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On the Eneström-Kakeya Theorem

O twierdzeniu Eneströma—Kakeya

О теореме Энестрёма-Какэйя

The following result is well known in the theory of the distribution of zeros of polynomials.

Theorem A. (Eneström-Kakeya). *If* $p(z) = \sum_{k=0}^{n} a_k z^k \neq 0$ *is a polynomial of degree n such that*

(1) $$a_n \geqslant a_{n-1} \geqslant a_{n-2} \geqslant \ldots \geqslant a_1 \geqslant a_0 > 0,$$

then $p(z)$ *does not vanish in* $|z| > 1$.

There already exist in literature ([1], [2, Theorem 1, 2, 3, 4], [3, Theorem 3], [4]) some extensions of Eneström-Kakeya theorem. Govil and Rahman [2, Theorem 2] proved the following generalization of Eneström-Kakeya theorem.

Theorem B. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k \neq 0$ *be a polynomial of degree n with complex coefficients such that*

$$|\arg a_k - \beta| \leqslant a \leqslant \pi/2, \; k = 0, 1, \ldots, n$$

for some real β, *and*

(2) $$|a_n| \geqslant |a_{n-1}| \geqslant |a_{n-2}| \geqslant \ldots \geqslant |a_0|,$$

then $p(z)$ *has all its zeros on or inside the circle*

(3) $$|z| = \cos a + \sin a + \frac{2 \sin a}{|a_n|} \sum_{k=0}^{n-1} |a_k|.$$

For $a = \beta = 0$ *this reduces to Eneström-Kakeya theorem.*

Here we prove the following refinement of Theorem B.

Theorem 1. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k \neq 0$ *be polynomial of degree* n *with complex coefficients such that*

$$|\arg a_k - \beta| \leqslant \alpha \leqslant \pi/2, \quad k = 0, 1, \ldots, n \tag{4}$$

for some real β, *and*

$$|a_n| \geqslant |a_{n-1}| \geqslant |a_{n-2}| \geqslant \ldots \geqslant |a_0|, \tag{5}$$

then $p(z)$ *has all its zeros in the ring shaped region given by*

$$\frac{1}{R^{n-1}\left[\dfrac{2R|a_n|}{|a_0|} - (\cos\alpha + \sin\alpha)\right]} \leqslant |z| \leqslant R, \tag{6}$$

where

$$R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{k=0}^{n-1}|a_k|. \tag{7}$$

The example $p(z) = 1 + z + z^2 + \ldots + z^n$ shows that the result is best possible.

We need the following lemma.

Lemma. *If* $|\arg a_k - \beta| \leqslant \alpha \leqslant \pi/2$; $|a_k| \geqslant |a_{k-1}|$, *then*

$$|a_k - a_{k-1}| \leqslant \{(|a_k| - |a_{k-1}|)\cos\alpha + (|a_k| + |a_{k-1}|)\sin\alpha\}. \tag{8}$$

The above lemma is due to Govil and Rahman [2, Inequality 6].

Proof of Theorem 1. We may plainly assume $\beta = 0$.

In view of Theorem B, it is sufficient to prove that $p(z) \neq 0$ if

$$|z| < R^{n-1}\frac{1}{\left[2R\dfrac{|a_n|}{|a_0|} - (\cos\alpha + \sin\alpha)\right]}. \tag{9}$$

Consider

$$g(z) = (1-z)p(z) = a_0 + \sum_{k=1}^{n}(a_k - a_{k-1})z^k - a_n z^{n+1} \tag{10}$$
$$= a_0 + f(z), \text{ say}.$$

Let

$$M(r) = \max_{|z|=r}|f(z)|.$$

Then $M(R) \geqslant a_0$ where R is defined in (7).

Clearly

$$|f(z)| \leqslant |a_n|\,|z|^{n+1} + \sum_{k=1}^{n} |a_k - a_{k-1}|\,|z|^k,$$

and $R \geqslant 1$. Hence

$$M(R) = \max_{|z|=R} |f(z)| \leqslant |a_n| R^{n+1} + \sum_{k=1}^{n} |a_k - a_{k-1}| R^k$$

$$\leqslant |a_n| R^{n+1} + R^n \sum_{k=1}^{n} |a_k - a_{k-1}| \leqslant |a_n| R^{n+1}$$

$$+ R^n \Big[\sum_{k=1}^{n} (|a_k| - |a_{k-1}|)\cos a + \sum_{k=1}^{n} (|a_k| + |a_{k-1}|)\sin a\Big],$$

by the lemma

$$= |a_n| R^{n+1} + R^n \Big[|a_n|(\cos a + \sin a) + 2\sum_{k=0}^{n-1} |a_k| \sin a - |a_0|(\cos a + \sin a)\Big]$$

$$= |a_n| R^{n+1} + |a_n| R^n \Big[R - \frac{|a_0|}{|a_n|}(\cos a + \sin a)\Big]$$

$$(11) \qquad = |a_n| R^n \Big[2R - \frac{|a_0|}{|a_n|}(\cos a + \sin a)\Big].$$

Since $f(0) = 0$, hence for $|z| \leqslant R$ we have by Schwarz's lemma,

$$(12) \qquad |f(z)| \leqslant \frac{|z|\, M(R)}{R}.$$

Combining (10), (11) and (12) we get for $|z| \leqslant R$,

$$|g(z)| \geqslant |a_0| - |z| R^{n-1}[2R|a_n| - |a_0|(\cos a + \sin a)] > 0$$

if

$$|z| < \frac{1}{R^{n-1}\Big[2R\dfrac{|a_n|}{|a_0|} - (\cos a + \sin a)\Big]}.$$

From this the theorem follows.

We may apply Theorem 1 to $z^n p\left(\frac{1}{z}\right)$ to obtain the following

Corollary 1. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k \neq 0$ *be a polynomial of degree* n *with complex coefficients such that*

$$|\arg a_k - \beta| \leqslant a \leqslant \pi/2, \ k = 0, 1, \ldots, n$$

for some real β, *and*

$$|a_0| \geqslant |a_1| \geqslant |a_2| \geqslant \ldots \geqslant |a_n|$$

then $p(z)$ *has all its zeros in the ring*

$$\frac{1}{R_1} \leqslant |z| \leqslant R_1^{n-1}\left[2R_1 \frac{|a_0|}{|a_n|} - (\cos a + \sin a)\right]$$

where R_1 *is given by*

$$R_1 = \cos a + \sin a + \frac{2\sin a}{|a_0|} \sum_{k=1}^{n} |a_k|. \tag{13}$$

We shall briefly indicate how we can prove

Theorem 2. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k \neq 0$ *be a polynomial of degree* n. *If* $\operatorname{Re} a_k = a_k$, $\operatorname{Im} a_k = \beta_k$ *for* $k = 0, 1, \ldots, n$ *and*

$$a_n \geqslant a_{n-1} \geqslant a_{n-2} \geqslant \ldots \geqslant a_0 \geqslant 0, \ a_n > 0$$

then $p(z)$ *has all its zeros in the ring*

$$\frac{|a_0|}{R_1^{n-1}[R_1(2a_n + |\beta_n|) - (a_0 + |\beta_0| + |\beta_n|)]} \leqslant |z| \leqslant R_1 \tag{14}$$

where

$$R_1 = 1 + \frac{2}{a_n} \sum_{k=0}^{n} |\beta_k|. \tag{15}$$

Proof of Theorem 2.

Consider

$$g(z) = (1-z)p(z) = a_0 + \sum_{k=1}^{n} (a_k - a_{k-1}) z^k - a_n z^{n+1}$$
$$= a_0 + f^*(z), \text{ say.}$$

Obviously $R_1 \geqslant 1$, hence on $|z| = R_1$,

$$|f^*(z)| \leqslant |a_n| R_1^{n+1} + \sum_{k=1}^{n} |a_k - a_{k-1}| R_1^k$$

$$\leqslant |a_n| R_1^{n+1} + R_1^n \sum_{k=1}^{n} |a_k - a_{k-1}|$$

$$\leqslant |a_n| R_1^{n+1} + R_1^n \left[\sum_{k=1}^{n} (a_k - a_{k-1}) + \sum_{k=1}^{n} (|\beta_k| + |\beta_{k-1}|) \right]$$

$$\leqslant (|a_n| + |\beta_n|) R_1^{n+1} + R_1^n \left[a_n - a_0 + 2 \sum_{k=0}^{n} |\beta_k| - |\beta_0| - |\beta_n| \right]$$

$$= R_1^n [2a_n R_1 + |\beta_n| R_1 - (a_0 + |\beta_0|) - |\beta_n|]$$

Therefore

$$M(R_1) = \max_{|z|=R_1} |f^*(z)| \leqslant R_1^n [2a_n R_1 + (R_1 - 1)|\beta_n| - (a_0 + |\beta_0|)]. \tag{16}$$

Since $f^*(0) = 0$ hence by Schwarz's lemma

$$|f^*(z)| \leqslant \frac{M(R_1)|z|}{R_1} \leqslant R_1^{n-1} |z| [2a_n R_1 + (R_1 - 1)|\beta_n| - (a_0 + |\beta_0|)] \tag{17}$$

or $|z| \leqslant R_1$. Consequently for $|z| \leqslant R_1$,

$$|g(z)| \geqslant |a_0| - |f^*(z)|$$

$$\geqslant |a_0| - R_1^{n-1} |z| [2a_n R_1 + (R_1 - 1)|\beta_n| - (a_0 + |\beta_0|)],$$

by (17) if

$$|z| \leqslant \frac{|a_0|}{R_1^{n-1} [2a_n R_1 + (R_1 - 1)|\beta_n| - (a_0 + |\beta_0|)]}. \tag{18}$$

Combining (18) with [2, Theorem 4] the conclusion follows immediately.

In particular, when the coefficients are non-negative, monotonic non-decreasing, we get

Theorem 3. *If $p(z) = \sum_{k=0}^{n} a_k z^k \neq 0$ is a polynomial of degree n such that*

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_0 \geqslant 0,$$

then $p(z)$ has all its zeros in the ring-shaped region given by

$$\frac{a_0}{(2a_n - a_0)} \leqslant |z| \leqslant 1. \tag{19}$$

This is clearly a refinement of Eneström-Kakeya theorem.

The example $p(z) = z^n + z^{n-1} + \ldots + z + 1$ shows that the above result is the best possible.

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STRESZCZENIE

Jeśli $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ jest wielomianem o współczynnikach rzeczywistych i dodatnich, tworzących ciąg rosnący, to jak wiadomo, wszystkie zera $p(z)$ leżą w kole jednostkowym. Ten klasyczny rezultat Eneströma i Kakeyi został uogólniony przez autorów którzy wyznaczają pierścień zawierający wszystkie zera wielomianu $p(z)$ ze współczynnikami zespolonymi a_k w następujących przypadkach: (i) moduły a_k tworzą ciąg rosnący a ponadto

$$|\arg a_k - \beta| \leqslant a \leqslant \pi/2, \quad k = 0, 1, \ldots, n,$$

(ii) części rzeczywiste współczynników a_k tworzą ciąg rosnący.

РЕЗЮМЕ

Если $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ является полином с положительным и действительным коэффициентами, образующими возрастающую последовательность, то как известно, все нули $p(z)$ лежат в единичном круге. Это классическое утверждение Энестрёма-Кекэия было обобщено авторами, которые дают кольцо, внутри которого лежат все нули полинома $p(z)$ с комплексными коэффициентами a_k в следующих случаях:

(i) модуля a является возрастающей последовательностью; кроме того

$$|\arg a_k - \beta| \leqslant a \leqslant \pi/2, \quad k = 0, 1, \ldots, n,$$

(ii) действительные части коэффициентов a_k, образующие возрастающую прогрессию.