will be changed.

5. Combination of cross and nested classifications $C(A)\times B$ Let us consider a model in which the fixed treatment A and the the random treatment B form the cross classification and the levels of the random treatment C are nested in the levels of A. The observations are assumed to have the following structure

(5.1) $$y_{ijkl} = \mu+a_i+b_j+ab_{ij}+c(a_i)_{k(i)}+bc(a_i)_{jk(i)}+e_{ijk(i)l(ijk)},$$
$$i=1,2,\ldots,a;\ j=1,2,\ldots,b;\ k(i)=1,2,\ldots,n_i\geqslant 2;$$
$$l(ijk)=1,2,\ldots,p_{ijk}\geqslant 2,$$

where μ is a population mean, a_i is the fixed effect of A_i, b_j is the random effect of B_j, ab_{ij} is the random effect of A_iB_j, $c(a_i)_{k(i)}$ is the random effect of the k-th level of the treatment C within the i-th level of A, $bc(a_i)_{jk(i)}$ is the random effect of interaction between the j-th level of B and the k-th od C within the i-th level of A, and $e_{ijk(i)l(ijk)}$ is the random effect of error associated with a given observation. Suppose the random effects satisfy the following assumptions

a. The $\{b_j\}$ are random variables that are independent and have 0 means and constant variances σ_b^2.

b. The $\{ab_{ij}\}$ are random variables that have 0 means, constant variances $(a-1)a^{-1}\sigma_{ab}^2$, and covariances

(5.2) $$\operatorname{Cov}(ab_{ij}, ab_{rs}) = \begin{cases} -\dfrac{1}{a}\sigma_{ab}^2, & \text{if } i \neq r, j = s, \\ 0, & \text{otherwise.} \end{cases}$$

c. The $\{c(a_i)_{k(i)}\}$ are random variables that are independent and have 0 means and constant variances σ_c^2.

d. The $\{bc(a_i)_{jk(i)}\}$ are random variables that are independent and have 0 means and constant variances σ_{bc}^2.

e. The $\{e_{ijkl}\}$ are random variables that are independent and have 0 means and constant variances σ_e^2.

f. The random vectors $\{b_j\}$, $\{ab_{ij}\}$, $\{c(a_i)_{k(i)}\}$, $\{bc(a_i)_{jk(i)}\}$ and $\{e_{ijkl}\}$ are independent of one another.

It is easily shown that

$$\operatorname{Var}(y_{ijkl}) = \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2 + \sigma_{bc}^2 + \sigma_e^2.$$

From the assumptions (5.2), we get

a. $E(y_{ijkl}y_{ijku}) = (\mu+a_i)^2 + \sigma_b^2 + \dfrac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2 + \sigma_{bc}^2$, if $l \neq u$.

b. $E(y_{ijk(i).}\, y_{ist(i).}) = (\mu+a_i)^2 + \sigma_b^2 + \dfrac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2 + \sigma_{bc}^2 + p_{ijk}^{-1}\sigma_e^2$, if $j = s$, $k(i) = t(i)$,

(5.3) $$(\mu+a_i)^2 + \sigma_b^2 + \frac{a-1}{a}\sigma_b^2, \text{ if } j = s,\ k(i) \neq t(i),$$

$$(\mu+a_i)^2 + \sigma_c^2, \text{ if } j \neq s,\ k(i) = t(i),$$

$$(\mu+a_i)^2 \quad \text{if } j \neq s,\ k(i) \neq t(i).$$

c. $E(y_{ij..}\, y_{rs..}) = (\mu+a_i)(\mu+a_r) + \sigma_b^2 - \dfrac{1}{a}\sigma_{ab}^2$, if $i \neq r, j = s$,

$$= (\mu+a_i)(\mu+a_r), \quad \text{if } i \neq r, j \neq s.$$

Let us form the following normalized sums

$$S_1 = \frac{1}{ab} \sum_{i=1}^{a} n_i^{-1} \sum_{j=1}^{b} \sum_{k(i)=1}^{n_i} y_{ijk(i).}^2,$$

$$S_2 = \frac{1}{ab} \sum_{i=1}^{a} n_i^{-1} \sum_{j=1}^{b} \sum_{k=1}^{n_i} (p_{ijk}^2 - p_{ijk})^{-1} \sum_{l \neq u} y_{ijkl} y_{ijku},$$

$$S_3 = \frac{1}{ab} \sum_{i=1}^{a} (n_i^2 - n_i)^{-1} \sum_{j=1}^{b} \sum_{k \neq t} y_{ijk.} y_{ijt.},$$

$$S_4 = \frac{1}{a(b^2 - b)} \sum_{i=1}^{a} n_i^{-1} \sum_{k=1}^{n_i} \sum_{j \neq s} y_{ijk.} y_{isk.},$$

(5.4)
$$S_5 = \frac{1}{a(b^2 - b)} \sum_{i=1}^{a} (n_i^2 - n_i)^{-1} \sum_{j \neq s} \sum_{k \neq t} y_{ijk.} y_{ist.},$$

$$S_6 = \frac{1}{(a^2 - a)b} \sum_{j=1}^{b} \sum_{i \neq r} y_{ij..} y_{rj..},$$

$$S_7 = \frac{1}{(a^2 - a)(b^2 - b)} \sum_{i \neq r} \sum_{j \neq s} y_{ij..} y_{rs..}.$$

Now, for example, we shall calculate the expectation of S_7. According to (5.3c)

$$E\Big(\sum_{j \neq s} \sum_{i \neq r} y_{ij..} y_{rs..}\Big) = \sum_{j \neq s} \sum_{i \neq r} [\mu^2 + \mu(a_i + a_r) + a_i a_r]$$
$$= (b^2 - b)(a^2 - a)\mu^2 + (b^2 - b) \sum_{i \neq r} [\mu(a_i + a_r) + a_i a_r].$$

Averaging this expression, we obtain $E(S_7)$. Similarly we can get the remaining expectations. They are as follows

$$E(S_1) = \mu^2 + \frac{1}{a} \sum_{i=1}^{a} (a_i^2 + 2\mu a_i) + \sigma_b^2 + \frac{a-1}{a} \sigma_{ab}^2 + \sigma_c^2 + \sigma_{bc}^2 +$$
$$+ h_p \sigma_e^2, \text{ where } h_p = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} n_i^{-1} \sum_{k=1}^{n_i} p_{ijk}^{-1},$$

$$E(S_2) = \mu^2 + \frac{1}{a} \sum_{i=1}^{a} (a_i^2 + 2\mu a_i) + \sigma_b^2 + \frac{a-1}{a} \sigma_{ab}^2 + \sigma_c^2 + \sigma_{bc}^2,$$

$$E(S_3) = \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(a_i^2 + 2\mu a_i) + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2,$$

$$E(S_4) = \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(a_i^2 + 2\mu a_i) + \sigma_c^2,$$

$$E(S_5) = \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(a_i^2 + 2\mu a_i),$$

$$E(S_6) = \mu^2 + \frac{1}{a^2-a}\sum_{i\neq r}[a_i a_r + \mu(a_i + a_r)] + \sigma_b^2 - \frac{1}{a}\sigma_{ab}^2,$$

$$E(S_7) = \mu^2 + \frac{1}{a^2-a}\sum_{i\neq r}[a_i a_r + \mu(a_i + a_r)].$$

If S_w are equated to their respective expectations the unbiased estimators of the variance components are obtained by solving the resulting equations. This gives

(5.5)
$$\hat{\sigma}_e^2 = h_p^{-1}(S_1 - S_2),\ \hat{\sigma}_{bc}^2 = S_2 - S_3 - S_4 + S_5,$$
$$\hat{\sigma}_c^2 = S_4 - S_5,\ \frac{a-1}{a}\hat{\sigma}_{ab}^2 = \frac{a-1}{a}(S_3 - S_5 - S_6 + S_7),$$
$$\hat{\sigma}_b^2 = \frac{a-1}{a}(S_6 - S_7) + \frac{1}{a}(S_3 - S_5).$$

Imposing the restriction (3.7), we can obtain the unbiased estimators

$$\hat{\mu}^2 = \frac{a-1}{a}S_7 + \frac{1}{a}S_5,\ \widehat{\frac{1}{a-1}\sum_{i=1}^{a}a_i^2} = S_5 - S_7. \tag{5.6}$$

It can be proved that for balanced data the estimators (5.5) are identical with those obtained by the analysis of variance method.

The computation formulas for S_w are as follows

$$S_2 = \frac{1}{ab}\sum_{i=1}^{a}n_i^{-1}\sum_{j=1}^{b}\sum_{k=1}^{n_i}(p_{ijk}^2 - p_{ijk})^{-1}\Big[\Big(\sum_{l=1}^{p_{ijk}}y_{ijkl}\Big)^2 - \sum_{l=1}^{p_{ijk}}y_{ijkl}^2\Big],$$

$$S_3 = \frac{1}{ab}\sum_{i=1}^{a}(n_i^2 - n_i)^{-1}\sum_{j=1}^{b}\Big[\Big(\sum_{k=1}^{n_i}y_{ijk\cdot}\Big)^2 - \sum_{k=1}^{n_i}y_{ijk\cdot}^2\Big],$$

$$S_4 = \frac{1}{a(b^2-b)}\sum_{i=1}^{a}n_i^{-1}\sum_{k=1}^{n_i}\Big[\Big(\sum_{j=1}^{b}y_{ijk\cdot}\Big)^2 - \sum_{j=1}^{b}y_{ijk\cdot}^2\Big],$$

$$S_5 = \frac{1}{a(b^2-b)} \sum_{i=1}^{a} (n_i^2 - n_i)^{-1} \Big[\Big(\sum_{j=1}^{b} \sum_{k=1}^{n_i} y_{ijk.} \Big)^2 - \sum_{j=1}^{b} \Big(\sum_{k=1}^{n_i} y_{ijk.} \Big)^2 - \\ - \sum_{k=1}^{n_i} \Big(\sum_{j=1}^{b} y_{ijk.} \Big)^2 + \sum_{j=1}^{b} \sum_{k=1}^{n_i} y_{ijk.}^2 \Big],$$

$$S_6 = \frac{1}{(a^2-a)b} \sum_{j=1}^{b} \Big[\Big(\sum_{i=1}^{a} y_{ij..} \Big)^2 - \sum_{i=1}^{a} y_{ij..}^2 \Big],$$

$$S_7 = \frac{1}{(a^2-a)(b^2-b)} \Big[\Big(\sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij..} \Big)^2 - \sum_{i=1}^{a} \Big(\sum_{j=1}^{b} y_{ij..} \Big)^2 - \\ - \sum_{j=1}^{b} \Big(\sum_{i=1}^{a} y_{ij..} \Big)^2 + \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij..}^2 \Big].$$

If we subject the observations to the tranlation: $y_{ijkl} \to y_{ijkl} + \beta$, then the S_w will be transformed as follows

$$S'_w = S_w + \frac{2\beta}{ab} \sum_{j=1}^{b} n_i^{-1} \sum_{j=1}^{b} \sum_{k=1}^{n_i} y_{ijk.} + \beta^2, \quad w = 1, 2, 3, 4, 5$$

$$S'_w = S_w + \frac{2\beta}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij..} + \beta^2, \quad w = 6, 7.$$

It is easy to show that replacing the S_w by the corresponding S'_w does not change the estimators (5.5), except $\hat{\sigma}_b^2$.

5. Combination of cross and nested classifications $A \times C(B)$ Now, we consider a model in which the fixed treatment A and the random treatment B form a cross classification and the levels of the random treatment C are nested in the levels of B. The observation have the following structure

(6.1) $$y_{ijkl} = \mu + a_i + b_j + ab_{ij} + c(b_j)_{k(j)} + ac(b_j)_{ik(j)} + e_{ijkl(ijk)},$$

$$i = 1, 2, \ldots, a; j = 1, 2, \ldots, b; k(j) = 1, 2, \ldots, n_j \geqslant 2;$$
$$l(ijk) = 1, 2, \ldots, p_{ijk} \geqslant 2,$$

where μ is a population mean, a_i is the fixed effect of A_i b_j is the random effect of B_j, ab_{ij} is the random effect of $A_j B_j$, $c(b_j)_{k(j)}$ is the random effect of $C^j_{k(j)}$, $ac(b_j)_{ik(j)}$ is the random effect of interaction between the i-th level of A and the-th level of C within the j-th level of B, and e_{ijkl} is the random effect of error. We assume that

a. The $\{b_j\}$ are random variables that are independent and have 0 means and constant variances σ_b^2.

b. The $\{ab_{ij}\}$ are random variables that have 0 means, constant variances $(a-1)a^{-1}\sigma_{ab}^2$, and covariances

$$\mathrm{Cov}(ab_{ij}, ab_{rs}) = \begin{cases} -\frac{1}{a}\sigma_{ab}^2, & \text{if } i \neq r, j = s \\ 0, & \text{otherwise.} \end{cases}$$

(6.2)

c. The $\{c(b_j)_{k(j)}\}$ are random variables that are independent and have 0 means and constant variances σ_c^2.

d. The $\{ac(b_j)_{ik(j)}\}$ are random variables that have 0 means, constant variances $(a-1)a^{-1}\sigma_{ac}^2$, and covariances

$$\mathrm{Cov}\left(ac(b_j)_{ik(j)}, ac(b_j)_{rt(j)}\right) = \begin{cases} -\frac{1}{a}\sigma_{ac}^2, & \text{if } i \neq r, k(j) = t(j) \\ 0, & \text{otherwise.} \end{cases}$$

e. The $\{e_{ijkl}\}$ are random variables that are independent and have 0 means and constant variances σ_e^2.

f. The random vectors $\{b_j\}$, $\{ab_{ij}\}$, $\{c(b_j)_{k(j)}\}$, $\{ac(b_j)_{ik(j)}\}$ and $\{e_{ijkl}\}$ are independent one of another.

From these assumptions we obtain

$$\mathrm{Var}(y_{ijkl}) = \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2 + \frac{a-1}{a}\sigma_{ac}^2 + \sigma_e^2.$$

To obtain the estimators of variance components we need the following expectations:

(6.3)

a. $$E(y_{ijkl}y_{ijku}) = (\mu+a_i)^2 + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2 + \frac{a-1}{a}\sigma_{ac}^2, \quad \text{if } l \neq u.$$

b. $E(y_{ijk(j).}\,y_{ijt(j).})$

$$= \begin{cases} (\mu+a_i)^2 + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2 + \sigma_c^2 + \frac{a-1}{a}\sigma_{ac}^2 + p_{ijk}^{-1}\sigma_e^2, & \text{if } k(j) = t(j), \\ (\mu+a_i)^2 + \sigma_b^2 + \frac{a-1}{a}\sigma_{ab}^2, & \text{if } k(j) \neq t(j). \end{cases}$$

c. $E(y_{ijk(j).}\,y_{rjt(j).})$

$$= \begin{cases} (\mu+a_i)(\mu+a_r) + \sigma_b^2 - \frac{1}{a}\sigma_{ab}^2 + \sigma_c^2 - \frac{1}{a}\sigma_{ac}^2, & \text{if } i \neq r, k(j) = t(j), \\ (\mu+a_i)(\mu+a_r) + \sigma_b^2 - \frac{1}{a}\sigma_{ab}^2, & \text{if } i \neq r, k(j) \neq t(j). \end{cases}$$

d. $E(y_{ij..}\,y_{rs..}) = (\mu+a_i)^2$ if $i = r,\ j \neq s$,

$(\mu+a_i)(\mu+a_r)$, if $i \neq r,\ j \neq s$.

Let us form the following normalized sums

$$S_1 = \frac{1}{ab}\sum_{j=1}^{b} n_j^{-1}\sum_{i=1}^{a}\sum_{k=1}^{n_j} y_{ijk.}^2,$$

$$S_2 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_j^{-1}\sum_{k=1}^{n_j}(p_{ijk}^2-p_{ijk})^{-1}\sum_{l\neq u} y_{ijkl}\,y_{ijku},$$

$$S_3 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}(n_j^2-n_j)^{-1}\sum_{k\neq t} y_{ijk.}\,y_{ijt.},$$

(6.4)
$$S_4 = \frac{1}{(a^2-a)b}\sum_{j=1}^{b} n_j^{-1}\sum_{k=1}^{n_j}\sum_{i\neq r} y_{ijk.}\,y_{rjk.},$$

$$S_5 = \frac{1}{(a^2-a)b}\sum_{j=1}^{b}(n_j^2-n_j)^{-1}\sum_{i\neq r}\sum_{k\neq t} y_{ijk.}\,y_{rjt.},$$

$$S_6 = \frac{1}{a(b^2-b)}\sum_{i=1}^{a}\sum_{j\neq s} y_{ij..}\,y_{is..},$$

$$S_7 = \frac{1}{(a^2-a)(b^2-b)}\sum_{i\neq r}\sum_{j\neq s} y_{ij..}\,y_{rs..}.$$

Now, we calculate the expectation of S_3

$$E(S_3) = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}(n_j^2-n_j)^{-1}\sum_{k\neq t}\left[(\mu+a_i)^2+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2\right]$$

$$= \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\mu^2+2\mu a_i+a_i^2+\frac{a-1}{a}\sigma_{ab}^2+\sigma_b^2\right)$$

$$= \mu^2+\frac{1}{a}\sum_{i=1}^{a}(a_i^2+2\mu a_i)+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2.$$

In the similar way we can obtain the expectations of the remaining S_w. We have

$$E(S_1) = \mu^2+\frac{1}{a}\sum_{i=1}^{a}(a_i^2+2\mu a_i)+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2+\sigma_c^2+\frac{a-1}{a}\sigma_{ac}^2+$$

$$+h_p\sigma_e^2, \text{ where } h_p = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_j^{-1}\sum_{k=1}^{n_j} p_{ijk}^{-1}.$$

$$E(S_2) = \mu^2+\frac{1}{a}\sum_{i=1}^{a}(a_i^2+2\mu a_i)+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2+\sigma_c^2+\frac{a-1}{a}\sigma_{ac}^2,$$

$$E(S_3) = \mu^2+\frac{1}{a}\sum_{i=1}^{a}(\sigma_i^2+2\mu a_i)+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2,$$

$$E(S_4) = \mu^2+\frac{1}{a}\sum_{i=1}^{a}(a_i^2+2\mu a_i)+\sigma_c^2-\frac{1}{a}\sigma_{ac}^2,$$

$$E(S_5) = \mu^2+\frac{1}{a}\sum_{i=1}^{a}(a_i^2+2\mu a_i)+\sigma_b^2-\frac{1}{a}\sigma_{ab}^2,$$

$$E(S_6) = \mu^2+\frac{1}{a}\sum_{i=1}^{a}(a_i^2+2\mu a_i),$$

$$E(S_7) = \mu^2+\frac{1}{a^2-a}\sum_{i\neq r}[a_i a_r+\mu(a_i+a_r)].$$

Using the method of previous chapter we obtain the unbiased estimators of the variance components

(6.5)

$$\hat{\sigma}_e^2 = h_p^{-1}(S_1-S_2),\quad \frac{a-1}{a}\hat{\sigma}_{ac}^2 = \frac{a-1}{a}(S_2-S_3-S_4+S_6),$$

$$\hat{\sigma}_c^2 = \frac{a-1}{a}(S_4-S_6)+\frac{1}{a}(S_2-S_3),$$

$$\frac{a-1}{a}\hat{\sigma}_{ab}^2 = \frac{a-1}{a}(S_3-S_5-S_6+S_7),$$

$$\hat{\sigma}_b^2 = \frac{a-1}{a}(S_5-S_7)+\frac{1}{a}(S_3-S_5).$$

If the restriction (3.7) is true, then we can obtain the following unbiased estimators:

$$\text{(6.6)}\qquad \hat{\mu}^2 = \frac{a-1}{a}S_7+\frac{1}{a}S_6,\quad \widehat{\frac{1}{a-1}\sum_{i=1}^{a}a_i^2} = S_6-S_7.$$

In balanced case estimators (6.5) coincide with those obtained by the analysis of variance method. The computation formulas for S_w are as follows:

$$S_2 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_j^{-1}\sum_{k=1}^{n_j}(p_{ijk}^2-p_{ijk})^{-1}\Big[\Big(\sum_{l=1}^{p_{ijk}} y_{ijkl}\Big)^6-\sum_{l=1}^{p_{ijk}} y_{ijkl}^2\Big],$$

$$S_3 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}(n_j^2-n_j)^{-1}\Big[\Big(\sum_{k=1}^{n_j} y_{ijk.}\Big)^2-\sum_{k=1}^{n_j} y_{ijk.}^2\Big].$$

$$S_4 = \frac{1}{(a^2-a)b}\sum_{j=1}^{b} n_j^{-1}\sum_{k=1}^{n_j}\Big[\Big(\sum_{i=1}^{a} y_{ijk.}\Big)^2-\sum_{i=1}^{a} y_{ijk}^2\Big],$$

$$S_5 = \frac{1}{(a^2-a)b}\sum_{j=1}^{b}(n_i^2-n_j)^{-1}\Big[\Big(\sum_{i=1}^{a}\sum_{k=1}^{n_j} y_{ijk.}\Big)^2-\sum_{i=1}^{a}\Big(\sum_{k=1}^{n_j} y_{ijk.}\Big)^2-$$

$$-\sum_{k=1}^{n_j}\Big(\sum_{i=1}^{a} y_{ijk.}\Big)^2+\sum_{i=1}^{a}\sum_{k=1}^{n_j} y_{ijk.}^2\Big],$$

$$S_6 = \frac{1}{a(b^2-b)}\sum_{i=1}^{a}\Big[\Big(\sum_{j=1}^{b} y_{ij..}\Big)^2-\sum_{j=1}^{b} y_{ij..}^2\Big],$$

$$S_7 = \frac{1}{(a^2-a)(b^2-b)}\Big[\Big(\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}\Big)^2-\sum_{i=1}^{a}\Big(\sum_{j=1}^{b} y_{ij..}\Big)^2-$$

$$-\sum_{j=1}^{b}\Big(\sum_{i=1}^{a} y_{ij..}\Big)^2+\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}^2\Big].$$

A translation of the observations: $y_{ijkl}\to y_{ijkl}+\beta$, causes the corresponding transformation of S_w

$$S'_w = S_w+\frac{2\beta}{ab}\sum_{j=1}^{b} n_j^{-1}\sum_{i=1}^{a}\sum_{k=1}^{n_j} y_{ijk.}+\beta^2,\ w = 1, 2, 3, 4, 5;$$

$$S'_w = S_w+\frac{2\beta}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}+\beta^2,\ w = 6, 7.$$

Replacing S_w by S'_w does not change the estimators (6.5) except $\hat{\sigma}_b^2$, and the second estimator (6.6) either.

7. Three-way cross classification $A \times B \times C$ with one fixed treatment Let us consider a model in which the fixed treatment A and two random treatments B and C form a three-way cross classification. We assume the following model of observations

$$(7.1)\quad \begin{aligned} &y_{ijkl} = \mu + a_i + b_j + c_k + ab_{ij} + ac_{ik} + bc_{jk} + abc_{ijk} + e_{ijkl}, \\ &i = 1, 2, \ldots, a;\ j = 1, 2, \ldots, b;\ k = 1, 2, \ldots, c;\ l = 1, 2, \ldots \\ &\ldots, n_{ijk} \geqslant 2, \end{aligned}$$

where μ is a population mean, a_i is the fixed effect of A_i, b_j and c_k are the random effects of B_j and C_k respectively, ab_{ij}, ac_{ik}, bc_{jk} and abc_{ijk} are the random effects of A_iB_j, A_iC_k, B_jC_k and $A_iB_jC_k$ respectively, and e_{ijkl} is the random effect of error. Suppose that

a. The $\{b_j\}$ are random variables that are independent and have 0 means and constant variances σ_b^2.

b. The $\{ab_{ij}\}$ are random variables that have 0 means, constant variances $(a-1)a^{-1}\sigma_{ab}^2$, and covariances

$$\mathrm{Cov}(ab_{ij}, ab_{rs}) = \begin{cases} -\dfrac{1}{a}\sigma_{ab}^2, & \text{if } i \neq r,\ j = s \\ 0, & \text{otherwise.} \end{cases}$$

(7.2) **c.** The $\{c_k\}$ are random variables that are independent and have 0 means and constant variances σ_c^2.

d. The $\{ac_{ik}\}$ are random variables that have 0 means, constant variances $(a-1)a^{-1}\sigma_{ac}^2$, and covariances

$$\mathrm{Cov}(ac_{ik}, ac_{rt}) = \begin{cases} -\dfrac{1}{a}\sigma_{ac}^2, & \text{if } i \neq r,\ k = t \\ 0, & \text{otherwise.} \end{cases}$$

e. The $\{bc_{jk}\}$ are random variables that are independent and have 0 means and constant variances σ_{bc}^2.

f. The $\{abc_{ijk}\}$ are random variables that have 0 means, constant variances $(a-1)a^{-1}\sigma_{abc}^2$, and covariances

g. The $\{e_{ijkl}\}$ are random variables that are independent and have 0 means and constant variances σ_e^2.

h. The random vectors $\{b_j\}$, $\{c_k\}$, $\{ab_{ij}\}$, $\{ac_{ik}\}$, $\{bc_{jk}\}$, $\{abc_{ijk}\}$ and $\{e_{ijkl}\}$ are independent one of another.

It follows from (7.2) that

$$\mathrm{Var}(y_{ijkl}) = \sigma_b^2 + \sigma_c^2 + \frac{a-1}{a}\sigma_{ab}^2 + \frac{a-1}{a}\sigma_{ac}^2 + \sigma_{bc}^2 + \frac{a-1}{a}\sigma_{abc}^2 + \sigma_e^2,$$

To find the estimators of variance components we need the following expectations that can be obtained by using (7.2).

$$\text{a. } E(y_{ijkl}y_{ijku}) = (\mu+a_i)^2+\sigma_b^2+\sigma_c^2+\frac{a-1}{a}\sigma_{ab}^2+\frac{a-1}{a}\sigma_{ac}^2+\sigma_{bc}^2 + \frac{a-1}{a}\sigma_{abc}^2, \text{ if } l\neq u.$$

$$\text{b. } E(y_{ijk.}\,y_{rst.}) = (\mu+a_i)^2+\sigma_b^2+\sigma_c^2+\frac{a-1}{a}\sigma_{ab}^2+\frac{a-1}{a}\sigma_{ac}^2+\sigma_{bc}^2 + \frac{a-1}{a}\sigma_{abc}^2+n_{ijk}^{-1}\sigma_e^2, \text{ if } (i,j,k)=(r,s,t),$$

$$(7.3)\qquad (\mu+a_i)^2+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2, \text{ if } (i,j)=(r,s),\ k\neq t,$$

$$(\mu+a_i)^2+\sigma_c^2+\frac{a-1}{a}\sigma_{ac}^2, \text{ if } j\neq s,\ (i,k)=(r,t),$$

$$(\mu+a_i)^2, \qquad \text{if } i=r, j\neq s, k\neq t,$$

$$(\mu+a_i)(\mu+a_r)+\sigma_b^2+\sigma_c^2-\frac{1}{a}\sigma_{ab}^2-\frac{1}{a}\sigma_{ac}^2+\sigma_{bc}^2-\frac{1}{a}\sigma_{abc}^2, \quad \text{if } i\neq r, (j,k)=(s,t),$$

$$(\mu+a_i)(\mu+a_r)+\sigma_b^2-\frac{1}{a}\sigma_{ab}^2, \qquad \text{if } i\neq r, j=s, k\neq t,$$

$$(\mu+a_i)(\mu+a_r)+\sigma_c^2-\frac{1}{a}\sigma_{ac}^2, \qquad \text{if } i\neq r, j\neq s, k=t,$$

$$(\mu+a_i)(\mu+a_r), \qquad \text{if } i\neq r, j\neq s, k\neq t.$$

Now consider the normalized sums

$$S_1 = \frac{1}{abc}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2,$$

$$S_2 = \frac{1}{abc}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(n_{ijk}^2-n_{ijk})^{-1}\sum_{l\neq u} y_{ijkl}y_{ijku},$$

$$S_3 = \frac{1}{ab(c^2-c)}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq t} y_{ijk.}\,y_{ijt.},$$

$$S_4 = \frac{1}{a(b^2-b)c}\sum_{i=1}^{a}\sum_{j\neq s}\sum_{k=1}^{c} y_{ijk.}\,y_{isk.},$$

$$S_5 = \frac{1}{a(b^2-b)(c^2-c)} \sum_{i=1}^{a} \sum_{j\neq s} \sum_{k\neq t} y_{ijk.}\, y_{ist.},$$

(7.4) $$S_6 = \frac{1}{(a^2-a)bc} \sum_{i\neq r} \sum_{j=1}^{b} \sum_{k=1}^{c} y_{ijk.}\, y_{rjk.},$$

$$S_7 = \frac{1}{(a^2-a)b(c^2-c)} \sum_{i\neq r} \sum_{j=1}^{b} \sum_{k\neq t} y_{ijk.}\, y_{rsk.},$$

$$S_8 = \frac{1}{(a^2-a)(b^2-b)c} \sum_{i\neq r} \sum_{j\neq s} \sum_{k=1}^{c} y_{ijk.}\, y_{rsk.},$$

$$S_9 = \frac{1}{(a^2-a)(b^2-b)(c^2-c)} \sum_{i\neq r} \sum_{j\neq s} \sum_{k\neq t} y_{ijk.}\, y_{rst.},$$

Using (7.3) we can obtain the expectations of S_w. For example

$$E(S_8) = \frac{1}{(a^2-a)(b^2-b)c} \sum_{i\neq r} \sum_{j\neq s} \sum_{k=1}^{c} \left[(\mu+a_i)(\mu+a_r)+\sigma_c^2 - \frac{1}{a}\sigma_{ac}^2\right]$$

$$= \frac{1}{a^2-a} \sum_{i\neq r} (\mu+a_i)(\mu+a_r)+\sigma_c^2 - \frac{1}{a}\sigma_{ac}^2.$$

By a similar method we can obtain the remaining expectations.

$$E(S_1) = \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(a_i^2+2\mu a_i)+\sigma_b^2+\sigma_c^2+\frac{a-1}{a}\sigma_{ab}^2+\frac{a-1}{a}\sigma_{ac}^2$$

$$+\sigma_{bc}^2+\frac{a-1}{a}\sigma_{abc}^2+h_n\sigma_e^2, \text{ where } h_n = \frac{1}{abc}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} n_{ijk}^{-1},$$

$$E(S_2) = \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(a_i^2+2\mu a_i)+\sigma_b^2+\sigma_c^2+\frac{a-1}{a}\sigma_{ab}^2+\frac{a-1}{a}\sigma_{ac}^2$$

$$+\sigma_{bc}^2+\frac{a-1}{a}\sigma_{abc}^2,$$

$$E(S_3) = \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(a_i^2+2\mu a_i)+\sigma_b^2+\frac{a-1}{a}\sigma_{ab}^2,$$

$$E(S_4) = \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(a_i^2+2\mu a_i)+\sigma_c^2+\frac{a-1}{a}\sigma_{ac}^2,$$

$$E(S_5) = \mu^2 + \frac{1}{a}\sum_{i=1}^{a}(a_i^2 + 2\mu a_i),$$

$$E(S_6) = \mu^2 + \frac{1}{a^2-a}\sum_{i\neq r}[a_i a_r + \mu(a_i + a_r)] + \sigma_b^2 + \sigma_c^2 - \frac{1}{a}\sigma_{ab}^2$$

$$-\frac{1}{a}\sigma_{ac}^2 + \sigma_{bc}^2 - \frac{1}{a}\sigma_{abc}^2,$$

$$E(S_7) = \mu^2 + \frac{1}{a^2-a}\sum_{i\neq r}[a_i a_r + \mu(a_i + a_r)] + \sigma_b^2 - \frac{1}{a}\sigma_{ab}^2,$$

$$E(S_8) = \mu^2 + \frac{1}{a^2-a}\sum_{i\neq r}[a_i a_r + \mu(a_i + a_r)] + \sigma_c^2 - \frac{1}{a}\sigma_{ac}^2,$$

$$E(S_9) = \mu^2 + \frac{1}{a^2-a}\sum_{i\neq r}[a_i a_r + \mu(a_i + a_r)],$$

After equating these expectations to their observed values and solving the resulting equations we obtain the following unbiased estimators

$$\hat{\sigma}_e^2 = h_n^{-1}(S_1 - S_2),$$

$$\frac{a-1}{a}\hat{\sigma}_{abc}^2 = \frac{a-1}{a}(S_2 - S_3 - S_4 + S_5 - S_6 + S_7 + S_8 - S_9),$$

$$\hat{\sigma}_{bc}^2 = \frac{a-1}{a}(S_6 - S_7 - S_8 + S_9) + \frac{1}{a}(S_2 - S_3 - S_4 + S_5),$$

(7.5)
$$\frac{a-1}{a}\hat{\sigma}_{ac}^2 = \frac{a-1}{a}(S_4 - S_5 - S_8 + S_9),$$

$$\frac{a-1}{a}\hat{\sigma}_{ab}^2 = \frac{a-1}{a}(S_3 - S_5 - S_7 + S_9),$$

$$\hat{\sigma}_c^2 = \frac{a-1}{a}(S_8 - S_9) + \frac{1}{a}(S_4 - S_5),$$

$$\hat{\sigma}_b^2 = \frac{a-1}{a}(S_7 - S_9) + \frac{1}{a}(S_3 - S_5),$$

It can be proved that for balanced data these estimators coincide with those obtained by the analysis of variance method. Imposing the restriction $\Sigma a_i = 0$, we obtain the following two unbiased estimators:

(7.6)
$$\hat{\mu}^2 = \frac{a-1}{a}S_9 + \frac{1}{a}S_5, \quad \widehat{\frac{1}{a-1}\sum_{i=1}^{a}a_i^2} = S_5 - S_9.$$

The computing formulas for S_w are as follows:

$$S_2=\frac{1}{abc}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(n_{ijk}^2-n_{ijk})^{-1}\Big[\Big(\sum_{l=1}^{n_{ijk}}y_{ijkl}\Big)^2-\sum_{l=1}^{n_{ijk}}y_{ijkl}^2\Big],$$

$$S_3=\frac{1}{ab(c^2-c)}\sum_{i=1}^{a}\sum_{j=1}^{b}\Big[\Big(\sum_{k=1}^{c}y_{ijk.}\Big)^2-\sum_{k=1}^{c}y_{ijk.}^2\Big],$$

$$S_4=\frac{1}{a(b^2-b)c}\sum_{i=1}^{a}\sum_{k=1}^{c}\Big[\Big(\sum_{j=1}^{b}y_{ijk.}\Big)^2-\sum_{j=1}^{b}y_{ijk.}^2\Big],$$

$$S_5=\frac{1}{a(b^2-b)(c^2-c)}\sum_{i=1}^{a}\Big[\Big(\sum_{j=1}^{b}\sum_{k=1}^{c}y_{ijk.}\Big)^2-\sum_{j=1}^{b}\Big(\sum_{k=1}^{c}y_{ijk.}\Big)^2-\sum_{k=1}^{c}\Big(\sum_{j=1}^{b}y_{ijk.}\Big)^2+\sum_{j=1}^{b}\sum_{k=1}^{c}y_{ijk.}^2\Big],$$

$$S_6=\frac{1}{(a^2-a)bc}\sum_{j=1}^{b}\sum_{k=1}^{c}\Big[\Big(\sum_{i=1}^{a}y_{ijk.}\Big)^2-\sum_{i=1}^{a}y_{ijk.}^2\Big],$$

$$S_7=\frac{1}{(a^2-a)b(c^2-c)}\sum_{j=1}^{b}\Big[\Big(\sum_{i=1}^{a}\sum_{k=1}^{c}y_{ijk.}\Big)^2-\sum_{i=1}^{a}\Big(\sum_{k=1}^{c}y_{ijk.}\Big)^2-\sum_{k=1}^{c}\Big(\sum_{i=1}^{a}y_{ijk.}\Big)^2+\sum_{i=1}^{a}\sum_{k=1}^{c}y_{ijk.}^2\Big],$$

$$S_8=\frac{1}{(a^2-a)(b^2-b)c}\sum_{k=1}^{c}\Big[\Big(\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ijk.}\Big)^2-\sum_{i=1}^{a}\Big(\sum_{j=1}^{b}y_{ijk.}\Big)^2-\sum_{j=1}^{b}\Big(\sum_{i=1}^{a}y_{ijk.}\Big)^2+\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ijk.}^2\Big],$$

$$S_9=\frac{1}{(a^2-a)(b^2-b)(c^2-c)}\Big[\Big(\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}y_{ijk.}\Big)^2-\sum_{i=1}^{a}\Big(\sum_{j=1}^{b}\sum_{k=1}^{c}y_{ijk.}\Big)^2-\sum_{j=1}^{b}\Big(\sum_{i=1}^{a}\sum_{k=1}^{c}y_{ijk.}\Big)^2-\sum_{k=1}^{c}\Big(\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ijk.}\Big)^2+\sum_{i=1}^{a}\sum_{j=1}^{b}\Big(\sum_{k=1}^{c}y_{ijk.}\Big)^2+\sum_{i=1}^{a}\sum_{k=1}^{c}\Big(\sum_{j=1}^{b}y_{ijk.}\Big)^2+\sum_{j=1}^{b}\sum_{k=1}^{c}\Big(\sum_{i=1}^{a}y_{ijk.}\Big)^2-\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}y_{ijk.}^2\Big].$$

If we subject the observations to the translation: $y_{ijkl} \to y_{ijkl} + \beta$, then S_w will be transformed as follows:

$$S'_w = S_w + \frac{2\beta}{abc} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} y_{ijk.} + \beta^2, \; w = 1, 2, \ldots, 9.$$

Replacing S_w by S'_w does not change the estimators (7.5) and the second estimator (7.6). These estimators are translation invariant.

8. Remarks The methods of estimation presented in the previous chapters can be applied to every model based on a cross classification, nested classification or the combination of both.

In the present paper the models have been considered in which the interaction effects between the random and fixed treatments are correlated. Same authors, e.g. Furukawa [1] assume that these effects are independent. Such an assumption makes the problem of estimation easier. This paper does not consider the case of lack of correlation separately. The estimators for this case can be obtained by replacing in assumptions and derivations the coefficients $(a-1)a^{-1}$ and a^{-1} by 1 and 0 respectively.

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STRESZCZENIE

W pracy tej znaleziono estymatory nieobciążne komponentow wariacyjnych w modelach mieszanych nieortogonalnych. Rozpatrzono modele podwójnej i potrójnej klasyfikacji krzyżowej oraz trzy modele oparte na kombinacji podwojnej klasyfikacji krzyzowej z hierarchiczną. Niektóre ze znalezionych estymatorów są niezmiennikami translacji danych liczbowych.

РЕЗЮМЕ

В работе приводятся несмещенные оценки компонент дисперсии в несбалансированных смешанных моделях. Рассматриваются модели двухфакторной и трехфакторной перекрестных классификаций и три модели, основанные на комбинации двухфакторной перекрестной и иерархической классификаций. Полученные оценки являются в некоторых случаях инвариатными относительно трансляции численных данных.