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On the Radius of Convexity of Some Class of Analytic k-Symmetrical Functions

O promieniu wypukłości pewnej klasy funkcji analitycznych k-symetrycznych

Радиус выпуклости некоторого класса к-симметричных аналитических функций

Let a, $0 \leqslant a < 1$, be an arbitrary fixed number and let k be an arbitrary fixed natural number.

Denote by S_k the family of regular and univalent functions of the form

$$f(z) = z + \sum_{j=1}^{\infty} a_{jk}^{(k)} z^{jk+1} \tag{1}$$

defined in the circle $K = \{z: |z| < 1\}$ while $\overline{S_k^*(a)}$ stands for the subclass of the family S_k made up of all functions of form (1) of the family S_k which satisfy the condition

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - a}{1-a} - 1 \right| < 1 \tag{2}$$

i.e. which satisfy the condition

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - a.$$

Moreover we accept the following denotations:

$S_1 = S$ — the family of all regular and univalent functions of form (1) defined in the circle K,

S^* — the subclass of all starlike functions of the family S, i.e. the subclass of functions of form (1) which map the circle K onto starlike regions with respect to the origin,

S_k^* — the subclass of all starlike functions of the family S_k,

$S_k^*(a)$ — the family of all functions of form (1) which are starlike of order a i. e. satisfy the condition

$$\operatorname{re} \frac{zf'(z)}{f(z)} > a \quad \text{for every } z \in K.$$

Evidently the family $\widetilde{S_k^*}(a)$ is a subclass of the family $S_k^*(a)$. In fact, condition (2) means that

$$\zeta = \frac{\frac{zf'(z)}{f(z)} - a}{1-a} \in K(1,1) = \{z\colon |z-1| < 1\}$$

by which

$$\operatorname{re} \frac{\frac{zf'(z)}{f(z)} - a}{1-a} < 0$$

and thus

$$\operatorname{re} \frac{zf'(z)}{f(z)} > a.$$

Since $S_k^*(a) \subset S_k^*$ and $\widetilde{S_k^*}(a) \subset S_k^*(a)$

$$\widetilde{S_k^*}(a) \subset S_k^*.$$

The problem formulated in this paper consists in determining the radius of convexity r_0 of the family $\widetilde{S_k^*}(a)$, i.e. the radius of the largest circle $|z| < r < 1$ which is mapped by every function of the class $\widetilde{S_k^*}(a)$ onto a convex region. A function $f(z) \in S$ is convex, i.e. it maps the circle K onto a convex region if and only if

$$\operatorname{re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \text{ for every } z \in K.$$

Now we shall come back to the definition of the radius of convexity which is to be made more precise. Let for every fixed function $f = f(z) \in \widetilde{S_k^*}(a)$

$$r(f) = \sup\left\{r\colon \operatorname{re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ |z| < r\right\}$$

$$r_0 = \inf_{f \in \widetilde{S_k^*}(a)} r(f).$$

Since the family $\widetilde{S_k^*}(a)$ is compact and since it is a subclass of the family S, r_0 is the radius of the largest circle which is mapped onto a convex region by every function of the class $\widetilde{S_k^*}(a)$, or which is the same, r_0 is the smallest root of the equation $w(r) = 0$ contained in the interval $(0, 1\rangle$ where

(3) $$w(r) = \min_{|z| = r < 1,\, f(z) \in \widetilde{S_k^*}(a)} \operatorname{re}\left[1 + \frac{zf''(z)}{f'(z)}\right].$$

Next denote by $\tilde{\mathscr{P}}_k(a)$ the family of all regular functions of the form

$$P(z) = 1 + \sum_{j=1}^{\infty} b_{jk}^{(k)} z^{jk} \tag{4}$$

defined in the circle K which satisfy the condition

$$\left|\frac{P(z)-a}{1-a} - 1\right| < 1 \text{ for every } z \in K$$

and by $\mathscr{P}_k(a)$ the family of all functions $p(z)$ of form (4) such that

$$\operatorname{re} p(z) > a \text{ for every } z \in K.$$

It follows from what has been said above that $\mathscr{P}_1(0) = \mathscr{P}$, where $\mathscr{P}$ is the family of Carathéodory functions, and that $\tilde{\mathscr{P}}_k(a) \subset \mathscr{P}_k(a)$. It follows from the definitions of the families $\tilde{S}_k^*(a)$ and $\tilde{\mathscr{P}}_k(a)$ that $f(z) \in \tilde{S}_k^*(a)$ if and only if $\frac{zf'(z)}{f(z)} \in \tilde{\mathscr{P}}_k(a)$. Let $f(z)$ be an arbitrary function of the class $\tilde{S}_k^*(a)$. Then

$$\frac{zf'(z)}{f(z)} = P(z) \tag{5}$$

for some function $P(z) \in \tilde{\mathscr{P}}_k(a)$. Hence by differentiating we easily obtain equation (5) and after simple transformations the relationship

$$1 + \frac{zf''(z)}{f'(z)} = P(z) + \frac{zP'(z)}{P(z)}. \tag{6}$$

Thus by (3) and (6) we have

$$w(r) = \min_{|z|=r<1,\, P(z) \in \tilde{\mathscr{P}}_k(a)} \operatorname{re}\left[P(z) + \frac{zP'(z)}{P(z)}\right].$$

Let $p(z) \in \mathscr{P}_k(a)$ then, as it is easily seen, the function

$$P(z) = \frac{(1+\beta)p(z) + 1 - \beta}{p(z)+1}, \quad \beta = 1-a, \tag{7}$$

belongs to the family $\tilde{\mathscr{P}}_k(a)$, the converse being also true. In fact, the function $P(z)$ defined by formula (7) is the superposition of the function $\zeta = p(z)$ which maps the circle K onto the semiplane $\operatorname{re}\zeta > a$ and of the homograph function $w(\zeta) = \frac{(1+\beta)\zeta + (1-\beta)}{\zeta+1}$ which maps the semiplane $\operatorname{re}\zeta > a$ on to the circle $|w-1| < \beta$. Thus $|P(z)-1| < 1-a$

and consequently $\operatorname{re} P(z) > a$. The function $P(z)$ defined by formula (7) is regular in the circle K as the superposition of regular functions, we also have $P(0) = 1$. Consider the functional

$$F(P) = P(\zeta), \quad P(z) \in \tilde{\mathscr{P}}_k(a). \tag{8}$$

Lemma 1. *The set of values of functional* (8) *is the closed circle* $K(C, \varrho)$ *with the centre at* C *and the radius* ϱ, *where* $C = 1$ *and* $\varrho = \beta r^k$, $r = |z|$.

Proof. Every boundary function $P_0(z)$ of the family $\tilde{\mathscr{P}}_k(a)$ with respect to functional (8) is of form (7) where

$$p_0(z) = \frac{1+\varepsilon z^k}{1-\varepsilon z^k}, \quad |\varepsilon| = 1 \text{ (comp. [1])}. \tag{9}$$

Thus

$$P_0(z) = 1 + \beta\varepsilon z^k.$$

Consequently for $z = re^{i\varphi}$, $0 \leqslant \varphi \leqslant 2\pi$,

$$P_0(z) = C + \varrho\cdot\eta_0,$$

where

$$\eta_0 = \varepsilon e^{ik\varphi}$$

which ends the proof. Further denote by $\tilde{\mathscr{P}}_{k,2}(a)$ the subclass of the family $\tilde{\mathscr{P}}_k(a)$ consisting of all functions of form (7) with

$$p(z) = \frac{1+\lambda}{2}\, p_1(z) + \frac{1-\lambda}{2}\, p_2(z), \tag{10}$$

$$p_j(z) = \frac{1+\varepsilon_j z^k}{1-\varepsilon_j z^k}, \quad |\varepsilon_j| = 1, \; j = 1, 2, \; -1 \leqslant \lambda \leqslant 1. \tag{11}$$

Next let $F(u, v)$ be an arbitrary analytic function defined in the semiplane $\operatorname{re} u > 0$ and in the plane v and let $|F'_u|^2 + |F'_v|^2 > 0$ at every point (u, v). Then it is known that every boundary function $p(z)$ with respect to the functional $F\big(p(z), zp'(z)\big)$, $|z| = r$ is of form (10) [1]. Thus every boundary function with respect to the functional

$$F\big(P(z), zP'(z)\big), \quad P(z) \in \tilde{\mathscr{P}}_k(a), \; |z| = r$$

is of form (7) where $p(z)$ is of form (10). Therefore

$$w(r) = \min_{|z| = r \leqslant 1,\, P(z) \in \tilde{\mathscr{P}}_{k,2}(a)} \operatorname{re}\left[P(z) + \frac{zP'(z)}{P(z)}\right].$$

Now we shall prove the following lemma:

Lemma 2. *If* $P(z)\in\tilde{\mathscr{P}}_{k,2}(a)$ *and* $z=re^{i\varphi}$, $0\leqslant\varphi\leqslant 2\pi$, *then*

$$zP'(z)=k[P(z)-1]-ka[\varrho^2-|P(z)-1|^2]\cdot\eta, \tag{12}$$

with

$$\varrho=\varrho(r^k)=\beta r^k,\ a=a(r)=\frac{1}{\beta(1-r^{2k})}\ \text{and}\ |\eta|=1. \tag{13}$$

Proof. Differentiating function (7) with respect to z and then multiplying the result by z we get

$$zP'(z)=\frac{2\beta zp'(z)}{(p(z)+1)^2} \tag{14}$$

According to formula (11) we have

$$p_j(re^{i\varphi})-\frac{1+r^{2k}}{1-r^{2k}}=\frac{2r^k}{1-r^{2k}}\cdot\frac{\varepsilon_j e^{ik\varphi}-r^k}{1-\varepsilon_j r^k e^{ik\varphi}},$$

thus

$$p_j(re^{i\varphi})=\frac{1+r^{2k}}{1-r^{2k}}+\frac{2r^k}{1-r^{2k}}\cdot\frac{\varepsilon_j e^{ik\varphi}-r^k}{1-\varepsilon_j r^k e^{ik\varphi}}.$$

If $p_j(z)$ is of form (11), we have for $z=re^{i\varphi}$

$$p_j(re^{i\varphi})=c^*+\varrho^*\gamma_j,\quad j=1,2 \tag{15}$$

with

$$c^*=\frac{1+r^{2k}}{1-r^{2k}},\quad \varrho^*=\frac{2r^k}{1-r^{2k}}, \tag{16}$$

$$\gamma_j=\varepsilon_j e^{ik\varphi}\cdot\frac{1-\bar{\varepsilon}_j r^k e^{-ik\varphi}}{1-\varepsilon_j r^k e^{ik\varphi}},\ |\gamma_j|=1,\ j=1,2.$$

Let now $p(z)$ be of form (10), then taking into account formula (15) we obtain

$$p(z)=\frac{1+\lambda}{2}(c^*+\varrho^*\gamma_1)+\frac{1-\lambda}{2}(c^*+\varrho^*\gamma_2),\ z=re^{i\varphi}, \tag{17}$$

c^*, ϱ^* and γ_j, $j=1,2$ being defined by formulas (16). By (17) we find that for $z=re^{i\varphi}$

$$p(z)=c^*+\varrho^*\cdot\left(\frac{1+\lambda}{2}\gamma_1+\frac{1-\lambda}{2}\gamma_2\right)$$

holds.
Let

$$\varkappa\mu_1=\varrho^*\left(\frac{1+\lambda}{2}\gamma_1+\frac{1-\lambda}{2}\gamma_2\right) \tag{18}$$

with

(19) $$\varkappa = \varrho^* \cdot \left|\frac{1+\lambda}{2}\gamma_1 + \frac{1-\lambda}{2}\gamma_2\right|, \quad |\mu_1| = 1.$$

Multiplying both sides of (18) by $\overline{\varkappa\mu_1}$ we get the formula

$$\varkappa^2 = \frac{\varrho^{*2}}{4}[(1+\lambda)^2 + (1-\lambda)^2 + (1-\lambda^2)\cdot(\gamma_1\overline{\gamma}_2 + \overline{\gamma}_1\gamma_2)].$$

Putting

(20) $$\gamma_j = e^{i\beta_j}, \quad j = 1, 2$$

we get

$$\varkappa^2 = \varrho^{*2}\left[1 - (1-\lambda^2)\sin^2\frac{\beta_1-\beta_2}{2}\right],$$

i.e.

(21) $$\varkappa^2 = \varrho^{*2} - \varrho^{*2}(1-\lambda^2)\sin^2\frac{\beta_1-\beta_2}{2}.$$

It follows from formula (21) that

$$0 \leqslant \varkappa \leqslant \varrho^*.$$

Thus if $p(z)$ is of form (10), then according to formulas (18) and (19) we have

(22) $$p(re^{i\varphi}) = c^* + \varkappa\mu_1.$$

Now we shall evaluate the expression $zp'(z)$ for $z = re^{i\varphi}, p(z)$ being of form (10) and then multiplying both sides of the result by z we get on some transformations the formula

(23) $$zp'(z) = \frac{k}{2}[p^2(z) - 1] + \frac{k}{2}\frac{1-\lambda^2}{4}[p_1(z) - p_2(z)].$$

Further applying formula (15) to the function $p_j(z)$, $j = 1, 2$ for $z = re^{i\varphi}$ we find, with the denotations of (20) that

(24) $$[p_1(z) - p_2(z)]^2 = \varrho^{*2}\gamma_1\gamma_2\cdot[2\cos(\beta_1 - \beta_2) - 2].$$

Denoting

(25) $$\gamma_1\gamma_2 = e^{i(\beta_1+\beta_2)} = \eta.$$

We reduce formula (24) to the form

(26) $$[p_1(z) - p_2(z)]^2 = -4\varrho^{*2}\cdot\eta\sin^2\frac{\beta_1-\beta_2}{2}.$$

Taking into account formula (21) in formula (26) we obtain

$$\frac{1-\lambda^2}{8}[p_1(z)-p_2(z)]^2 = -\frac{\eta}{2}[\varrho^{*2}-\varkappa^2].$$

Thus formula (23) becomes

$$zp'(z) = \frac{k}{2}[p^2(z)-1]-\frac{k}{2}\eta[\varrho^{*2}-\varkappa^2],\ z = re^{i\varphi}. \tag{27}$$

From formula (22) we have

$$|\varkappa| = |p(re^{i\varphi})-c^*|.$$

Substituting the obtained value for $|\varkappa|$ into formula (27) we get ultimately

$$zp'(z) = \frac{k}{2}[p^2(z)-1]-\frac{k}{2}[\varrho^{*2}-|p(z)-c^*|^2]\cdot\eta,\ |\eta| = 1. \tag{28}$$

Thus taking into account (14) and (28) we have for $|z| = r$

$$zP'(z) = \frac{\beta}{(p(z)+1)^2}\cdot\{k[p^2(z)-1]-k[\varrho^{*2}-|p(z)-c^*|^2]\eta\}.$$

From formula (7) we obtain

$$p(z) = \frac{1-\beta-P(z)}{P(z)-(1+\beta)}. \tag{29}$$

Hence

$$p(z)+1 = \frac{-2\beta}{P(z)-(1+\beta)},\ p(z)-1 = \frac{2(1-P(z))}{P(z)-(1+\beta)},\ \frac{p(z)-1}{p(z)+1} = \frac{P(z)-1}{\beta}. \tag{30}$$

Then we get for $z = re^{i\varphi}$, $0 \leqslant \varphi \leqslant 2\pi$

$$\varrho^{*2}-|p(z)-c^*|^2 = 4\frac{\varrho^2-|P(z)-1|^2}{(1-r^{2k})|P(z)-(1+\beta)|^2}. \tag{31}$$

By (29) — (31) we obtain ultimately formula (12) which ends the proof of lemma 2. According to lemma 2 we have

$$w(r) = \min_{\substack{|z|=r<1\\ P(z)\in\mathscr{P}_{k,2}(a)}} \operatorname{re}\left[P(z)+\frac{zP'(z)}{P(z)}\right] \tag{32}$$

$$= \min_{\substack{|z|=r<1\\ P(z)\in\mathscr{P}_{k,2}(a)}} \operatorname{re}\left\{P(z)+k\left[1-\frac{1}{P(z)}\right]-ka[\varrho^2-|P(z)-1|^2]\frac{\eta}{P(z)}\right\}.$$

Let

$$P(re^{i\varphi}) = se^{it},\ s > 0,\ \operatorname{im} t = 0.$$

By lemma 1 s and t satisfy the conditions

$$1-\varrho \leqslant s \leqslant 1+\varrho \text{ and } -\Psi(s) \leqslant t \leqslant \Psi(s),$$

with

$$\Psi(s) = \arccos \frac{1+s^2-\varrho^2}{2s}. \tag{33}$$

Moreover we introduce the denotations

$$G = \{(s,t):\ 1-\varrho < s < 1+\varrho,\ -\Psi(s) < t < \Psi(s)\},$$
$$\partial G = \{(s,t):\ 1-\varrho \leqslant s \leqslant 1+\varrho,\ t = \pm\Psi(s)\},$$
$$I = \{s:\ 1-\varrho < s < 1+\varrho\}.$$

Then formula (44) becomes

$$w(r) = \min_{\substack{|z|=r<1\\(s,t)\in G\cup\partial G}} \left\{ s\cos t + k - \frac{k\cos t}{s} - ka[-s^2+2\cos t-(1-\varrho^2)]\,\mathrm{re}\,\frac{\eta}{P(z)} \right\},$$

where $2s\cos t - s^2-(1-\varrho^2) \geqslant 0$ for $(s,t)\in G\cup\partial G$. Since

$$\mathrm{re}\frac{\eta}{P(z)} \leqslant \frac{1}{|P(z)|}, \tag{34}$$

(35) $w(r) =$

$$\min_{|r|=r<1,\ P(z)\in\mathscr{P}_{k,2}(a)} \mathrm{re}\left[P(z)+\frac{zP'(z)}{P(z)}\right] \geqslant \min_{|z|=r<1,\ (s,t)\in G\cup\partial G} B(s,t) = \omega(r)$$

where

$$B(s,t) = \left[\left(s-\frac{k}{s}\right)\cos t + k\right] + ka\left[s-2\cos t+\frac{1-\varrho^2}{s}\right]. \tag{36}$$

Now we proceed to determining the minimum of the function $B(s,t)$ and then we shall find the radius of convexity r_0 of the family $S_k^*(a)$. We consider two cases: I $(s,t)\in G$, II $(s,t)\in\partial G$.

I. $(s,t)\in G$. Consider the system of equations

$$B'_t(s,t) = \left(-s+\frac{k}{s}+2ak\right)\sin t = 0$$

$$B'_s(s,t) = \frac{1}{s^2}\left[(1+ka)s^2+k\big(1-a(1-\varrho^2)\big)\right] = 0.$$

Finding that $-s+\dfrac{k}{s}+2ak \neq 0$ for $s\in I$ we get that $\sin t = 0$ and because of $\cos t > 0$, we have $\cos t = 1$. Thus

$$\omega(r) = \min_{|z|=r<1,\ (s,t)\in G} B(s,t) = \min_{|z|=r<1,\ s\in I} C(s),$$

where

$$C(s) = B(s, 0) = s - \frac{k}{s} + k + ka\left[s - 2 + \frac{1-\varrho^2}{s}\right].$$

Since

$$C'(s) = \frac{1}{s^2}\left[(1+ka)s^2 - k\left(a(1-\varrho^2)-1\right)\right]$$

and

$$C''(s) = \frac{2k[a(1-\varrho^2)-1]}{s^3}$$

the function $C(s)$ attains a local minimum at the point

$$s_1 = \sqrt{k\,\frac{a(1-\varrho^2)-1}{1+ka}}, \tag{37}$$

if $s_1 \in I$.

Now we shall find out for what values of $r \in (0, 1\rangle$, $s_1 \in I$. It is easily verified that the inequality $s_1 < 1+\varrho$ always holds. In order to determine the values of r for which $1-\varrho < s_1$ holds we assume the following notation

$$l(r) = (1-\varrho)^2 = (1-\beta r^k)^2$$

$$m(r) = s_1^2(r) = k(1-\beta)\,\frac{1+\beta r^{2k}}{(\beta+k)-\beta r^{2k}}. \tag{38}$$

Then $1-\varrho < s_1$ if

$$l(r) - m(r) < 0.$$

Since

$$l(0) = 1,\ l(1) = (1-\beta)^2$$

and

$$l'(r) = 2(1-\beta r^k)\cdot(-k\beta r^{k-1}) < 0 \text{ for } r \in (0, 1\rangle,$$

$l(r)$ is a decreasing function for $r \in (0, 1\rangle$. By an analogous argument we obtain

$$m(0) = \frac{k(1-\beta)}{k+\beta},\ m(1) = 1-\beta^2$$

and

$$m'(r) = 2k^2\beta(1-\beta)r^{2k-1}\cdot\frac{k+\beta+1}{[(\beta+k)-\beta r^{2k}]^2} > 0$$

thus $m(r)$ is an increasing function in the interval $(0, 1\rangle$. Moreover taking into account that

$$(1-\beta)^2 < \frac{k(1-\beta)}{k+\beta} < 1-\beta^2 < 1$$

we get $1-\varrho < s_1$ for $r > r^*$ where r^* is the only root, $0 < r^* < 1$, of the equation

$$l(r)-m(r) = 0. \tag{39}$$

Now we shall transform equation (39). Employing in it denotations (38), (37) and (13) we obtain

$$l(r)-m(r) = \frac{-\beta}{k+\beta(1-r^{2k})}\cdot h(r^k) = 0,$$

with

$$h(r^k) = \beta^2 r^{4k} - 2\beta r^{3k} + \\ +[(1-2\beta)k+(1-\beta^2)]r^{2k}+2(k+\beta)r^k-(k+1). \tag{40}$$

Since

$$\frac{-\beta}{k+\beta(1-r^{2k})} < 0 \text{ for } r\in(0,1)$$

r^*, $0<r^*<1$ is the only root of the equation

$$h(r^k) = 0 \text{ for } r\in(0,1). \tag{41}$$

It follows from the above considerations that

$$h(r^k) > 0 \text{ for } r^* < r < 1$$

and that

$$h(r^k) \leqslant 0 \text{ for } 0 < r \leqslant r^* \tag{42}$$

Summing up we find that $s_1 \in I$ for $r\in(r^*, 1)$ and then

$$\operatorname{loc} \min_{|z|=r<1,(s,t)\in G} B(s,t) = \operatorname{loc} \min_{|z|=r<1,s\in I} C(s)$$

$$= (1+ak)s_1 - k(2a-1) + \frac{k}{s_1}[a(1-\varrho^2)-1].$$

By (37) and (13)

$$C(s_1) = \min_{|z|=r<1,s\in I} \operatorname{loc} C(s) \tag{43}$$

$$= \frac{kU(r^{2k})}{\beta(1-r^{2k})^2[2(1+ak)s_1+k(2a-1)]} \text{ for } r^* < r < 1$$

where

$$U(r^{2k}) = -\beta[k+4(1-\beta)]r^{4k} - \\ -2[k\beta+2(1-\beta)^2]r^{2k}-[k\beta-4(1-\beta)] \tag{44}$$

and

$$\beta(1-r^{2k})^2[2(1+ka)s_1+k(2a-1)] > 0 \text{ for } r\in(0,1)$$

We have

$$U(0) > 0 \text{ for } k < k_1(\beta),$$

where

$$k_1(\beta) = \frac{4(1-\beta)}{\beta}. \tag{45}$$

It is easily verified that if

$$k < k_1(\beta)$$

then function (44) of the variable r^{2k} has in the interval $(0, 1)$ exactly one root given by the formula

$$X = \frac{2(1-\beta)\sqrt{2\beta(k+2)+(1-\beta)^2} - k\beta - 2(1-\beta)^2}{\beta[k+4(1-\beta)]} \tag{46}$$

while if $k > k_1(\beta)$, then $U(r^{2k}) < 0$ for $0 < r < 1$. Accepting $r_1 = \sqrt[2k]{X}$ we have by (46)

$$r_1 = \sqrt[2k]{\frac{2(1-\beta)\sqrt{2\beta(k+2)+(1-\beta)^2} - k\beta - 2(1-\beta)^2}{\beta[k+4(1-\beta)]}} \tag{47}$$

with, according to (43)

$$\min_{|s|=r<1,\,(s,t)\in G} B(s, t) = C(s_1) = 0 \text{ for } r = r_1 > r^*.$$

II. $(s, t) \in \partial G$. Then we obtain from formula (33)

$$\cos t = \frac{1+s^2-\varrho^2}{2s}$$

and substituting this value for $\cos t$ in formula (36) we get

$$B(s, \Psi(s)) = H(s) = \frac{s^4+(k+1-\varrho^2)s^2 - k(1-\varrho^2)}{2s^2},$$

Hence

$$H'(s) = s + \frac{k(1-\varrho^2)}{s^3} > 0 \text{ for } s \in \bar{I} = \langle 1-\varrho, 1+\varrho\rangle.$$

Thus $H(s)$ is an increasing function in the interval $\bar{I}$ and thus it attains its minimum at the point $s_2 = s_2(r^k)$, $s_2(r^k) = 1-\varrho(r^k)$ equal to

$$\min_{|s|=r<1,\,s\in\bar{I}} H(s) = H(1-\varrho) = \frac{\varrho^2-(k+2)\varrho+1}{1-\varrho}.$$

By $\varrho(r^k) = \beta r^k$

$$\min_{|s|=r<1,\,(s,t)\in\partial G} B(s, t) = H(s_2) = \frac{F(r^k)}{1-\beta r^k}, \tag{48}$$

where

(49) $$F(r^k) = \beta^2 r^{2k} - (k+2)\beta r^k + 1.$$

We have

$$F(0) > 0.$$

It is easily verified that if

$$k > k_2(\beta)$$

with

(50) $$k_2(\beta) = \frac{(1-\beta)^2}{\beta}$$

then function (49) of the variable r^k has exactly one root given by the formula

(51) $$y = \frac{k+2-\sqrt{k(k+4)}}{2\beta}$$

in the interval (0,1), while if $k < k_2(\beta)$, then $F(r^k) > 0$ for $0 < r < 1$. Accepting $r_2 = \sqrt[k]{y}$, by (51) we have

(52) $$r_2 = \sqrt[k]{\frac{k+2-\sqrt{k(k+4)}}{2\beta}} \quad \text{when } k \geqslant k_2(\beta).$$

We sum up the results obtained. According to the performed considerations the function $C(s) = B(s, 0)$ attains its local minimum at the point $s_1(r)$; this minimum is equal zero for $r = r_1$ only if $r_1 > r^*$. Next the function $H(s) = B\big(s, \Psi(s)\big)$ attains its local minimum at the point $s_2(r^k)$; this minimum is equal zero for $r = r_2$ independently of the position of the number r_2 relatively to r^*. Moreover if $r_2 < r^*$, then the function $B(s, t)$ defined in the region $G \cup \partial G$ attains its absolute minimum equal zero at the point r_2. It is easily verified that $H(s_2) = C(s_2) > C(s_1)$. In fact for $s \in \bar{I}$ we have

$$C(s) - C(s_1) = (s-s_1)C'(s_1) + \frac{(s-s_1)^2}{2} C''\big(s_1 + (s-s_1)\theta\big), \; 0 < \theta < 1.$$

Thus taking into consideration that $C'(s_1) = 0$ and $C''(s) > 0$ for $s \in \bar{I}$ we obtain $C(s) - C(s_1) \geqslant 0$ for every $s \in \bar{I}$, thus $C(s_2) \geqslant C(s_1)$. Hence it immediately follows that if $r > r^*$ then minimum $B(s, t) = C(s_1)$, thus if $r_1 > r^*$, the function $B(s, t)$ attains its absolute minimum at the point r_1. Since

$$H'_{r^k}\big(s_2(r^k)\big) = \frac{-k\beta - \beta(1-\beta r^k)^2}{(1-\beta r^k)^2} < 0 \text{ for } r \in (0, 1)$$

we have moreover that $r_1 < r_2$ for $r_1 > r^*$. Thus, because of the definition of the radius of convexity r_0 and inequality (35) we have proved

Lemma 3. *The radius of convexity r_0 of the family $\tilde{S}_k^*(a)$ satisfies the inequalities*

(53) $$r_0 \geqslant \begin{cases} r_2 \text{ when } 0 < r_2 \leqslant r^* \text{ and } k > k_2(\beta) \\ r_1 \text{ when } r_1 > r^* \text{ and } k < k_1(\beta) \end{cases}$$

where r_1 and r_2 are defined by formulars (47) *and* (52) *and r^* is the only root of equation* (41) *which belongs to the interval* (0,1) Now we shall prove

Lemma 4. *The radius of convexity r_0 of the family $\tilde{S}_k^*(a)$ satisfies the inequalities*

$$r_0 \leqslant \begin{cases} r_2 \text{ when } 0 < r_2 \leqslant r^* \text{ and } k \geqslant k_2(\beta) \\ r_1 \text{ when } r^* < r_1 < 1 \text{ and } k < k_1(\beta). \end{cases}$$

By which, because of lemma 3 we will prove that $r_0 = r_2$ or $r_0 = r_1$ respecitvely.

Proof. We distinguish two cases:

A. $r_2 \leqslant r^*$ and $k > k_2(\beta)$, B. $r^* < r_1$ and $k < k_1(\beta)$.

A. Let $P(z)$ be a function of the family $\wp_{k,2}(a)$ such that for $z = r_2 e^{i\varphi}$, $0 \leqslant \varphi \leqslant 2\pi$, $B(s, t)$ attains its minimum equal zero. Since this minimum is attained at the point $t = 0$, $s = s_2(r_2)$ where $s_2(r) = 1 - \varrho(r^k)$, $\varrho(r^k) = \beta r^k$,

(54) $$P(r_2 e^{i\varphi}) = 1 - \varrho(r_2^k).$$

Formula (7) assigns uniquely some function $p(z)$ of the family $\wp_k(a)$ to the function $P(z)$, $p(z)$ being uniquely defined by the formulas (10) and (11). By (54) and (31) we have for $z = r_2 e^{i\varphi}$ and $r = r_2$

$$\varrho^{*2} - |p(z) - c^*|^2 = 0.$$

Thus by formula (22) we have

$$\varkappa(r_2) = \varrho^*(r_2).$$

Therefore according to formula (22)

(55) $$p(r_2 e^{i\varphi}) = C^*(r_2) + \varrho^*(r_2)\mu_1, \quad |\mu_1| = 1$$

Hence it follows that

(56) $$p(z) = \frac{1 + \varepsilon z^k}{1 - \varepsilon z^k}, \quad |\varepsilon| = 1,$$

and consequently

$$P(z) = 1 + \beta\varepsilon z^k.$$

We have to determine ε.

From formula (54) it follows that $\operatorname{im} P(r_2 e^{i\varphi}) = 0$ thus by (41) also $\operatorname{im} p(r_2 e^{i\varphi}) = 0$. Consequently (55) implies $\mu_1^2 = 1$. On the other hand by (54) and (29) we have

$$p(r_2 e^{i\varphi}) = \frac{1 - r_2^k}{1 + r_2^k}. \tag{57}$$

Thus because of (16), (55) and (57) we find that $\mu_1 = -1$. Accepting $z = r_2 e^{i\varphi}$ in (56) we get by (57) $\varepsilon e^{ik\varphi} = -1$, hence

$$\varepsilon = -e^{-ik\varphi}.$$

Thus

$$P(z) = 1 + \beta\varepsilon z^k = 1 - \beta e^{-ik\varphi} z^k. \tag{58}$$

Denote by $\hat{f}(z)$ a function of the class $\tilde{S}_k^*(\alpha)$ which satisfies the equation

$$\frac{z\hat{f}'(z)}{\hat{f}(z)} = P(z)$$

with $P(z)$ defined by formula (58). This equation is equivalent to the following

$$\frac{\hat{f}'(z)}{\hat{f}(z)} - \frac{1}{z} = -\beta e^{-ik\varphi} z^{k-1}.$$

Hence

$$\log \frac{\hat{f}(z)}{z} = -\frac{\beta e^{-ik\varphi}}{k} z^k, \quad \log 1 = 0.$$

Thus

$$\hat{f}(z) = z \exp\left(-\frac{\beta e^{-ik\varphi}}{k} z^k\right). \tag{59}$$

We have for the function (59)

$$1 + \frac{z\hat{f}''(z)}{\hat{f}'(z)} = \frac{F(e^{-ik\varphi} z^k)}{1 - \beta e^{-ik\varphi} z^k}$$

with $F(r^k)$ given by (49). Thus at the point $z = r_2 e^{i\varphi}$

$$\operatorname{re}\left(1 + \frac{z\hat{f}''(z)}{\hat{f}'(z)}\right) = 0$$

holds. Thus the function $\hat{f}(z)$ is not convex in the circle $|z| < r$ for $r > r_2$. Consequently $r_0 \leqslant r_2$ and by $r_0 \geqslant r_2$ [comp. (53)] we find

$$r_0 = r_2 \text{ when } 0 < r_1 \leqslant r^* \text{ and } k \geqslant k_2(\beta).$$

B. Let now $P(z)$ be a function of the family $\tilde{\wp}_{k,2}(\alpha)$ such that for $z = r_1 e^{i\varphi}$, $0 \leqslant \varphi \leqslant 2\pi$, $B(s, t)$ attains its minimum equal zero. Since this minimum is attained at a point $t = 0$, $s = s_1(r_1)$,

$$P(r_1 e^{i\varphi}) = s_1(r_1) \text{ when } r^* < r_1 < 1 \text{ and } k < k_1(\beta). \tag{60}$$

Since $\eta = 1$ [comp. (34)], by (25)

$$\gamma_2 = \bar{\gamma}_1.$$

Thus taking into account (16) we obtain

$$\varepsilon_2 e^{ik\varphi} \cdot \frac{1 - \bar{\varepsilon}_2 r_1^k e^{-ik\varphi}}{1 - \varepsilon_2 r_1^k e^{ik\varphi}} = \bar{\varepsilon}_1 e^{-ik\varphi} \cdot \frac{1 - \varepsilon_1 r_1^k e^{ik\varphi}}{1 - \bar{\varepsilon}_1 r_1^k e^{-ik\varphi}}.$$

Hence we have

$$\varepsilon_1 \varepsilon_2 = e^{-2ik\varphi} \tag{61}$$

and because of (7), (10) and (11) the function $P(z)$ becomes

$$P(z) = 1 + \beta + 2\beta \cdot \frac{\varepsilon_1 \varepsilon_2 z^{2k} - (\varepsilon_1 + \varepsilon_2) z^k + 1}{[(\varepsilon_1 + \varepsilon_2) - \lambda(\varepsilon_1 - \varepsilon_2)] z^k - 2}. \tag{62}$$

Therefore

$$P(re^{i\varphi}) = 1 + \beta + 2\beta \cdot \frac{\varepsilon_1 \varepsilon_2 e^{2ik\varphi} r^{2k} - (\varepsilon_1 + \varepsilon_2) e^{ik\varphi} r^k + 1}{[(\varepsilon_1 + \varepsilon_2) - \lambda(\varepsilon_1 - \varepsilon_2)] e^{ik\varphi} r^k - 2},$$

thus because of (60) and (61)

$$s_1(r_1) = 1 + \beta + 2\beta \cdot \frac{r_1^{2k} - (\varepsilon_1 + \varepsilon_2) e^{ik\varphi} r^k + 1}{[(\varepsilon_1 + \varepsilon_2) - \lambda(\varepsilon_1 - \varepsilon_2)] e^{ik\varphi} r_1^k - 2}.$$

By (61) we have

$$\varepsilon_1 + \varepsilon_2 = e^{-ik\varphi} (\varepsilon_1 e^{ik\varphi} + \bar{\varepsilon}_1 e^{-ik\varphi}).$$

Accept further

$$d = \varepsilon_1 e^{ik\varphi} + \bar{\varepsilon}_1 e^{-ik\varphi} = 2\operatorname{re}(\varepsilon_1 e^{ik\varphi}), \tag{63}$$

then

$$s_1 = 1 + \beta + 2\beta \cdot \frac{r_1^{2k} - d r_1^k + 1}{[d - \lambda(\varepsilon_1 - \varepsilon_2) e^{ik\varphi}] r_1^k - 2}. \tag{64}$$

It follows from (64) that

$$\operatorname{im}\{\lambda(\varepsilon_1 - \varepsilon_2) e^{ik\varphi}\} = 0. \tag{65}$$

By (61)

(66) $$(\varepsilon_1-\varepsilon_2)e^{ik\varphi} = \varepsilon_1 e^{ik\varphi}-\bar{\varepsilon}_1 e^{-ik\varphi}$$

holds, thus condition (65) because of (66) becomes

$$\lambda(\varepsilon_1 e^{ik\varphi}-\bar{\varepsilon}_1 e^{-ik\varphi}) = 0$$

hence

(67) $$\lambda(\varepsilon_1^2 e^{2ik\varphi}-1) = 0.$$

By (67) we have

$$1^\circ \ \varepsilon_1^2 e^{2ik\varphi}-1 = 0,$$

or

$$2^\circ \ \lambda = 0.$$

We shall prove that case 1° does not occur. In fact, assuming for the sake of proof, that the opposite holds we would have

(68) $$\varepsilon_1 = \chi e^{-ik\varphi} \text{ where } \chi = \pm 1$$

and then by (68) we would get from (61)

$$\varepsilon_2 = \chi e^{-ik\varphi}$$

and thus

$$\varepsilon_1 = \varepsilon_2.$$

The function $p(z)$ would then be of form (9), thus we would have

$$P(r_1 e^{i\varphi}) = 1+\beta\varepsilon r_1^k.$$

Hence because of $\operatorname{Im} P(r_1 e^{i\varphi}) = 0$ [comp. (60)] and $|\varepsilon| = 1$ we would have $\varepsilon = 1$ or $\varepsilon = -1$ which is impossible because of

$$P(r_1 e^{i\varphi}) = 1+\beta r_1^k \neq s_1(r_1)$$

as well as

$$P(r_1 e^{i\varphi}) = 1-\beta r_1^k \neq s_1(r_1).$$

Thus

$$\lambda = 0.$$

Then formula (64) becomes

$$s_1(r_1) = 1+\beta+2\beta\frac{r^{2k}-dr_1^k+1}{dr_1^k-2}.$$

Hence we get

(69) $$d = 2\frac{\beta r_1^{2k}+s_1(r_1)-1}{[s_1(r_1)-(1-\beta)]r_1^k}.$$

Now we can determine the function $P(z)$ which satisfies condition (60). By formulas (61), (62) and (63) we find

$$(70)\qquad P(z) = \frac{2\beta e^{-2ik\varphi}z^{2k} + (1+\beta)e^{-ik\varphi}z^k - 2}{e^{-ik\varphi}dz^k - 2}$$

with d defined by formula (69). Similarly as in case A denote by $\tilde{f}(z)$ a function of the class $\tilde{S}_k^*(a)$ which satisfies the equation

$$\frac{z\tilde{f}'(z)}{\tilde{f}(z)} = P(z),$$

$P(z)$ being a function defined by formula (70) with $d \neq 0$. This equation is equivalent to

$$\frac{\tilde{f}'(z)}{\tilde{f}(z)} - \frac{1}{z} = \frac{2\beta}{d} e^{-ik\varphi}z^{k-1} + \frac{\beta(4-d^2)}{d}\cdot\frac{e^{-ik\varphi}z^{k-1}}{e^{-ik\varphi}dz^k - 2} \text{ with } d \neq 0.$$

Hence

$$\log\frac{\tilde{f}(z)}{z} = \beta\frac{4-d^2}{kd^2}\log\left(1 - \frac{d}{2}e^{-ik\varphi}z^k\right) + \frac{2\beta}{kd}e^{-ik\varphi}z^k,\ \log 1 = 0.$$

Thus

$$(71)\ \tilde{f}(z) = z\cdot\exp\left[\beta\frac{4-d^2}{kd^2}\log\left(1-\frac{d}{2}e^{-ik\varphi}z^k\right) + \frac{2\beta}{kd}e^{-ik\varphi}z^k\right] \text{ with } d \neq 0.$$

For function (71) we have

$$1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)} = \frac{kU(e^{-2ik\varphi}z^{2k})}{\beta(1-e^{-2ik\varphi}z^{2k})^2[2(1+ak)s_1 + k(2a-1)]},$$

with $U(r^{2k})$ given by formula (44). Thus at the point $z = r_1e^{i\varphi}$ we have

$$\mathrm{re}\left(1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)}\right) = 0.$$

So the function $\tilde{f}(z)$ is not convex in the circle $|z| < r$ for $r > r_1$. Thus $r_0 \leqslant r_1$ and by $r_0 \geqslant r_1$ [comp. (53)] we obtain

$$r_0 = r_1 \text{ when } r^* < r_1 < 1,\ k < k_1(\beta) \text{ and } d \neq 0.$$

Let further $d = 0$. Then

$$P(z) = -\beta e^{-2ik\varphi} z^{2k} - \frac{1+\beta}{2} e^{-ik\varphi} z^k + 1$$

thus

$$\frac{\tilde{f}'(z)}{\tilde{f}(z)} - \frac{1}{z} = -\beta e^{-2ik\varphi} z^{2k-1} - \frac{1+\beta}{2} e^{-ik\varphi} z^{k-1}.$$

Hence

$$\log \frac{\tilde{f}(z)}{z} = -\frac{1}{2k} e^{-ik\varphi} z^k [\beta e^{-ik\varphi} z^k + (1+\beta)], \quad \log 1 = 0,$$

and consequently

$$\tilde{f}(z) = \frac{z}{\exp\left\{\frac{1}{2k} e^{-ik\varphi} z^k \cdot [\beta e^{-ik\varphi} z^k + (1+\beta)]\right\}} \text{ with } d = 0.$$

Similarly as before we find that in the case $d = 0$ we also have

$$r_0 = r_1, \text{ when } r^* < r < 1,\ k < k_1(\beta) \text{ and } d = 0.$$

In lemmas 3 and 4 inequalities are given which being satisfied imply $r_0 = r_2$ or $r_0 = r_1$ respectively. They do not specify explicitely the conditions for β and k under which the radius of convexity is determined by one or the other formula. Such conditions will be found now.

Lemma 5. *Let*

$$D_1 = \{(\beta, k)\colon 0 < \beta \leqslant 1,\ k \geqslant k_1(\beta)\},$$

$$D_2 = \{(\beta, k)\colon 0 < \beta \leqslant 1,\ k_2(\beta) < k < k_1(\beta)\},$$

$$D_3 = \{(\beta, k)\colon 0 < \beta \leqslant 1,\ k \leqslant k_2(\beta)\},$$

with $k_1(\beta)$ and $k_2(\beta)$ defined by the formulas (45) and (50). Then

$$r_0 = \begin{cases} r_2 \text{ when } (\beta, k) \in D_1 \text{ or } (\beta, k) \in D_2 \text{ and } r_2 \leqslant r^* \\ r_1 \text{ when } (\beta, k) \in D_2 \text{ and } r_2 > r^* \text{ or } (\beta, k) \in D_3. \end{cases}$$

Proof. Retaining the denotations accepted earlier, by (48) and (43) we have

$$(72) \quad w(r) = \min_{|z| = r < 1,\ f(z) \in \tilde{S}^*_k(a)} \operatorname{re}\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

$$= \begin{cases} \dfrac{F(r^k)}{1-\beta r^k} \text{ for } 0 < r \leqslant r^* \\[2ex] \dfrac{kU(r^{2k})}{\beta(1-r^{2k})^2[2(1+ka)s_1 + k(2a-1)]} \text{ for } r^* < r < 1. \end{cases}$$

By (49) we find that if $(\beta, k) \in D_1 \cup D_2$, then the function $F(r^k)$ is positive for $0 < r < r_2$, negative when $r_2 < r < 1$ and equal zero at the point $r = r_2$. Similarly, it follows from (44) that the function $U(r^{2k})$ is positive in the interval $(0, r_1)$, negative in the interval $(r_1, 1)$ and equal zero for $r = r_1$. Hence by (72) and by the definition of the radius of convexity we obtain the assertion of the lemma.

Lemma 6. *Let* $(\beta, k) \in D_2$ *and let*

$$S(\beta, k) = \beta^2(k+2)^3 - 2\beta(\beta^2 - \beta + 1)(k+2)^2 - (1-\beta^2)^2(k+2) - 2(1-\beta)^4.$$

The condition $r_2 \leqslant r^*$ *is satisfied if and only if* $S(\beta, k) \geqslant 0$.

Proof. Let $r_2 \leqslant r^*$. Then by (40) and (42)

$$h(y) = \beta^2 y^4 - 2\beta y^3 + [(1-2\beta)k + (1-\beta^2)]y^2 + 2(k+\beta)y - (k+1) \leqslant 0, \quad y = r_2^k \tag{73}$$

holds. We have

$$F(y) = \beta^2 y^2 - (k+2)\beta y + 1 = 0. \tag{74}$$

Thus

$$(1-y^2)F(y) = -\beta^2 y^4 + \beta(k+2)y^3 - (1-\beta^2)y^2 - \beta(k+2)y + 1 = 0. \tag{75}$$

Adding side-wise (73) and (75), then dividing by k and finally adding to both sides (74) we obtain

$$\beta y^3 + (1-\beta)^2 y^2 + (2 - k\beta - 3\beta)y \leqslant 0.$$

Ultimately we multiply both sides of this inequality by β/y and then subtract $F(y)$. In this way we obtain the inequality

$$\beta[(k+2) + (1-\beta)^2]y - [(1-\beta)^2 + \beta^2(k+2)] \leqslant 0.$$

Thus if $r_2 \leqslant r^*$, then

$$r_2^k \leqslant \frac{\beta^2(k+2) + (1-\beta)^2}{\beta[(k+2) + (1-\beta)^2]}.$$

Hence we get the inequality $S(\beta, k) \geqslant 0$. It follows from the above argument that if the last inequality is satisfied, then $r_2 \leqslant r^*$.

Corollary. $r_2 > r^*$ *if and only if* $S(\beta, k) < 0$.

Lemma 7. *The equation* $S(\beta, k) = 0$ *with unknown* k *has one solution* $k(\beta)$ *for every* β, $0 < \beta \leqslant 1$; *this solution satisfies the condition* $k_2(\beta) < k(\beta) < k_1(\beta)$, *with* $k_2(\beta)$ *and* $k_1(\beta)$ *defined by the formulas* (50) *and* (45).

Proof. Since

$$S(\beta, k_1(\beta)) > 0 \text{ and } S(\beta, k_2(\beta)) < 0 \text{ for } 0 < \beta \leqslant 1 \tag{76}$$

the equation $S(\beta, k) = 0$ has at least one solution in the interval $(k_2(\beta), k_1(\beta))$. Then we have

$$S'_k(\beta, k) = 3\beta^2(k+2)^2 - 4\beta(\beta^2-\beta+1)(k+2) - (1-\beta^2)^2 \tag{77}$$

and

$$S''_{kk}(\beta, k) = 6\beta^2 k - 4\beta(\beta^2 - 4\beta + 1).$$

By $S''_{kk}(\beta, k) > 0$ for $k_2(\beta) \leqslant k \leqslant k_1(\beta)$ and $0 < \beta \leqslant 1$ the derivative (77) is an increasing function of the variable k for every $\beta \in (0, 1]$. Moreover we have $S'_k(0, k_2(0)) < 0$ and $S'_k(1, k_2(1)) > 0$, thus there exists a number β^*, $0 < \beta^* < 1$ such that for every $\beta \in (0, \beta^*)$ $S'_k(\beta, k_2(\beta)) < 0$ holds, while $S'_k(\beta, k_2(\beta)) > 0$ for $\beta \in (\beta^*, 1]$. In the first case since the derivative (77) increases there exists $k^*(\beta)$ such that for $k_2(\beta) < k < k^*(\beta)$ the function $S(\beta, k)$ of the variable k decreases, while it increases in the interval $(k^*(\beta), k_1(\beta))$, because of (76) the lemma has been proved in this case. In the other case i.e. if $\beta^* < \beta \leqslant 1$ we have $S'_k(\beta, k_2(\beta)) > 0$ and since $S'_k(\beta, k)$ increases, $S(\beta, k)$ is an increasing function of the variable k defined in the interval $(k_2(\beta), k_1(\beta))$. Consequently because of (76) the lemma has been proved in the second case. The lemmas (6) and (7) imply:

Corollary. *If $k \geqslant k(\beta)$, then $r_2 \leqslant r^*$, while if $k < k(\beta)$ then $r_2 > r^*$.*

Lemmas 4 – 7 immediately imply the following

Theorem. *Let*

$$k_1(\beta) = \frac{4(1-\beta)}{\beta}, \quad k_2(\beta) = \frac{(1-\beta)^2}{\beta} \text{ for } 0 < \beta \leqslant 1,$$

$$S(\beta, k) = \beta^2(k+2)^3 - 2\beta(\beta^2-\beta+1)(k+2)^2 - (1-\beta^2)^2(k+2) - 2(1-\beta)^4$$

and let $k(\beta)$ be the only solution of the equation $S(k, \beta) = 0$ with the unknown k in the interval $(k_2(\beta), k_1(\beta))$. Accept

$$E_1 = \{(\beta, k)\colon\ 0 < \beta \leqslant 1,\ k < k(\beta)\}$$

$$E_2 = \{(\beta, k)\colon\ 0 < \beta \leqslant 1,\ k \geqslant k(\beta)\}.$$

Then the radius of convexity of the family $\tilde{S}^*_k(a)$

$$\mathrm{r.c}\,\tilde{S}^*_k(a) = \begin{cases} r_2 & \text{if } (\beta, k) \in E_2 \\ r_1 & \text{if } (\beta, k) \in E_1, \end{cases}$$

with

$$r_2 = \sqrt[k]{\frac{k+2-\sqrt{k(k+4)}}{2\beta}}$$

$$r_1 = \sqrt[2k]{\frac{2(1-\beta)\sqrt{2\beta(k+2)+(1-\beta)^2} - k\beta - 2(1-\beta)^2}{\beta[k+4(1-\beta)]}}$$

and

$$\beta = 1-a, \ a \in \langle 0, 1).$$

With r.c$\{\tilde{f}(z)\} = r_2$ *and* r.c$\{\hat{f}(z)\} = r_1$ *where*

$$\hat{f}(z) = \frac{z}{\exp\left(\frac{\beta}{k} e^{-ik\varphi} z^k\right)}$$

and

$$\tilde{f}(z) = \begin{cases} z\exp\left\{\frac{\beta}{kd}\left[\frac{4-d^2}{d}\log\left(1-\frac{d}{2}e^{-ik\varphi}z^k\right)+2e^{-ik\varphi}z^k\right]\right\}, \ \log 1 = 0 & \textit{with } d \neq 0 \\ \dfrac{z}{\exp\left\{\frac{1}{2k}e^{-ik\varphi}z^k[\beta e^{-ik\varphi}z^k+(1+\beta)]\right\}} & \textit{with } d = 0 \end{cases}$$

and

$$d = 2\frac{\beta r_1^{2k}+s_1-1}{[s_1-(1-\beta)]r_1^k}, \ s_1 = \sqrt{k(1-\beta)\frac{1+\beta r_1^{2k}}{(k+\beta)-\beta r_1^{2k}}}.$$

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STRESZCZENIE

Niech a, $0 \leqslant a < 1$, będzie dowolną ustaloną liczbą i niech k będzie dowolną ustaloną liczbą naturalną. Oznaczmy przez $\overline{S_k^*}(a)$ rodzinę wszystkich funkcji postaci

$$f(z) = z + \sum_{j=1}^{\infty} a_{jk+1}^{(k)} z^{jk+1}$$

holomorficznych, jednolistnych i gwiaździstych w kole $K = \{z\colon |z| < 1\}$ spełniających warunek

$$\left| \frac{\frac{zf'(z)}{f(z)} - a}{1-a} - 1 \right| < 1 \text{ dla każdego } z \in K.$$

Oznaczmy następnie przez $\tilde{\mathscr{P}}_k(\alpha)$ rodzinę wszystkich funkcji postaci

(1) $$P(z) = 1 + \sum_{j=1}^{\infty} b_{jk}^{(k)} z^{jk}$$

holomorficznych w kole K, spełniających warunek

$$\left| \frac{P(z) - \alpha}{1 - \alpha} - 1 \right| < 1 \text{ dla każdego } z \in K$$

oraz przez $\mathscr{P}_k(\alpha)$ rodzinę wszystkich funkcji $p(z)$ postaci (1) takich że

$$\operatorname{re} p(z) > \alpha \text{ dla każdego } z \in K.$$

Z powyższego wynika, że $\mathscr{P}_1(0) = \mathscr{P}$, gdzie $\mathscr{P}$ jest rodziną funkcji Carathéodory'ego oraz że $\tilde{\mathscr{P}}_k(\alpha) \subset \mathscr{P}_k(\alpha)$. Korzystając z własności rodziny $\mathscr{P}_k(\alpha)$ oraz ze związków, jakie zachodzą między odpowiednimi funkcjami rodzin $\widetilde{S_k^*}(\alpha)$, $\tilde{\mathscr{P}}_k(\alpha)$ i $\mathscr{P}_k(\alpha)$ wyznaczam dokładną wartość promienia wypukłości rodziny funkcji $\widetilde{S_k^*}(\alpha)$.

РЕЗЮМЕ

Пусть $\alpha, 0 \leqslant \alpha < 1$ будет произвольным фиксированным числом, а k — произвольным фиксированным натуральным числом.

Пусть $\widetilde{S_k^*}(\alpha)$ обозначает семейство всех функций вида

$$f(z) = z + \sum_{j=1}^{\infty} a_{jk+1}^{(k)} z^{jk+1}$$

голоморфных, однолистных и звездных в круге $k = \{z : |z| < 1\}$ удовлетворяющих условию

$$\left| \frac{\frac{zf'(z)}{f(z)} - \alpha}{1 - \alpha} - 1 \right| < 1 . \quad \bigwedge_{z \in K}$$

Пусть $\tilde{\mathscr{P}}_k(\alpha)$ обозначает семейство всех функций вида

(1) $$P(z) = 1 + \sum_{j=1}^{\infty} b_{jk}^{(k)} z^{jk}$$

голоморфных в круге K, удовлетворяющих условию

$$\left| \frac{P(z) - \alpha}{1 - \alpha} - 1 \right| < 1. \quad \bigwedge_{z \in K}$$

а $\mathscr{P}_k(a)$ — семейство всех функций $p(z)$ вида (1), таких, что

$$\operatorname{re} p(z) > a. \bigwedge_{z \in K}$$

Из вышесказанного следует, что $\mathscr{P}_1(0) = \mathscr{P}$ где $\mathscr{P}$ — семейство функций Каратеодори и $\tilde{\mathscr{P}}_k(a) \subset \mathscr{P}_k(a)$. Используя свойства семейства $\mathscr{P}_k(a)$ а также свойства, которые возникают между соответствующими функциями семейств $\widetilde{S_k^*}(a)$, $\tilde{\mathscr{P}}_k(a)$ и $\mathscr{P}_k(a)$ определяется точная величина радиуса выпуклости семейства функций $\widetilde{S_k^*}(a)$.