ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XXVI, 5 SECTIO A 1972

Zespół Matematyki, Wyższa Szkoła Inżynierska, Lublin
Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

ANNA SZYNAL, JAN SZYNAL, JADWIGA ZYGMUNT

On the Coefficients of Functions whose Real Part is Bounded

O współczynnikach funkcji, których część rzeczywista jest ograniczona

О коэффициентах функций, вещественная часть которых ограничена

1. In this note we are going to give sharp estimates for the coefficients of regular functions of some classes. As some special cases one can easily obtain the estimates given in [2] — [4], [6], [9].

Let $P(z)$ be a function regular in $K_1 = \{z: |z| < 1\}$ and such that $P(0) = 1$, i.e.

(1) $$P(z) = 1 + p_1 z + p_2 z^2 + \ldots, \quad z \in K_1.$$

We say that $P(z) \in \mathcal{P}[a, t, m, M]$ if and only if there exists a function $\omega(z)$, $|\omega(z)| \leqslant |z|$, regular in K_1 such that

(2) $$P(z) = \frac{A\omega(z) + M}{B\omega(z) + M},$$

the parameters a, t, m, M should satisfy the following conditions:

(3) $$\begin{gathered} A = (M^2 - m^2) + m(1-2a) + a(1-a) - t^2 + it, \\ B = 1 - m - a + it, \; D = D[a, t, m, M] = (A - B) > 0, \\ -\infty < a < 1, \; -\infty < t < +\infty, \; m > 1/2, \; M > 0. \end{gathered}$$

The condition (2) denotes that $P(z)$ is subordinate to $P_0(z)$, where $P_0(z) = (Az + M)/(Bz + M)$ is a function mapping the disk K_1 onto the disk $K(m + a + it, M)$ so that $0 \leftrightarrow 1$. The condition $D > 0$ means that $1 \in K(m + a + it, M)$.

The following particular cases were investigated earlier:

1. If $t = 0$, $m = \infty$, $M = \infty$, then $Re P(z) > a$, where $-\infty < a < 1$, [9].
2. If $t = 0$, $a = 0$ and $m = M$, where $M \geqslant 1$, then $|P(z) - M| < M$, for $z \in K_1$, [4].

3. If $a = 0$, $t = 0$ and $(m, M) \in D$, where $D = D_1 \cup D_2$ with $D_1 = \{(m, M): 1/2 < m < 1,\ 1-m < M \leqslant m\}$ $D_2 = \{m, M: 1 \leqslant m,\ m-1 < M \leqslant m\}$, then we obtain the class introduced in [2].
4. If $0 \leqslant a < 1$ and $m = M$ then one obtains the class introduced in [6].

2. Using the method of Clunie [1], we obtain

Theorem 1. *If* $P(z) \in P[a, t, m, M]$, $P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$, *then*

$$|p_n| \leqslant \frac{D}{M}, \quad n = 1, 2, \ldots. \tag{4}$$

The sign of equality holds for the function

$$P(z) = (A\varepsilon z^n + M)/(B\varepsilon z^n + M), \quad |\varepsilon| = 1.$$

Proof. From the representation formula for $P(z)$ we obtain

$$\Big\{(A-B) - B\sum_{k=1}^{\infty} p_k z^k\Big\}\,\omega(z) = M\sum_{k=1}^{\infty} p_k z^k$$

hence

$$\Big\{(A-B) - B\sum_{k=1}^{n-1} p_k z^k\Big\}\,\omega(z) = M\sum_{k=1}^{n} p_k z^k + \sum_{k=n+1}^{\infty} q_k z^k.$$

Putting $z = re^{i\theta}$, $0 > r < 1$, $0 \leqslant \theta \leqslant 2\pi$ after integration we obtain

$$M^2\sum_{k=1}^{n}|p_k|^2 r^{2k} + \sum_{k=n+1}^{\infty}|q_k|^2 r^{2k} \leqslant |A-B|^2 + |B|^2\sum_{k=1}^{n-1}|p_k|^2 r^{2k}.$$

If $r \to 1$, we get

$$M^2\sum_{k=1}^{n}|p_k|^2 + \sum_{k=n+1}^{\infty}|q_k|^2 \leqslant |A-B|^2 + |B|^2\sum_{k=1}^{n-1}|p_k|^2.$$

Hence

$$M^2|p_n|^2 \leqslant |A-B|^2 + (|B|^2 - M^2)\sum_{k=1}^{n-1}|p_k|^2,$$

which gives (4).

Remark. If $P(z) \in \mathscr{P}[a, t, m, M]$, then

$$\sum_{k=1}^{\infty}|p_k|^2 \leqslant \frac{D^2}{M^2 - |B|^2}. \tag{5}$$

Proof: If $[A - BP(z)]\,\omega(z) = M[P(z) - 1]$, then the condition $|\omega(z)| < 1$ gives

$$\Big|M\sum_{k=1}^{\infty} p_k z^k\Big| < \Big|(A-B) - B\sum_{k=1}^{\infty} p_k z^k\Big|.$$

Thus

$$M^2 \int_0^{2\pi} \Big| \sum_{k=1}^{\infty} p_k r^k e^{ik\varphi} \Big|^2 d\varphi \leqslant \int_0^{2\pi} \Big| (A-B) - B \sum_{k=1}^{\infty} p_k r^k e^{ik\varphi} \Big|^2 d\varphi .$$

Hence, after integration we obtain by letting $r \to 1$:

$$M^2 \sum_{k=1}^{\infty} |p_k|^2 \leqslant D^2 + |B|^2 \sum_{k=1}^{\infty} |p_k|^2,$$

which gives (5).

Corollary. *If* $P(z) \in \mathscr{P}[0, 0, m, M]$, *then*

$$|p_n| \leqslant \frac{M^2-(m-1)^2}{M}, \quad n = 1, 2, \ldots, \ [2].$$

Let $f(z) = z + \sum_{k=1}^{\infty} a_k z^k$ be a regular function in the unit disk K_1.

We say that $f(z) \in \mathscr{R}[a, t, m, M]$ if $f'(z) \in P[a, t, m, M]$, i.e.

$$f'(z) = \frac{A\omega(z)+M}{B\omega(z)+M},$$

where $\omega(z)$ is a regular function in K_1 such that $|\omega(z)| \leqslant |z|$ and the parameters a, t, m, M satisfy the conditions (3).

Theorem 1 implies

Theorem 2. *If* $f(z) \in \mathscr{R}[a, t, m, M]$, *then the sharp estimates*

(6) $$|a_n| \leqslant \frac{D}{nM}, \quad n = 2, 3, \ldots,$$

and

(7) $$\sum_{k=2}^{\infty} k^2 |a_n|^2 \leqslant \frac{D^2}{M^2-|B|^2}$$

hold, and the sign of equality in (6) *takes place for the function*

$$f(z) = \int_0^z \frac{A\varepsilon\zeta^n + M}{B\varepsilon\zeta^n + M}\, d\zeta, \quad |\varepsilon| = 1.$$

Corollary. *If* $P(z) \in \mathscr{R}[0, 0, m, M]$, *then*

$$|a_n| \leqslant \frac{M^2-(m-1)^2}{nM}, \quad n = 2, 3, \ldots \ [2].$$

3. Denote by $\mathscr{S}^*[a, t, m, M, \beta]$ the family of regular functions

(8) $$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in K_1$$

satisfying the equality

$$(9)\qquad \frac{e^{i\beta}\dfrac{zf'(z)}{f(z)}-i\sin\beta}{\cos\beta}=\frac{A\omega(z)+M}{B\omega(z)+M},$$

where a, t, m, M are as in (3), and $\beta\in(-\pi/2,\pi/2)$.

We have

Theorem 3. *Let $f(z)\in\mathcal{S}^*[a, t, m, M, \beta]$. If the parameters a, t, m, M satisfy the conditions* (3), $\beta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *and* $E = E[a, t, m, M, \beta] = M\cos\beta\times\times(\tau\sin\beta-v\cos\beta)+\cos\beta\sqrt{M^2(\tau\sin\beta-v\cos\beta)^2+D-1}\leqslant 0$, *where* $a = A/M = u+i\tau$, $b = B/M = v+i\tau$, *then*

$$(10)\qquad |a_n|\leqslant\frac{D\cos\beta}{M(n-1)},\ n = 2, 3, \ldots$$

If the parameters a, t, m, M obey the conditions (3) *and* $M\cos\beta(\tau\sin-v\cos\beta)+ +\cos\beta\sqrt{M^2(\tau\sin\beta-v\cos\beta)^2+D-1}>0$, *then*

$$(11)\qquad |a_n|\leqslant\frac{1}{(n-1)!}\prod_{k=1}^{n-1}|\mu_k|,\ n = 2,\ldots, N$$

and

$$(12)\qquad |a_n|\leqslant\frac{1}{(n-1)(N-2)!}\prod_{k=1}^{N-1}|\mu_k|\ \ n = N+1,\ldots$$

where $N = [2+M\cos\beta(\tau\sin\beta-v\cos\beta)+\cos\beta\sqrt{M^2(\tau\sin\beta-v\cos\beta)^2+D}]$ $\mu_k = e^{-i\beta}(a\cos\beta+ib\sin\beta)-kb = \frac{1}{M}[De^{-i\beta}\cos\beta-(k-1)B]$. *The estimates* (10) *and* (11) *are sharp and equality holds for the functions*

$$(13)\qquad f(z) = z\exp\left(\varepsilon\frac{De^{i\beta}\cos\beta}{M(n-1)}z^{n-1}\right),\ |\varepsilon| = 1\ \textit{when}\ E\leqslant 0$$

and $B = 0$

$$(14)\qquad f(z) = z\left(1+\varepsilon\frac{B}{M}z^{n-1}\right)^{\frac{De^{-i\beta}\cdot\cos\beta}{B(n-1)}},\ |\varepsilon| = 1\ \textit{when}$$

$E\leqslant 0$ *and* $B\neq 0$

$$(15)\qquad f(z) = z\left(1+\varepsilon\frac{B}{M}z\right)^{\frac{De^{-i\beta}\cdot\cos\beta}{B}},\ |\varepsilon| = 1\ \textit{when}\ E>0,$$

respectively. Moreover,

$$\sum_{k=1}^{\infty}\{(k-1)^2-|\mu_k|^2\}|a_k|^2 \leqslant \frac{D^2\cos^2\beta}{M^2}. \tag{16}$$

Proof. From (9) we get

$$zf'(z)-f(z) = \omega(z)[e^{-i\beta}(a\cos\beta+ib\sin\beta)f(z)-bzf'(z)], \tag{17}$$

where $a = A/M$, $b = B/M$. Hence using (8) and taking into (17) we have

$$\sum_{k=1}^{\infty}(k-1)a_kz^k = \omega(z)\Big[\mu_1 z+\sum_{k=2}^{\infty}\mu_k a_k z^k\Big], \tag{18}$$

where $\mu_k = \dfrac{1}{M}[De^{-i\beta}\cdot\cos\beta-(k-1)B]$, $k = 1, 2, \ldots$. We rewrite (18) as follows

$$\sum_{k=1}^{n}(k-1)a_kz^k+\sum_{k=n+1}^{\infty}d_kz^k = \omega(z)\Big[\mu_1 z+\sum_{k=2}^{n-1}\mu_k a_k z^k\Big],$$

where the sum $\sum\limits_{k=n+1}^{\infty} d_kz^k$ is convergent in K_1. Put $z = re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, $0<r<1$, then since $|\omega(z)|<1$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{k=1}^{n}(k-1)a_k(re^{i\theta})^k+\sum_{k=n+1}^{\infty}d_k(re^{i\theta})^k\Big|^2 d\varphi$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi}\Big|\mu_1 re^{i\theta}+\sum_{k=2}^{n-1}\mu_k a_k(re^{i\theta})^k\Big|^2 d\varphi.$$

Upon integration, we get

$$\sum_{k=1}^{n}(k-1)^2|a_k|^2r^{2k}+\sum_{k=n+1}^{\infty}|d_k|^2r^{2k} \leqslant |\mu_1|^2r^2+\sum_{k=2}^{n-1}|\mu_k|^2|a_k|^2r^{2k} \tag{19}$$

In particular (19) with $r\to 1$ implies

$$\sum_{k=1}^{n}(k-1)^2|a_k|^2 \leqslant |\mu_1|^2+\sum_{k=2}^{n-1}|\mu_k|^2|a_k|^2.$$

This inequality is equivalent to

$$(n-1)^2|a_n|^2 \leqslant |\mu_1|^2+\sum_{k=2}^{n-1}[|\mu_k|^2-(k-1)^2]|a_k|^2. \tag{20}$$

If for each $k = 2, \ldots, n-1$, $|\mu_k|^2-(k-1)^2 \leqslant 0$, i.e. if $E \leqslant 0$, then $(n-1)^2|a_n|^2 \leqslant |\mu_1|^2$ holds. By the last inequality we have (10). Therefore the extremal functions have the form as in (13) or (14). Since $|\mu_{n-1}|^2 - (n-1)^2 > 0$, if and only if $n \leqslant N$, where $N = [2 + M\cos\beta(\tau\sin\beta - v\cos\beta) + \cos\beta\sqrt{M^2(\tau\sin\beta - v\cos\beta)^2 + D}]$, the term in the square brac-k ets is $\max(N_1, N_2)$, where N_1 and N_2 are the roots of equation $|\mu_{n-1}|^2 - (n-1)^2 = 0$. The induction argument gives (11). Of course, for $n = 2$ we have $a_2 = \mu_1 c_1$, where $\omega(z) = c_1 z + \ldots$ and $|c_1| \leqslant 1$, therefore $|a_2| = |\mu_1||c_1| \leqslant |\mu_1|$. Let us suppose that (11) holds. We must prove that

$$|a_{n+1}| \leqslant \frac{1}{n!}\prod_{k=1}^{n-1} |\mu_k|.$$

From (20), we obtain:

$$n^2|a_{n+1}|^2 \leqslant |\mu_1|^2 + (|\mu_2|^2 - 1^2)|a_2|^2 + \ldots + (|\mu_n|^2 - (n-1)^2)|a_n|^2$$

$$\leqslant |\mu_1|^2 + \frac{|\mu_1|^2}{(1!)^2}(|\mu_2|^2 - 1) + \frac{|\mu_1|^2|\mu_2|^2}{(2!)^2}(|\mu_3|^2 - 2^2) + \ldots +$$

$$\frac{|\mu_1|^2|\mu_2|^2\ldots|\mu_{n-1}|^2}{[(n-1)!]^2}[|\mu_n|^2 - (n-1)^2] = \frac{|\mu_1|^2|\mu_2|^2\ldots|\mu_n|^2}{[(n-1)!]^2}.$$

Now the extremal function (15) and the estimate (12) may be easily obtained from the above consideration. The result (16) follows immediately by integration from (18).

Corollary 1. *If $a = 0$, $t = 0$ we find the results published in the paper* [3] *for the class* $\mathcal{S}^*[0, 0, m, M, \beta]$.

Corollary 2. *If $t = 0$, $\beta = 0$, $M = m = \infty$ we also obtain from* (11) *the coefficient estimates for starlike functions of order a, $-\infty < a < 1$* [5], or $0 \leqslant a < 1$ [8].

$$|a_n| \leqslant \frac{1}{(n-1)}\prod_{k=1}^{n-1} |[(k+1) - 2a]|.$$

Finally, we give the sharp estimates of the functional $|a_3 - va_2^2|$ for the function $f(z) \in \mathcal{S}^*[a, t, m, M, \beta]$.

We have

Theorem 4. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*[a, t, m, M, \beta]$ and if v is a complex number, then*

$$|a_3 - va_2^2| \leqslant \frac{|\mu_1|^2}{2}\max(1, |2\mu_1 v - \mu_2|). \tag{21}$$

For each v, there exists a function $f(z) \in \mathcal{S}^*[a, t, m, M, \beta]$ for which the equality holds in (21).

Proof. If $f(z) \in \mathcal{S}^*[a, t, m, M, \beta]$, then from (9) by equating the coefficients we obtain

$$c_1 = a_2/\mu_1, \tag{22}$$

$$c_2 = \frac{2a_3\mu_1 - a_2^2\mu_2}{\mu_1^2}, \tag{23}$$

where $\omega(z) = c_1 z + c_2 z^2 + \ldots$ is regular and $|\omega(z)| < 1$ for $z \in K_1$. Since $|c_1| \leqslant 1$, $|c_2| \leqslant 1 - |c_1|^2$, therefore for every complex number s we have the sharp estimates

$$|c_2 - sc_1^2| \leqslant \max(1, |s|) \ [7]. \tag{24}$$

Using (22), (23), (24) we obtain (21), where $\nu = \dfrac{\mu_2 + s}{2\mu_1}$. Again (21) is sharp which follows from the fact that (24) is sharp.

Corollary. *If $t = 0$, $M = m = \infty$, then $f(z)$ is β-spiral-like function of order a, $\beta \in (-\pi/2, \pi/2)$, $-\infty < a < 1$. Moreover, if ν is complex number, we have*

$$|a_3 - \nu a_2^2| \leqslant (1-a)\cos\beta \max\left(1, |2\cos\beta(1-a)(2\nu-1) - e^{i\beta}|\right). \tag{25}$$

For each ν, there is a function in $S^*[a, 0, \infty, \infty, \beta]$ for which equality holds. The estimate (25) is identical if for β-spirallike functions of order a, $\beta \in (-\pi/2, \pi/2)$, $0 \leqslant a < 1$, [7].

REFERENCES

[1] Clunie, J., *On Meromorphis Schlicht Functions*, J. London Math Soc., 34 (1959), 215-216.

[2] Jakubowski, Z. J., *On the Coefficients of Caratheodory Functions*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 19 (1971), 805-809.

[3] —, *On the Coefficients of Starlike Functions of Some Classes*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 19 (1971), 811-815.

[4] Janowski, W., *Extremal Problems for a Family of Functions with Positive Real Part and for Some Related Families*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 17 (1969), 633-637.

[5] Klein, M., *Functions Starlike of Order a*, T.A.M.S., 131 (1) (1968), 99-106.

[6] Libera, R. J., Livingston, A. E., *Bounded Functions with Positive Real Part*, (to appear).

[7] Keogh, F. R., Merkes, E. P., *A Coefficients Inequality for Certain Classes of Analytic Functions*, Proc. Amer. Math. Soc., 20 (1969), 1, 8-12.

[8] Robertson, M. S., *On the Theory of Univalent Function*, Ann. of Math. 37 (1936), 374-408.

[9] Tonti, N. E., Trahan, D. H., *Analytic Functions whose Real Parts are Bounded Below*, Math. Z., 115 (1970), 252-258.

STRESZCZENIE

Niech $P(z)$ będzie funkcją regularną w $K_1 = \{z\colon |z| < 1\}$ postaci $P(z) = 1 + p_1 z + \ldots, z \in K_1$.

Mówimy, że $P(z) \in \mathscr{P}[a, t, m, M]$ wtedy i tylko wtedy, gdy istnieje funkcja $\omega(z)$, $|\omega(z)| \leqslant |z|$ regularna w K_1 i taka, że

$$P(z) = \frac{A\omega(z) + M}{B\omega(z) + M};$$

parametry a, t, m, M spełniają warunki

$A = (M^2 - m^2) + m(1-2a) + a(1-a) - t^2 + it$,

$B = 1 - m - a + it$,

$-\infty < a < 1, \ -\infty < t < +\infty, \ m > 1/2, M > 0$.

Stosując metodę Clunie [1] otrzymano dokładne oszacowania współczynników dla funkcji $P(z)$ i dla funkcji $f(z)$ postaci $f(z) = z + a_2 z^2 + \ldots$, $z \in K_1$, $f'(z) = P(z)$ i $zf'(z)/f(z) = P(z)$.

РЕЗЮМЕ

Пусть $P(z)$ — регулярная функция в единичном круге $K_1 = \{z\colon |z| < 1\}$ вида $P(z) = 1 + p_1 z + \ldots, z \in k_1$.

Говорим, что $P(z) \in \mathscr{P}[a, t, m, M]$ тогда и только тогда, когда существует функция $\omega(z)$, $|\omega(z)| \leqslant |z|$ регулярная в K_1 и такая, что

$$P(z) = \frac{A\omega(z) + M}{B\omega(z) + M};$$

где параметры a, t, m, M удовлетворяют условиям

$A = (M^2 - m^2) + m(1-2a) + a(1-a) - t^2 + it$,

$B = 1 - m - a + it$,

$-\infty < a < 1, \quad -\infty < t < +\infty, \quad m > 1/2, \quad M > 0$.

Введенный класс касается и некоторых ранее рассматриваемых классов, [9], [4], [6]. Применяя метод Клуни [1], получаем точные оценки коэффициентов для функций $P(z)$ и для функций $f(z)$ вида $f(z) = z + a_2 z^2 + \ldots, z \in K_1$, $f'(z) = P(z)$ и $[zf'(z)/f(z)] = P(z)$.