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# A Coefficient Inequality for Bazilevič Functions

Nierówności na współczynniki dla funkcji Bazylewicza

Неравенства на коэффициенты для функций Базилевича

**Introduction.** Sheil-Small [7] has recently characterized Bazilevič functions [1] in terms of a certain integral inequality. More specifically, let $f(z)$ be Bazilevič of type $(a, b)$. Then, for each $r(0 < r < 1)$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}[1+zf''(z)/f'(z)+(a-1+ib)zf'(z)/f(z)]\,d\theta > -\pi \tag{1}$$

whenever $\theta_2 > \theta_1$. Conversely, if $f$ is analytic in $|z| < 1$, with $f(0) = 0$, $f(z) \neq 0$ $(0 < |z| < 1)$, and $f'(z) \neq 0$ for $|z| < 1$, and if $f$ satisfies (1) for $0 < r < 1$ where $a > 0$, $b$ real, then $f$ is Bazilevič of type $(a, b)$.

Let $B(a, b)$ denote the class of normalized functions satisfying (1). For a given complex number $\mu$, we wish to maximize $|a_3 - \mu a_2^2|$ over a fixed class of functions. We are unable to do this for the entire class $B(a, b)$; this paper is concerned with the solution of the above extremal problem over certain subclasses of $B(a, b)$, which are defined below.

**Definition.** The normalized univalent function $f$ is said to be $a-\lambda$-spiral-like, $a \geq 0$, $|\lambda| < \pi/2$, if

$$\operatorname{Re}\left[(e^{i\lambda}-a)zf'(z)/f(z)+a\left(1+zf''(z)/f'(z)\right)\right] > 0, \tag{2}$$

for $|z| < 1$. Let $M_a^\lambda$ denote the class of such functions.

Note that for $a > 0$, (2) is obtained by requiring the integrant in (1) to be positive, replacing $a$ and $b$ by $a^{-1}\cos\lambda$ and $a^{-1}\sin\lambda$, respectively, and then multiplying through by $a$. The reason for this parameter change

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is that (2) implies $f$ is $\lambda$-spiral-like [8], and thus we have facilitated comparison with known results.

Sheil-Small [7] has shown that $f \in B(a, 0)$ if and only if there exists a starlike function $g$, $|g'(0)| = 1$, such that

(3) $$\operatorname{Re}\left[\frac{zf'(z)}{f(z)^{1-a}g(z)^{a}}\right] > 0, \quad |z| < 1.$$

Let $B(a)$ denote those functions satisfying (3) with a **normalized** $g$, and let $C$ denote $B(1, 0)$. $C$ is the well known class of close-to-convex functions. In this paper we maximize $|a_3 - \mu a_2^2|$ over each of the three classes $M_a^\lambda$, $B(a)$, and $C$. Keogh and Merkes [3] solved the extremal problem (with $\mu$ real) over $B(1)$, and we show that their result holds also for the larger class $C$. In each of the three cases, the method we use, namely, application of the lemma below, is due to Keogh and Merkes [3]. The three results we obtain can be found in Theorems A, B, and C.

**Lemma:** Let $\omega(z) = \sum_{1}^{\infty} c_n z^n$ be analytic with $|\omega(z)| < 1$ for $|z| < 1$. If $\nu$ is any complex number then

(4) $$|c_2 - \nu c_1^2| \leqslant \max\{1, |\nu|\}.$$

Equality may be attained with the functions $\omega(z) = z^2$ and $\omega(z) = z$.

For a proof of this we refer the reader to [3].

**Theorem A:** *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_a^\lambda$ $(a \geqslant 0,\ |\lambda| < \pi/2)$ *and* $\mu$ *is any complex number, then*

(5) $$|a_3 - \mu a_2^2| \leqslant \frac{\cos\lambda}{|e^{i\lambda} + 2a|}\max\{1, |v|\}.$$

*where*

$$v = \frac{4\mu(e^{i\lambda}+2a)\cos\lambda + 4e^{i\lambda}\cos\lambda - (a+e^{i\lambda})(a+e^{i\lambda}+6\cos\lambda)}{(a+e^{i\lambda})^2}.$$

For each $\mu$, there exists an $a-\lambda$-spiral-like function for which equality holds in (5).

**Proof.** If $f(z) \in M_a^\lambda$, then there exists an analytic function $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ such that $|\omega(z)| < 1$ $(|z| < 1)$ for which

(6) $$(e^{i\lambda} - a)\frac{zf'(z)}{f(z)} + a\left(\frac{(f''(z)}{f'(z)} + 1\right) = \frac{e^{i\lambda} + e^{-i\lambda}\omega(z)}{1 - \omega(z)} \quad (|z| < 1).$$

By expanding (6) and equating coefficients we have

(7) $$c_1 = \frac{(a + e^{i\lambda})}{2} a_2 \quad \sec\lambda$$

and

$$(8)\qquad c_2 = (e^{i\lambda}+2a)\sec\lambda a_3 + \frac{[4e^{i\lambda}\sec\lambda - (a+e^{i\lambda}+6\cos\lambda)(a+e^{i\lambda})\sec^2\lambda]}{4}a_2^2.$$

Using (4), (7) and (8) we obtain (5), where

$$\mu = \frac{(a+e^{i\lambda})(a+e^{i\lambda}+6\cos\lambda)+(a+e^{i\lambda})^2 v - 4e^{i\lambda}\cos\lambda}{4(e^{i\lambda}+2a)\cos\lambda}.$$

The sharpness of (5) follows from that of (4).

**Corollary 1.** *If $f(z)$ is $a-\lambda$-spiral-like then*

$$(9)\qquad |a_2| \leqslant \frac{2\cos\lambda}{|a+e^{i\lambda}|}.$$

$$(10)\qquad |a_3| \leqslant \frac{\cos\lambda\,|(a+e^{i\lambda})^2+2\cos\lambda(e^{i\lambda}+3a)|}{|a+e^{i\lambda}|^2\,|e^{i\lambda}+2a|}$$

**Proof.** The inequalities (9) and (10) follow directly from (7) and (5), respectively.

**Corollary 2.** *If $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ is $a$-convex* (i.e., $f\in M_a^0$) *and $\mu$ is any complex number, then*

$$|a_3-\mu a_2^2| \leqslant \frac{1}{1+2a}\max\left\{1, \frac{|4\mu(1+2a)+4-(1+a)(7+a)|}{(1+a)^2}\right\}.$$

**Proof:** This result follows immediately upon substituting $\lambda = 0$ in (5). Further, corollary 2 agrees with a result of Szynal [9].

**Corollary 3.** *If $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ is $\lambda$-spiral-like* ($|\lambda| < \pi|2$) *and and $\mu$ is any complex number, then*

$$|a_3-\mu a_2^2| \leqslant \cos\lambda\max\{1, |2\cos\lambda(2\mu-1)-e^{i\lambda}|\}.$$

**Proof:** By substituting $a = 0$ in (5) we obtain this result, which is due to Keogh and Merkes [3].

**Remarks.** The proof of the theorem did not use the fact that $a$ was real. For $a = e^{i\lambda}$ the expression in (2) becomes $e^{i\lambda}\left(\frac{zf''(z)}{f'(z)}+1\right)$, and $M_a^\lambda$ corresponds to the class of analytic functions for which $zf'(z)$ is $\lambda$-spiral-like. This class was defined by Robertson [6]. Also, by substituting $a = e^{i\lambda}$ in (5) we obtain the following result of Libera and Ziegler [4].

**Corollary 4.** *If $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ is an analytic function for which $zf'(z)$ is $\lambda$-spiral-like* ($|\lambda| < \pi/2$) *and $\mu$ is any complex number, then*

$$|a_3-\mu a_2^2| \leqslant \tfrac{1}{3}\cos\lambda\max\{1, |e^{i\lambda}-(3\mu-2)\cos\lambda|\}.$$

**Theorem B:** *If $f \in B(\alpha)$ and $\mu$ is real,*

(11)
$$|a_3-\mu a_2^2| \leqslant \begin{cases} 3-4\mu & \text{if } \mu \leqslant \dfrac{1}{2+\alpha} \\ 1+\dfrac{4\alpha^2}{(1+\alpha)^2}(\mu_0-\mu)+\dfrac{8\alpha^2(\mu_0-\mu)^2}{(1+\alpha)^2(2\mu+\alpha-1)} & \text{if } \dfrac{1}{2+\alpha} \leqslant \mu \leqslant \mu_0 \\ 1 & \text{if } \mu_0 \leqslant \mu \leqslant 1 \\ 4\mu-3 & \text{if } \mu \geqslant 1 \end{cases}$$

*where* $\mu_0 = \dfrac{3+\alpha}{2(2+\alpha)}$. *Each estimate is sharp.*

**Proof:** We have from (3) the existence of a normalized starlike $g$, $g(z) = z+\sum_{n=2}^{\infty} b_n z^n$ such that

$$\operatorname{Re}\left[\frac{zf'(z)^{\alpha}}{f(z)^{1-\alpha}g(z)^{\alpha}}\right] > 0.$$

Hence,

(12)
$$\frac{f'(z)-\left(\dfrac{f(z)}{z}\right)^{1-\alpha}\left(\dfrac{g(z)}{z}\right)^{\alpha}}{f'(z)+\left(\dfrac{f(z)}{z}\right)^{1-\alpha}\left(\dfrac{g(z)}{z}\right)^{\alpha}} = \sum_{n=1}^{\infty} c_n z^n$$

satisfies the condition of the Lemma.

By expanding (12) and equating coefficients, we have

$$a_2 = \frac{2c_1}{1+\alpha}+\frac{\alpha}{1+\alpha}b_2$$

and

$$a_3 = \frac{2c_2}{2+\alpha}+\frac{(3+\alpha)(4c_1^2+4c_1b_2\alpha+\alpha^2b_2^2)}{2(2+\alpha)(1+\alpha)^2}+\frac{\alpha(b_3-\frac{1}{2}b_2^2)}{2+\alpha},$$

so that

(13)
$$a_3-\mu a_2^2 = \frac{\alpha}{2+\alpha}\left[b_3-\left(\frac{1}{2}-\frac{\alpha(3+\alpha)}{2(1+\alpha)^2}+\frac{\mu\alpha(2+\alpha)}{(1+\alpha)^2}\right)b_2^2\right]$$
$$+\frac{2}{2+\alpha}\left[c_2+\left(\frac{3+\alpha}{(1+\alpha)^2}-\frac{2\mu(2+\alpha)}{(1+\alpha)^2}\right)c_1^2\right]$$
$$+\frac{4c_1b_2\alpha}{(1+\alpha)^2}\left(\frac{3+\alpha}{2(2+\alpha)}-\mu\right).$$

If $\mu = \mu_0$, the third term is zero, and (13) becomes

$$a_3 - \mu_0 a_2^2 = \frac{a}{2+a}\,(b_3 - \tfrac{1}{2} b_2^2) + \frac{2}{2+a}\, c_2 .$$

Since $g$ is starlike $|b_3 - \frac{1}{2} b_2^2| \leqslant 1$ [3], and $|c_2| \leqslant 1$ by the Lemma. Hence, $|a_3 - \mu_0 a_2^2| \leqslant 1$. Also, the area theorem [5] gives $|a_3 - a_2^2| \leqslant 1$. Combining these two inequalities, we have for $\mu_0 \leqslant \mu \leqslant 1$,

$$|a_3 - \mu a_2^2| \leqslant \frac{\mu - \mu_0}{1 - \mu_0}\, |a_3 - a_2^2| + \frac{1 - \mu}{1 - \mu_0}\, |a_3 - \mu_0 a_2^2| \leqslant 1 .$$

We now examine $0 \leqslant \mu < \mu_0$. Let $\beta$ denote the coefficient of $b_2^2$ in (13). One easily checks that $\beta \leqslant \frac{1}{2}$ so that the result of Keogh and Merkes [3] applies, giving

$$|b_3 - \beta b_2^2| \leqslant 3 - 4\beta = 1 + \frac{2a(3 + a - 4\mu - 2\mu a)}{(1+a)^2} . \tag{14}$$

Using the facts that $|c_2| \leqslant 1 - |c_1|^2$ and $|b_2| \leqslant 2$, the sum of the second and third terms of (13) is bounded by

$$\varphi(r) = \frac{2}{2+a}\left[1 + \frac{(2+a)(1-a-2\mu)}{(1+a)^2}\, r^2\right] + \frac{8a(\mu_0 - \mu)}{(1+a)^2}\, r, \quad r = |c_1|$$

Now, if $2\mu + a - 1 > 0 \left(\text{i.e., } \mu > \dfrac{1-a}{2}\right)$ then $\varphi$ attains its maximum value at $r^*(\mu) = 2a(\mu_0 - \mu)(2\mu + a - 1)^{-1}$. On the interval $\left(\dfrac{1-a}{2}, \mu_0\right]$, $r^*(\mu)$ decreases from $+\infty$ to zero. The requirement $r^*(\mu) \leqslant 1$ yields: for $\mu \in \left[\dfrac{1}{2+a}, \mu_0\right]$, $|a_3 - \mu a_2^2|$ is maximized by using the estimate in (14) on the first term of (13), and then replacing $c_1$ by $r^*(\mu)$, $c_2$ by $1 - c_1^2$, and $b_2$ by 2 in the other terms of (13). This bound on $|a_3 - \mu a_2^2|$ is attained for the function $f$ defined implicity in (12), where $g$ is the Köebe function and $\sum_{n=1}^{\infty} c_n z^n$ is defined as $z\big(z + r^*(\mu)\big)\big(1 + r^*(\mu) z\big)^{-1}$.

It remains to consider $0 \leqslant \mu \leqslant (2+a)^{-1}$. From (11)

$$\begin{aligned} |a_3 - \mu a_2^2| &\leqslant (2+a)\mu\, |a_3 - (2+a)^{-1} a_2^2| + \big(1 - (2+a)\mu\big)\, |a_3| \\ &\leqslant (2+a)\mu \left(\frac{2+3a}{2+a}\right) + 3\big(1 - (2+a)\mu\big) = 3 - 4\mu . \end{aligned}$$

The bounds in (11) for $\mu \notin [0, 1]$ are identical with those for the entire class of univalent function [2]. Except for $\mu \in [(2+a)^{-1}, \mu_0]$, the bounds in (11) are attained by a starlike function [3], and the class of starlike functions is contained in each $B(a)$. The proof of Theorem B is complete.

**Corollary 5:** *If $f \in \bigcup_{\alpha \geqslant 0} B(\alpha)$ and $\mu$ is real,*

$$|a_3 - \mu a_2^2| \leqslant \begin{cases} 3-4\mu & \text{if } \mu \leqslant 0 \\ 1+2\left(\dfrac{3-4\mu}{3-2\mu}\right)^3 & \text{if } 0 \leqslant \mu \leqslant 3/4 \\ 1 & \text{if } 3/4 \leqslant \mu \leqslant 1 \\ 4\mu - 3 & \text{if } \mu \geqslant 1 \end{cases}$$

*For $\mu \notin (0, 3/4)$ the bound is attained by a starlike function.* If $\mu \in (0, 3/4)$ *equality is attained only for a function in* $B\left(\dfrac{3}{\mu} - 4\right)$.

We omit the proof of Corollary 5.

**Theorem C:** *If $f \in C$ and $\mu$ is real,*

$$(15) \qquad |a_3 - \mu a_2^2| \leqslant \begin{cases} 3-4\mu & \text{if } \mu \leqslant \frac{1}{3} \\ \frac{1}{3} + \dfrac{4}{9\mu} & \text{if } \frac{1}{3} \leqslant \mu \leqslant \frac{2}{3} \\ 1 & \text{if } \dfrac{2}{3} \leqslant \mu \leqslant 1 \\ 4\mu - 3 & \text{if } \mu \geqslant 1 \end{cases}$$

*For each $\mu$, equality is attained by a function in $B(1)$.*

**Proof:** From (3) there exists a **normalized** starlike function $g$ such that

$$\mathrm{Re}\left[e^{i\gamma}\frac{zf'(z)}{g(z)}\right] > 0, \quad |z| < 1.$$

for some real $\gamma$, $|\gamma| < \pi/2$. Now, if $\mu \notin (0, 2/3)$ the estimates in (15) are those obtained by Keogh and Merkes [3]. Thus we consider only $0 \leqslant \mu \leqslant 2/3$, and we begin with (9) of [3]:

$$(16) \qquad a_3 - \mu a_2^2 = \tfrac{1}{3}\left(c_3 - \frac{3}{4}\mu c_2^2\right) + \frac{2}{3}\cos\gamma\left[a_2 + \left(e^{i\gamma} - \frac{3}{2}\mu\cos\gamma\right)a_1^2\right] + \left(\frac{2}{3} - \mu\right)\cos\gamma\, a_1 c_2,$$

where $\{c_j\}$ is the coefficient sequence of $g$, and $\{a_j\}$ is the coefficient sequence of the related function $\omega$, $|\omega| \leqslant 1$. Since $\frac{3}{4}\mu \leqslant \frac{1}{2}$, $\left|c_3 - \frac{3}{4}\mu c_2^2\right| \leqslant 3(1-\mu)$ [3]. Also, $|a_2| \leqslant 1 - |a_1|^2$ and $|c_2| \leqslant 2$. Thus,

$$(17) \qquad |a_3 - \mu a_2^2| \leqslant 1 - \mu + \frac{2}{3}\cos\gamma\,[1 + (|s| - 1)|a_1|^2] + 2\left(\frac{2}{3} - \mu\right)\cos\gamma\,|a_1|.$$

where $s = e^{i\gamma} - \frac{3}{2}\mu\cos\gamma$. As a function of $|a_1|$, the right-hand side of (17) is maximized when $|a_1| = \left(1-\frac{3}{2}\mu\right)(1-|s|)^{-1}$. Since we must have $|a_1| \leqslant 1$, this gives $|s| \leqslant 3/2\mu$, or equivalently,

(18) $$\cos^2\gamma \geqslant \left(1-\frac{9}{4}\mu^2\right)\left(3\mu - \frac{9}{4}\mu^2\right)^{-1},$$

for each fixed $\mu$, $\frac{1}{3} \leqslant \mu < 2/3$. Define $\gamma_0(\mu) \in [0, \pi/2)$ so that equality holds in (18). Then define $q_\mu$ on $[0, \gamma_0]$ by

(19) $$q_\mu(\gamma) = \cos\gamma(1 + a(\mu)(1+|s|)\sec^2\gamma)$$

where

$$a(\mu) = (2-3\mu)^2[3\mu(4-3\mu)]^{-1}.$$

Note that, upon replacing $|a_1|$ by $\left(1-\frac{3}{2}\mu\right)(1-|s|)^{-1}$ in the right-hand side of (17), we obtain

(20) $$|a_3 - \mu a_2^2| \leqslant 1 - \mu + \frac{2}{3} q_\mu(\gamma).$$

We now assert that for $\frac{1}{3} \leqslant \mu < \frac{2}{3}$, $\max_{[0,\gamma_0]} q_\mu(\gamma) = q_\mu(0)$. To verify this, note that $|s|$ is an increasing function of $\gamma$, so that

$$q'_\mu(\gamma) = \frac{\sin\gamma\big(|s|(a(\mu) - \cos^2\gamma) + a(\mu)\big)}{|s|\cos^2\gamma}$$

$$\leqslant \frac{\sin\gamma}{|s|\cos^2\gamma}\left[\left(1-\frac{3}{2}\mu\right)\left(\frac{-2(2-3\mu)}{4-3\mu}\right) + \frac{(2-3\mu)^2}{3\mu(4-3\mu)}\right]$$

$$= \frac{\sin\gamma(2-3\mu)^2}{|s|\cos^2\gamma(4-3\mu)}\left(\tfrac{1}{3}\mu - 1\right) < 0, \text{ for } \tfrac{1}{3} < \mu < \tfrac{2}{3}.$$

We now must examine, for $\frac{1}{3} \leqslant \mu < \frac{2}{3}$, the case $\gamma_0 \leqslant \gamma < \pi/2$. By (18), this is equivalent to $|s| \geqslant 3/2\mu$ which implies the right-hand side of (17) is maximized when $|a_1| = 1$. We then have

(21) $$|a_3 - \mu a_2^2| \leqslant 1 - \mu + \tfrac{2}{3} p_\mu(\gamma),$$

where

$$p_\mu(\gamma) = \cos\gamma[\,|s| + (2-3\mu)].$$

In the same manner as above, $p_\mu(\gamma)$ is decreasing on $[\gamma_0, \pi/2)$, so that $\max_{[\gamma_0,\pi/2]} p_\mu(\gamma) = p_\mu(\gamma_0) = q_\mu(\gamma_0) \leqslant q_\mu(0)$. Thus, from (20), for each $\mu$, $1/3 \leqslant \mu < 2/3$,

(22) $$|a_3 - \mu a_2^2| \leqslant 1 - \mu + \tfrac{2}{3} q_\mu(0) = \tfrac{1}{3} + \frac{4}{9\mu}.$$

For $0 \leqslant \mu \leqslant \frac{1}{3}$, it follows from (22) that

$$|a_3 - \mu a_2^2| \leqslant 3\mu |a_3 - \tfrac{1}{3} a_2^2| + (1-3\mu)|a_3| \leqslant 3\mu(5/3) + (1-3\mu)3 = 3-4\mu.$$

The fact that, for each $\mu$, equality in (15) is attained by a function in $B(1)$, is shown in [3]. The proof of Theorem C is complete.

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## STRESZCZENIE

Przedmiotem pracy jest znalezienie dokładnego oszacowania wyrażenia $|a_3 - \mu a_2^2|$ w pewnej klasie funkcji Bazylewicza.

## РЕЗЮМЕ

Предметом заметки является определение точной оценки функционала $|a_3 - \mu a_2^2|$ в некотором классе функций Базилевича.