ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XXVI, 4 SECTIO A 1972

Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

ZDZISŁAW RYCHLIK

On Some Problems Concerning of the Inflated Binomial Distribution

O pewnych problemach dotyczących „rozdętego" rozkładu dwumianowego

О некоторых проблемах искаженного биномиального распределения

Introduction

The generalized union of Bayes and Bernoulli problems were investigated in the paper [1]. They unite with the search for joint, conditiona and marginal distributions of the following three random variables:

P — Probability which is a random variable taking values p from the interval $(0, 1)$.

A_n, B_N — random variables taking respectively values $a = 0, 1, \ldots, n$ and $\beta = 0, 1, \ldots, N$, standing for the number of favourable events taking place in runs of n and N experiments, of composite experiment carried out in the following way:

1. The realization of random variable P is being obtained from the interval $(0, 1)$.

2a. The a realization of random variable A_n is being obtained from runs of experiments carried out according to Bernoulli scheme with the constant conditional probability p of a favourable event D_k $(k = 1, 2, \ldots, n)$ equal to p which was defined at the point 1.

It means that, $P(D_k|P = p) = p$, $(k = 1, 2, \ldots, n)$, where D_k is the result of the k-th experiment.

2b. The β realization of random variable B_N is being obtained in a run of N further experiments executed according to Bernoulli scheme with the same (conditional) probability of the favourable event D_l $(l = n+1, n+2, \ldots, n+N)$.

It means that, $P(D_l|P = p) = p$ $(l = n+1, n+2, \ldots, n+N)$, where D_l is the result of l-th experiment.

The random variable A_n specifies the number of favourable events D_k $(k = 1, 2, \ldots, n)$ taking place during the whole complex experiment taking into consideration 1. and 2a.

The random variable B_N specyfies the number of favourable events D_l $(l = n+1, n+2, \ldots, n+N)$ in the same experiment taking into consideration the points 1. and 2b.

A total probability formula allows to compute an unconditional probability of favourable event. For

$$D_k \ (k = 1, 2, \ldots, n) \quad \text{and} \quad D_l \ (l = n+1, n+2, \ldots, n+N)$$

we get

$$P(D_k) = \int_0^1 P(D_k | P = p) f(p)\, dp = \int_0^1 p f(p)\, dp = EP, \ k = 1, 2, \ldots, n,$$

$$P(D_l) = \int_0^1 P(D_l | P = p) f(p)\, dp = EP$$

$$= P(D_k), \ l = n+1, n+2, \ldots, n+N,$$

where $f(p)$ is a density function of random variable P.

From the construction of random variables results that conditional distributions

(1) $$p(a|p) = P[A_n = a | P = p] = \binom{n}{a} p^a q^{n-a} (a = 0, 1, 2, \ldots, n),$$

(2) $$p(\beta|p) = P[B_N = \beta | P = p] = \binom{N}{\beta} p^\beta q^{N-\beta} (\beta = 0, 1, 2, \ldots, N),$$

are Bernoulli ones.

The knowledge of these distributions allows to determine the unconditional distributions of random variables A_n and B_N. The distributions of these random variables and the conditional distribution $p(\beta|a) = P[B_N = \beta | A_n = a]$, as well as suggestions relative to their applications to a quality control were given in [1].

However, random experiments exist in which an examined phenomenon is being well described by Binomial distribution except case when the number of favourable events is zero (or more general k_0). So called "Inflated" Binomial distribution [3] describes precisely such the phenomenons. The probability function of that distribution in the case inflated at the point k_0, where k_0 is a positive integer is given by the formula

$$(3)\qquad P(X=k)=\begin{cases}1-s+s\binom{n}{k_0}p^{k_0}q^{n-k_0}, & k=k_0,\\ s\binom{n}{k}p^kq^{n-k}, & k=0,1,2,\ldots,k_0-1,\ k_0+1,\ldots,n,\end{cases}$$

where $P(X=k)$ is probability that X takes value k, being a number of favourable events, and s is the proportion of population which follows simple Binomial, and $p+q=1$. In the case $k_0=0$ the formula (3) takes the form

$$(3')\qquad P(X=k)=\begin{cases}1-s+sq^n, & k=0,\\ s\binom{n}{k}p^kq^{n-k}, & k=1,2,\ldots,n.\end{cases}$$

In that what follows we assume s and p being values of the random variables λ and P respectively, where λ and P are independent.

The generalization of Bayes problems and ones connected with above inflated Binomial distributions we would like to consider in this paper. The results given here under the assumption $\lambda=\text{const.}=1$ reduce to the results, relative to the generalized Bayes and Bernoulli problems from [1].

Moreover in further parts of this paper, the random variable P has such a property that conditional distributions of the remaining ones are independent with regard to its optional, possible value. It means that $P(A_n, B_N|P)=P(A_n|P)\cdot P(B_N|P)$.

Now let us introduce the following notations

$$(4)\qquad a=E\lambda$$

$$(5)\qquad b=1-E\lambda,$$

$$(6)\qquad c(\alpha)=\begin{cases}0 & \text{if } \alpha\neq k_0,\\ b & \text{if } \alpha=k_0;\end{cases}$$

$$(7)\qquad d(\beta)=\begin{cases}0 & \text{if } \beta\neq k_0,\\ b & \text{if } \beta=k_0.\end{cases}$$

Remark. In particular cases we assume that λ is a random variable uniformly distributed. However, it should be noted that, in general, one does not known what values λ takes [4]. Sometimes, taking λ as a constant, we can estimate its value experimentally. Then, one could consider particular cases: P being uniformly or beta distributed and λ being a constant. But, desirable results in these or other cases can be obtained from the general formulas in the same way as given below.

1. Probability distribution (unconditional and conditional ones) of three random variables P, A_n, B_N

First let us observe that in case of inflated binomial distributions the formulas (1) and (2) take on the forms

$$(1.1)\qquad p(a|p) = \begin{cases} 1-E\lambda+E\lambda\binom{n}{k_0}p^{k_0}q^{n-k_0}, & a = k_0,\\ E\lambda\binom{n}{a}p^a q^{n-a}, & a = 0,1,2,\ldots,k_0-1,\ k_0+1,\ldots,n;\end{cases}$$

$$(1.2)\qquad p(\beta|p) = \begin{cases} 1-E\lambda+E\lambda\binom{N}{k_0}p^{k_0}q^{N-k_0}, & \beta = k_0,\\ E\lambda\binom{N}{\beta}p^\beta q^{N-\beta}, & \beta = 0,1,2,\ldots,k_0-1,\ k_0+1,\ldots,N.\end{cases}$$

Thus by means of the notations (4)-(7), the formulas (1.1) and (1.2) are as follows

$$(1.3)\qquad p(a|p) = c(a)+a\binom{n}{a}p^a q^{n-a},\ (a = 0,1,2,\ldots,n),$$

$$(1.4)\qquad p(\beta|p) = d(\beta)+a\binom{N}{\beta}p^\beta q^{N-\beta},\ (\beta = 0,1,2,\ldots,N).$$

Theorem 1.1. *Unconditional distributions of random variables A_n and B_N are given by the formulas*

$$(1.5)\qquad p(a) = c(a)+a\binom{n}{a}\int_0^1 p^a q^{n-a}f(p)\,dp,\ (a = 0,1,2,\ldots,n),$$

$$(1.6)\qquad p(\beta) = d(\beta)+a\binom{N}{\beta}\int_0^1 p^\beta q^{N-\beta}f(p)\,dp,\ (\beta = 0,1,2,\ldots,N).$$

The proof results from the total probability formula.

Theorem 1.2. *A joint distribution of random variables A_n and B_N is given by the formula*

$$(1.7)\qquad p(a,\beta) = c(a)d(\beta)+ad(\beta)\binom{N}{a}\int_0^1 p^a q^{n-a}f(p)\,dp+$$
$$+ac(a)\binom{N}{\beta}\int_0^1 p^\beta q^{N-\beta}f(p)\,dp+a^2\binom{n}{a}\binom{N}{\beta}\int_0^1 p^{a+\beta}q^{N+n-a-\beta}f(p)\,dp.$$

where here and in what follows $\beta = 0,1,2,\ldots,N$; $a = 0,1,2,\ldots,n$.

Proof. From the assumption, we have

$$p(a,\beta) = \int_0^1 p(a,\beta|p)f(p)\,dp = \int_0^1 p(a|p)p(\beta|p)f(p)\,dp.$$

Now, profiting by the formulas (1.3) and (1.4) we get (1.7).

Let us observe that the formula (1.7), explicite is given as follows

$$p(a,\beta) = \begin{cases} E^2\lambda\binom{N}{\beta}\binom{n}{a}\int_0^1 p^{a+\beta}q^{N+n-a-\beta}f(p)\,dp, \quad a\neq k_0,\ \beta\neq k_0; \\ E\lambda(1-E\lambda)\binom{N}{\beta}\int_0^1 p^{\beta}q^{N-\beta}f(p)\,dp + E^2\lambda\binom{n}{k_0}\binom{N}{\beta}\int_0^1 \times \\ \quad\times p^{\beta+k_0}q^{N+n-\beta-k_0}f(p)\,dp, \quad a=k_0,\ \beta\neq k_0; \\ E\lambda(1-E\lambda)\binom{n}{a}\int_0^1 p^{a}q^{n-a}f(p)\,dp + E^2\lambda\binom{n}{a}\binom{N}{k_0}\int_0^1 \times \\ \quad\times p^{a+k_0}q^{N+n-k_0-a}f(p)\,dp, \quad a\neq k_0,\ \beta=k_0; \\ (1-E\lambda)^2 + E\lambda(1-E\lambda)\int_0^1 p^{k_0}q^{n-k_0}\left[\binom{n}{k_0}+\binom{N}{k_0}q^{N-n}\right]f(p)\,dp + \\ \quad+E^2\lambda\binom{n}{k_0}\binom{N}{k_0}\int_0^1 p^{2k_0}q^{N+n-2k_0}f(p)\,dp, \quad a=k_0,\ \beta=k_0. \end{cases}$$

By means of Bayes formula we get the following formulas for conditional distributions $f(p|a)$ and $f(p|\beta)$.

$$f(p|a) = \frac{\left[c(a)+a\binom{n}{a}p^aq^{n-a}\right]f(p)}{c(a)+a\binom{n}{a}\int_0^1 p^aq^{n-a}f(p)\,dp}, \tag{1.8}$$

$$f(p|\beta) = \frac{\left[d(\beta)+a\binom{N}{\beta}p^\beta q^{N-\beta}\right]f(p)}{d(\beta)+a\binom{N}{\beta}\int_0^1 p^\beta q^{N-\beta}f(p)\,dp}. \tag{1.9}$$

On the basis of (1.3) and (1.4) we get joint distributions (A_n, P) and (B_N, P)

$$f(a,p) = \left[c(a)+a\binom{n}{a}p^aq^{n-a}\right]f(p), \tag{1.10}$$

$$f(\beta,p) = \left[d(\beta)+a\binom{N}{\beta}p^\beta q^{N-\beta}\right]f(p). \tag{1.11}$$

Theorem 1.3. *Conditional distribution $p(\beta|a)$ is given by the formula*

(1.12) $$p(\beta|a) = d(\beta) + \frac{ac(a)\binom{N}{\beta}\int_0^1 p^\beta q^{N-\beta} f(p)\,dp + a^2\binom{N}{\beta}\binom{n}{a}\int_0^1 p^{a+\beta} q^{N+n-a-\beta} f(p)\,dp}{c(a) + a\binom{n}{a}\int_0^1 p^a q^{n-a} f(p)\,dp}.$$

Proof. By the virtue of theorem 1 from [1]

$$p(\beta|a) = \int_0^1 p(\beta|p) f(p|a)\,dp$$

which in connection with (1.4) and (1.8) gives (1.12).

The practical applications often requires the knowledge of $P[B_N \leqslant \beta|A_n = a]$. In the considered case, this probability is given by the formula:

(1.13) $$P[B_N \leqslant \beta|A_n = a] = \varepsilon(\beta) + \frac{ac(a)\sum_{k=0}^{\beta}\binom{N}{k}\int_0^1 p^k q^{N-k} f(p)\,dp}{c(a) + a\binom{n}{a}\int_0^1 p^a q^{n-a} f(p)\,dp} + \frac{a^2\binom{n}{a}\sum_{k=0}^{\beta}\binom{N}{k}\int_0^1 p^{a+k} q^{N+n-a-k} f(p)\,dp}{c(a) + a\binom{n}{a}\int_0^1 p^a q^{n-a} f(p)\,dp},$$

where

$$\varepsilon(\beta) = \begin{cases} 0 \text{ if } \beta < k_0 \\ 1 - E\lambda \text{ if } \beta \geqslant k_0. \end{cases}$$

In a special cases, when the random variables P and λ are uniformly distributed, i.e.

(1.14) $$f(p) = \begin{cases} 1 \text{ for } 0 < p < 1, \\ 0 \text{ for } p \geqslant 1 \text{ and } p \leqslant 0, \end{cases}$$

and

(1.15) $$g(s) = \begin{cases} 1 \text{ for } 0 < s \leqslant 1, \\ 0 \text{ for } s > 1 \text{ and } s \leqslant 0, \end{cases}$$

(1.16) $$a = E\lambda = \frac{1}{2} = 1 - E\lambda = b,$$

and therefore the formulas (1.5) – (1.13) take on the corresponding forms

$$p(a) = c(a) + \frac{1}{2(n+1)}, \tag{1.17}$$

$$p(\beta) = d(\beta) + \frac{1}{2(N+1)}, \tag{1.18}$$

(1.19)

$$p(a, \beta) = c(a)d(\beta) + \frac{d(\beta)}{2(n+1)} + \frac{c(a)}{2(N+1)} + \frac{\binom{n}{a}\binom{N}{\beta}}{4(N+n+1)\binom{N+n}{a+\beta}},$$

$$f(p|a) = \frac{(n+1)\left[2c(a) + \binom{n}{a} p^{\beta} q^{n-a}\right]}{1+2c(a)(n+1)}, \tag{1.20}$$

$$f(p|\beta) = \frac{(N+1)\left[2d(\beta) + \binom{N}{\beta} p^{\beta} q^{N-\beta}\right]}{1+2d(\beta)(N+1)}, \tag{1.21}$$

$$f(a, p) = c(a) + \frac{1}{2}\binom{n}{a} p^{a} q^{n-a}, \tag{1.22}$$

$$f(\beta, p) = d(\beta) + \frac{1}{2}\binom{N}{\beta} p^{\beta} q^{N-\beta}, \tag{1.23}$$

$$p(\beta|a) = d(\beta) + \frac{c(a)(n+1)}{(N+1)[2c(a)(n+1)+1]} + \tag{1.24}$$

$$+ \frac{\binom{n}{a}\binom{N}{\beta}(n+1)}{2\binom{N+n}{a+\beta}(N+n+1)[1+2c(a)(n+1)]},$$

$$P[B_N \leqslant \beta | A_n = a] = \delta(\beta) + \frac{c(a)(n+1)(\beta+1)}{2(N+1)[1+2c(a)(n+1)]} + \tag{1.25}$$

$$+ \frac{\binom{n}{a}}{2(N+n+1)[1+2c(a)(n+1)]} \sum_{k=0}^{\beta} \frac{\binom{N}{k}}{\binom{N+n}{a+k}},$$

here and in what follows

$$\delta(\beta) = \begin{cases} \frac{1}{2} & \text{if } \beta \geq k_0, \\ 0 & \text{if } \beta < k_0. \end{cases}$$

When the random variable P has the beta distribution

(1.26) $$f(p) = \frac{p^r q^{s-r}}{B(r+1, s-r+1)}, 0 < p < 1,$$

where $-1 < r < s+1$ all parameters and the random variable λ is as in above, then the unconditional distributions of A_n and B_N are given by the formulas

(1.27) $$p(a) = c(a) + \binom{n}{a} \frac{B(a+r+1, n+s-r-a+1)}{2B(r+1, s-r+1)},$$

(1.28) $$p(\beta) = d(\beta) + \binom{N}{\beta} \frac{B(\beta+r+1, N+s-r-\beta+1)}{2B(r+1, s-r+1)}.$$

Their joint distribution is

(1.29) $$p(a, \beta) = c(a)d(\beta) + \frac{\binom{n}{a} d(\beta) B(a+r+1, n-a+s-r+1)}{2B(r+1, s-r+1)} + \\ + \frac{2c(a)\binom{N}{\beta} B(\beta+r+1, N-\beta+s-r+1)}{4B(r+1, s-r+1)} + \\ + \frac{\binom{n}{a}\binom{N}{\beta} B(a+\beta+r+1, N+n-a-\beta+s-r+1)}{4B(r+1, s-r+1)}.$$

The conditional densities of random variable P given a or β are

(1.30)

$$f(p|a) = \frac{2c(a)p^r q^{s-r} + \binom{n}{a} p^{a+r} q^{n-a+s-r}}{2c(a)B(r+1, s-r+1) + \binom{n}{a} B(a+r+1, n-a+s-r+1)},$$

(1.31)

$$f(p|\beta) = \frac{2d(\beta)p^r q^{s-r} + \binom{N}{\beta} p^{\beta+r} q^{N-\beta+s-r}}{2d(\beta)B(r+1, s-r+1) + \binom{N}{\beta} B(\beta+r+1, N-\beta+s-r+1)}.$$

The joint unconditional densities of random variables A_n and B_N are

$$(1.32)\qquad f(a,p) = \frac{2c(a)p^r q^{s-r} + \binom{n}{a} p^{a+r} q^{n-a+s-r}}{2B(r+1,s-r+1)},$$

$$(1.33)\qquad f(\beta,p) = \frac{2d(\beta)p^r q^{s-r} + \binom{N}{\beta} p^{\beta+r} q^{N-\beta+s-r}}{2B(r+1,s-r+1)}.$$

The conditional distribution of random variable B_N given vale A_n is

(1.34)

$$p(\beta|a) = d(\beta) + \frac{c(a)\binom{N}{\beta} B(r+\beta+1, N-\beta+s-r+1)}{2c(a)B(r+1,s-r+1) + \binom{n}{a} B(a+r+1, n-a+s-r+1)} + \frac{\frac{1}{2}\binom{n}{a}\binom{N}{\beta} B(a+\beta+r+1, N-\beta+n-a+s-r+1)}{2c(a)B(r+1,s-r+1) + \binom{n}{a} B(a+r+1, n-a+s-r+1)}.$$

From (1.34), we have

(1.35) $P[B_N \leqslant \beta | A_n = a]$

$$= \delta(\beta) + \frac{c(a)\sum_{k=0}^{\beta}\binom{N}{k} B(k+r+1, N-k+s-r+1)}{2c(a)B(r+1,s-r+1) + \binom{n}{a} B(a+r+1, n-a+s-r+1)} + \frac{\frac{1}{2}\binom{n}{a}\sum_{k=0}^{\beta}\binom{N}{k} B(a+k+r+1, N-k+n-a+s-r+1)}{2c(a)B(r+1,s-r+1) + \binom{n}{a} B(a+r+1, n-a+s-r+1)}.$$

2. Extreme

First let us consider if the random variables P and λ are uniformly distributed.

Let us introduce the following notations

$$(2.1)\qquad h(\beta, a) = \frac{\binom{n}{a}\binom{N}{\beta}(n+1)}{2(N+n+1)\binom{N+n}{a+\beta}},$$

$$(2.2)\qquad \phi(\beta) = \frac{h(\beta+1, a)}{h(\beta, a)} = \frac{(a+\beta+1)(N-\beta)}{(\beta+1)(N+n-a-\beta)}.$$

From (1.24) and (2.1) we have

$$(2.3)\qquad p(\beta|a) = d(\beta) + \frac{h(\beta, a)}{2c(a)(n+1)+1} + \frac{c(a)(n+1)}{(N+1)[1+2c(a)(n+1)]}.$$

Let us now suppose that n, N, a are constants.

If $\phi(\beta) \geqslant 1$, then $h(\beta, a)$ is an increasing function of β. Because $\phi(\beta) \geqslant 1$, for $\beta \leqslant \frac{a(N+1)-n}{n}$, and for every β $h(\beta, a) \leqslant \frac{1}{2} + h(k_0, a)$, therefore

$$(2.4)\qquad \max_{\beta} p(\beta|a) = p(k_0|a)$$

$$= \frac{1}{2} + \frac{\binom{n}{a}\binom{N}{k_0}(n+1)(N+1) + 2c(a)(n+1)(N+n+1) + \binom{N+n}{a+k_0}}{2\binom{n+N}{a+k_0}(N+n+1)[1+2c(a)(n+1)]}.$$

(a_1) If $a(N+1) < n$

$$(2.5)\qquad \min_{\beta} p(\beta|a) = p(N|a)$$

$$= \frac{(n+1)\left[\binom{n}{a}(N+1) + 2c(a)(N+n+1)\binom{N+n}{a+N}\right]}{2\binom{N+n}{a+N}(N+n+1)[1+2c(a)(n+1)]}.$$

In case when $a(N+1) > n$, $h(\beta, a)$ reaches maximum at the greatest β, therefore e.g.

(b_1) When $a = n$

$$(2.6)\qquad \max_{\beta \neq k_0} p(\beta|a) = p(N|n) = \frac{n+1}{2(N+n+1)}.$$

(c_1) If $a \neq k_0$ and $\frac{a(N+1)-n}{n} = k$, and k is positive integer

$$(2.7)\qquad \max_{\beta \neq k_0} p(\beta|a) = p(k|a)$$

$$= \frac{(n+1)\left[\binom{n}{a}\binom{N}{k}(N+1) + 2c(a)(N+n+1)\binom{N+n}{a+k}\right]}{2\binom{N+n}{a+k}(N+n+1)[1+2c(a)(n+1)]}.$$

In case when k is not positive integer $\max_{\beta \neq k_0} p(\beta|a)$ reaches the greatest value at β equal to $[(N+1)/n]$ (here $[(N+1)/n]$ denotes the integral part of the real number $(N+1)/n$).

In case when the random variable λ is uniformly distributed and the random variable P has beta distribution, we introduce the following notations

$$(2.8)\qquad l(\beta, a, N, n) = \binom{N}{\beta}\binom{n}{a} B(a+\beta+r+1, N+n-a-\beta+s-r+1),$$

and by definition $\binom{0}{0} = 1$,

$$(2.9)\qquad \eta(\beta) = \frac{l(\beta+1, a, N, n)}{l(\beta, a, N, n)} = \frac{(N+\beta)(a+\beta+r+1)}{(\beta+1)(n-a+N-\beta+s-r)}.$$

By means of the formulas (1.34) and (2.8), we have

$$(2.10)\qquad p(\beta|a) = d(\beta) + \frac{c(a)l(\beta, 0, N, 0) + \frac{1}{2}l(\beta, a, N, n)}{2c(a)l(0, 0, 0, 0,) + l(0, a, 0, n)}.$$

It is easy to see that for every β $p(\beta|a) \leqslant p(k_0|a)$, therefore

$$(2.11)\qquad \max_\beta p(\beta|a) = p(k_0|a)$$

$$= \frac{1}{2} + \frac{\binom{N}{k_0} c(a)B(k_0+r+1, N-k_0+s-r+1)}{2c(a)B(r+1, s-r+1) + \binom{n}{a}B(a+r+1, n-a+s-r+1)} +$$

$$+ \frac{\frac{1}{2}\binom{n}{a}\binom{N}{k_0}B(a+k_0+r+1, N-k_0+n-a+s-r+1)}{2c(a)B(r+1, s-r+1) + \binom{n}{a}B(a+r+1, n-a+s-r+1)}.$$

From (2.9) it follows that $l(\beta, a, N, n)$ is an increasing function of β for $\beta \leqslant \frac{(N+1)(a+r)}{n+s} - 1$.

(a_2) If $(N+1)(a+r) < n+s$, $l(\beta, a, N, n)$ is a decreasing function of β, therefore when $N \neq k_0$

$$(2.12)\qquad \min_\beta p(\beta|a) = p(N|a)$$

$$= \frac{c(a)B(N+r+1, s-r+1)}{2c(a)B(r+1, s-r+1) + \binom{n}{a}B(a+r+1, n-a+s-r+1)} +$$

$$+ \frac{\frac{1}{2}\binom{n}{a}B(a+N+r+1, n-a+s-r+1)}{2c(a)B(r+1, s-r+1) + \binom{n}{a}B(a+r+1, n-a+s-r+1)},$$

$$(2.13)\quad \max_{\beta\neq k_0} p(\beta|a) = p(0|a)$$

$$= \frac{c(a)B(r+1, N+s-r+1)+\frac{1}{2}\binom{n}{a}B(a+r+1, N+n-a+s-r+1)}{2c(a)B(r+1, s-r+1)+\binom{n}{a}B(a+r+1, n-a+s-r+1)}.$$

(b_2) When $a+r = n+s$, $l(\beta, a, N, n)$ is an increasing function of β, therefore

$$(2.14)\quad \min_{\beta} p(\beta|a) = p(0|a)$$

$$= \frac{c(a)B(r+1, N+s-r+1)+\frac{1}{2}\binom{n}{a}B(a+r+1, N+n-a+s-r+1)}{2c(a)B(r+1, s-r+1)+\binom{n}{a}B(a+r+1, n-a+s-r+1)},$$

$$(2.15)\quad \max_{\beta\neq k_0} p(\beta|a) = p(N|a) \text{ (see the formula (2.12))}.$$

(c_2) If $a+r \neq n+s$ and $\frac{(N+1)(a+r)}{n+s} = k$, where k is a positive integer $\max_{\beta} l(\beta, a, N, n) = l(k, a, N, n)$. Hence

$$(2.16)\quad \max_{\beta\neq k_0} p(\beta|a)$$

$$= \frac{c(a)\binom{N}{k}B(k+r+1, N-k+s-r+1)+\frac{1}{2}\binom{n}{a}\binom{N}{k}}{2c(a)B(r+1, s-r+1)+\binom{n}{a}B(a+r+1, n-a+s-r+1)}$$

$$\frac{B(a+k+r+1, N-k+n-a+s-r+1)}{2c(a)B(r+1, s-r+1)+\binom{n}{a}B(a+r+1), n-a+s-r+1)}.$$

When $(N+1)(a+r)/(n+s)$ is not a positive integer $l(\beta, a, N, n)$ reaches its maximum at β equal to the greatest integer smaller then $(N+1)/(a+r)/(n+s)$.

3. A limit behaviour of $p(\beta|a)$ probability

Now, we shall deal with the limit behaviour of $p(\beta|a)$ probability. We shall consider a limit behaviour of $p(\beta|a)$ probability in case when N, β are constants and a, n tend to infinity in such a way that n/a = constant, and in case when n, a are constants, but β, N tend to infinity in such a way that β/N = constant.

For this purpose, we shall prove two theorems.

Theorem 3.1. *If N, β are constants, $a \to \infty$ and $n \to \infty$ in such a way that*

$$(3.1) \qquad \frac{a}{n} = p_0 = constant, \ (p_0 \neq 0, \ p_0 \neq 1), \ then$$

$$(3.2) \qquad \lim_{\substack{n\to\infty \\ p_0 = const.}} p(\beta|a) = d(\beta) + a\binom{N}{\beta} p_0^\beta q_0^{N-\beta},$$

where $q_0 = 1 - p_0$.

It is easy to see that if random variable λ = constant = 1, we get the formula (4.V) *from* [1].

Proof. The above limit ought to be computed

$$\lim_{\substack{n\to\infty \\ p_0 = const.}} p(\beta|a) = \lim_{\substack{n\to\infty \\ p_0 = const.}} \Bigg\{ d(\beta) + \frac{ac(a)\binom{N}{\beta}\int_0^1 p^\beta q^{N-\beta} f(p)\,dp + a^2\binom{n}{a}\binom{N}{\beta}\int_0^1 p^{a+\beta} q^{N+n-a-\beta} f(p)\,dp}{c(a) + a\binom{n}{a}\int_0^1 p^a q^{n-a} f(p)\,dp.} \Bigg\}.$$

Taking into account that $c(a) = 0$ for $a \neq k_0$ we have

$$(3.3) \quad \lim_{\substack{n\to\infty \\ p_0 = const.}} p(\beta|a) = \lim_{\substack{n\to\infty \\ p_0 = const.}} \Bigg\{ d(\beta) + \frac{a\binom{N}{\beta}\int_0^1 p^{a+\beta} q^{N+n-a-\beta} f(p)\,dp}{\int_0^1 p^a q^{n-a} f(p)\,dp} \Bigg\}.$$

In paper [1] it has been proved that

$$(3.4) \qquad \lim_{\substack{n\to\infty \\ p_0 = const.}} \Bigg\{ \frac{\int_0^1 p^{a+\beta} q^{N-\beta+n-a} f(p)\,dp}{\int_0^1 p^a q^{n-a} f(p)\,dp} \Bigg\} = p_0^\beta q_0^{N-\beta}$$

under the assumption that $f''(p)$ is limited in the neighbourhood of point p. From (3.3) and (3.4) we have

$$\lim_{\substack{n\to\infty \\ p_0 = const.}} p(\beta|a) = d(\beta) + a\binom{N}{\beta} p_0^\beta q_0^{N-\beta}.$$

What was to be proved.

Theorem 3.2. *If n, a are constants, $N\to\infty$, $\beta\to\infty$ in such a way that*

$$\frac{\beta}{N} = p_1 = constant\ (p_1\neq 0, p_1\neq 1),\ then \tag{3.5}$$

$$\lim_{\substack{N\to\infty\\ p_1=const.}} p(\beta|a) = 0, \tag{3.6}$$

$$\lim_{\substack{N\to\infty\\ p_1=const.}} \{Np(\beta|a)\} = \frac{\left[ac(a)+a^2\binom{n}{a}p_1^a q_1^{n-a}\right]f^{(p)}(p_1)}{c(a)+a\binom{n}{a}\int_0^1 p^a q^{n-a}f(p)\,dp}, \tag{3.7}$$

where $q_1 = 1-p_1$, and $f^{(p)}(p_1)$ implies that f is a function of variable p taken at the point p_1.

Proof. From the (1.12)

$$\lim_{\substack{N\to\infty\\ p_1=const.}} p(\beta|a) = \lim_{\substack{N\to\infty\\ p_1=const.}} \left\{d(\beta)+ \vphantom{\int}\right.$$

$$\left. + \frac{ac(a)\binom{N}{\beta}\int_0^1 p^\beta q^{N-\beta}f(p)\,dp + a^2\binom{n}{a}\binom{N}{\beta}\int_0^1 p^{a+\beta}q^{N-\beta+n-a}f(p)\,dp}{a\binom{n}{a}\int_0^1 p^a q^{n-a}f(p)\,dp + c(a)}\right\}.$$

Since $d(\beta) = 0$ for $\beta\neq k_0$, therefore

$$\lim_{\substack{N\to\infty\\ p_1=const.}} p(\beta|a) = \lim_{\substack{N\to\infty\\ p_1=const.}} \times$$

$$\times\left\{\frac{ac(a)\binom{N}{\beta}\int_0^1 p^\beta q^{N-\beta}f(p)\,dp + a^2\binom{N}{\beta}\binom{n}{a}\int_0^1 p^{a+\beta}q^{N-\beta+n-a}f(p)\,dp}{c(a)+a\binom{n}{a}\int_0^1 p^a q^{n-a}f(p)\,dp}\right\}.$$

To compute the value of this last limit, it is necessary to determine

$$\lim_{\substack{N\to\infty\\ p_1=const.}} \left\{\binom{N}{\beta}\binom{n}{a}\int_0^1 p^{a+\beta}q^{N+n-a-\beta}f(p)\,dp\right\} \text{ and } \lim_{\substack{N\to\infty\\ p_1=const.}} \left\{\binom{N}{\beta}\int_0^1 p^\beta q^{N-\beta}f(p)\,dp\right\}.$$

Taking into account that

$$\lim_{\substack{N\to\infty\\ p_1=const.}} \left\{\binom{N}{\beta}\int_0^1 p^{a+\beta}q^{N+n-a-\beta}f(p)\,dp\right\}$$

$$= \lim_{\substack{N\to\infty\\ p_1=const.}} \left\{\frac{1}{N}p_1^a q_1^{n-a}[f^{(p)}(p_1)+O(N^{-2\omega})]\right\}$$

and

$$\lim_{\substack{N\to\infty\\ p_1=const.}} \left\{\binom{N}{\beta}\int_0^1 p^\beta q^{N-\beta} f(p)\,dp\right\} = \lim_{\substack{N\to\infty\\ p_1=const.}} \left\{\frac{1}{N}\left[f^{(p)}(p_1)+O(N^{-2\omega})\right]\right\}$$

where $1/3 < \omega < 1/2$ and $f^{(p)}(p_1)$ denotes that f is a function of variable p taken at the point p_1 (see [1]), we get

(3.8)
$$\lim_{\substack{N\to\infty\\ p_1=const.}} p(\beta|a) = \lim_{\substack{N\to\infty\\ p_1=const.}} \left\{\frac{\left[ac(a)+a^2\binom{n}{a}p_1^a q_1^{n-a}\right]\left[f^{(p)}(p_1)+O(N^{-2\omega})\right]}{N\left[c(a)+a\binom{n}{a}\int_0^1 p^a q^{n-a} f(p)\,dp\right]}\right\}.$$

On the basis of the formula (3.8), it is easy to see that (3.6) and (3.7) are satisfied, what ends the proof of the theorem.

Let us consider a limit problem in particular cases.

It is easy to see that if random variables λ and P are uniformly distributed, then

(3.9)
$$\lim_{\substack{n\to\infty\\ p_0=const.}} p(\beta|a) = d(\beta)+\tfrac{1}{2}\binom{N}{\beta}p_0^\beta q_0^{N-\beta},$$

(3.10)
$$\lim_{\substack{N\to\infty\\ p_1=const.}} \{Np(\beta|a)\} = \frac{2c(a)(n+1)+\binom{n}{a}(n+1)p_1^a q_1^{n-a}}{4c(a)(n+1)+4}.$$

In case when the random variable P has the beta distribution and the random variable λ is as in above we have

(3.11)
$$\lim_{\substack{N\to\infty\\ p_1=const.}} \{Np(\beta|a)\} = \frac{c(a)p_1^r q_1^{s-r}+\tfrac{1}{2}\binom{n}{a}p_1^{a+r}q_1^{n-a+s-r}}{2c(a)B(r+1,s-r+1)+\binom{n}{a}B(a+r+1,n-a+s-r+1)}.$$

4. Expected values and variances

Theorem 4.1. *Expected values and variances of random variables A_n and B_N are given by the formulas:*

(4.1)
$$EA_n = naEP+k_0b,$$

(4.2)
$$EB_N = NaEP+k_0b,$$

$$EA_n^2 = naEP + n(n-1)aEP^2 + k_0^2 b, \tag{4.3}$$

$$EB_N^2 = NaEP + N(N-1)aEP^2 + k_0^2 b, \tag{4.4}$$

$$\sigma^2 A_n = ab(k_0 - nEP)^2 + n^2 a\sigma^2 P + an(EP - EP^2), \tag{4.5}$$

$$\sigma^2 B_N = ab(k_0 - NEP)^2 + N^2 a\sigma^2 P + Na(EP - EP^2). \tag{4.6}$$

Proof. The probability law (1.5) and (1.6) allows to obtain in the known way the mathematical expections and the variances of the random variable A_n and B_N. To compute these characteristic still it will be more convenient, to use of the formulas:

$$E[E(A_n|P)] = EA_n, \tag{4.7}$$

$$E[E(A_n^2|P)] = EA_n^2. \tag{4.8}$$

In case being considered $E(A_n|p) = nap + k_0 b$. (It follows from 1.3). Taking $E(A_n|p)$ as a function of random variable P, we have

$$E(A_n|P) = naP + k_0 b. \tag{4.9}$$

Hence and from (4.7) we get (4.1). Similarly we prove (4.2).

Since $E(A_n^2|P) = k_0^2 b + naP + an(n-1)P^2$, therefore on the basis of (4.8), we get (4.3), what ends the proof of the theorem, as we derive (4.4) similarly as (4.3) and the two left formulas follow immediately.

Now, we suppose that random variables λ and P are uniformly distributed. Then, the formulas (4.1) – (4.6) take on the corresponding forms:

$$EA_n = \frac{1}{4}(n + 2k_0), \tag{4.10}$$

$$EB_n = \frac{1}{4}(N + 2k_0), \tag{4.11}$$

$$EA_n^2 = \frac{1}{12}(2n^2 + n + 6k_0^2), \tag{4.12}$$

$$EB_N^2 = \frac{1}{12}(2N^2 + N + 6k_0^2), \tag{4.13}$$

$$\sigma^2 A_n = \frac{1}{48}(5n^2 + 12k_0^2 + 4n - 12nk_0), \tag{4.14}$$

$$\sigma^2 B_N = \frac{1}{48}(5N^2 + 12k_0^2 + 4N - 12Nk_0). \tag{4.15}$$

In case when the random variable λ is as in above, but the random variable P has beta distribution the formulas (4.1) — (4.6) are as follows:

$$EA_n = \frac{1}{2} k_0 + \frac{n(r+1)}{2(s+2)}, \tag{4.16}$$

$$EB_N = k_0 + \frac{N(r+1)}{2(s+2)}, \tag{4.17}$$

$$EA_n^2 = \frac{1}{2} k_0^2 + \frac{n(r+1)}{2(s+2)} + \frac{n(n-1)(r+2)(r+1)}{2(s+3)(s+2)}, \tag{4.18}$$

$$EB_N^2 = \frac{1}{2} k_0^2 + \frac{N(r+1)}{2(s+2)} + \frac{N(N-1)(r+2)(r+1)}{2(s+3)(s+2)}, \tag{4.19}$$

$$\sigma^2 An = \frac{1}{4} k_0^2 + \frac{n(n-1)(r+1)(r+2)}{2(s+2)(s+3)} + \frac{n(r+1)(2s+4-nr-n-2k_0s-4k_0)}{4(s+2)^2}, \tag{4.20}$$

$$\sigma^2 B_N = \frac{1}{4} k_0^2 + \frac{N(N-1)(r+1)(r+2)}{2(s+2)(s+3)} + \frac{N(r+1)(2s+4-Nr-N-2k_0s-4k_0)}{4(s+2)^2}. \tag{4.21}$$

It is easy to see that in case when P follows the rectangular distribution or when P follows beta distribution the variances of random variables A_n and B_N are $O(n^2)$ and $O(N^2)$ order respectively.

5. Conditional expected values and conditional variances

Theorem 5.1. *Conditional expected values and conditional variances are given by the following formulas:*

$$E(B_N|a) = NaE(P|a) + k_0 b, \tag{5.1}$$

$$E(B_N^2|a) = k_0^2 b + NaE(P|a) + aN(N-1)E(P^2|a), \tag{5.2}$$

$$\sigma^2(B_N|a) = ab(k_0 - NE(P|a))^2 + aN^2\sigma^2(P|a) + aN[E(P|a) - E(P^2(a)]. \tag{5.3}$$

Proof. Taking into account that

$$E(B_N|a) = \sum_{\beta=0}^{N} \beta p(\beta|a) = \sum_{\beta=0}^{N} \beta \int_0^1 p(\beta|p)f(p|a)\,dp,$$

we get

$$E(B_N|a) = \int_0^1 E(B_N|p)f(p|a)\,dp = aNE(P|a) + k_0 b,$$

because by means of the formula (1.4) it is easy to see that $E(B_N|p) = Nap + k_0 b$. Now, taking advantage of fact that $E(B_N^2|p) = Nap + k_0^2 b + N(N-1)ap^2$, we easly obtain the formula (5.2), which with (5.1) gives (5.3). It is easy to see that formula (5.3) is similar to (4.6).

If the random variable λ is uniformly distributed, and the random variable P has the rectangular or beta distribution, then the above formulas take on the corresponding forms

$$(5.4)\qquad E(B_N|a) = \frac{1}{2}k_0 + \frac{N(1+a) + Nc(a)(n+2)}{2(n+2) + 4c(a)(n+1)(n+2)},$$

$$(5.5)\qquad E(B_N^2|a) = \frac{1}{2}k_0^2 + $$

$$+\frac{Nc(a)(n+2)(n+3)[3+(N-1)(n+1)] + 3N(a+1)[1+(N-1)(a+2)]}{6[1+2c(a)(n+1)](n+2)(n+3)},$$

$$(5.6)\qquad E(B_N|a) = \frac{1}{2}k_0 + $$

$$+\frac{1}{2}N\frac{2c(a)B(r+2,s-r+1) + \binom{n}{a}B(a+r+2,n-a+s-r+1)}{2c(a)B(r+1,s-r+1) + \binom{n}{a}B(a+r+1,n-a+s-r+1)},$$

$$(5.7)\qquad E(B_N^2|a) = \frac{1}{2}k_0^2 + $$

$$+\frac{1}{2}N\frac{2c(a)B(r+2,s-r+1) + \binom{n}{a}B(a+r+2,n-a+s-r+1)}{2c(a)B(r+1,s-r+1) + \binom{n}{a}B(a+r+1,n-a+s-r+1)} + $$

$$+\frac{1}{2}N(N-1)\frac{2c(a)B(r+3,s-r+1) + \binom{n}{a}B(a+r+3,n-a+s-r+1)}{2c(a)B(r+1,s-r+1) + \binom{n}{a}B(a+r+1,n-a+s-r+1)}.$$

Since, for the random variable P uniformly distributed, by the formula (1.20), we get

$$(5.8)\qquad E(P|a) = \frac{1+a+c(a)(n+2)}{2+n+2c(a)(n+1)(n+2)},$$

$$(5.9)\qquad E(P^2|a) = \frac{2c(a)(n+1)(n+2)(n+3) + 3(a+1)(a+2)}{3(1+2c(a)(n+1))(n+2)(n+3)}$$

and for the random variables P having beta distribution take place equalities

(5.10) $E(P|a)$

$$=\frac{2c(a)B(r+2,s-r+1)+\binom{n}{a}B(a+r+2,n-a+s-r+1)}{2c(a)B(r+1,s-r+1)+\binom{n}{a}B(a+r+1,n-a+s-r+1)},$$

(5.11) $E(p^2|a)$

$$=\frac{2c(a)B(r+3,s-r+1)+\binom{n}{a}B(a+r+3,n-a+s-r+1)}{2c(a)B(r+1,s-r+1)+\binom{n}{a}B(a+r+1,n-a+s-r+1)},$$

therefore we have (5.4) – (5.7).

Now, we shall deal with the variance of conditional expectation $E(B_N|A_n)$ (we shall consider as a random variable assuming values $E(B_N|a)$) and with the expected value of conditional variance $\sigma^2(B_N|A_n)$ (we shall consider as the random variable assuming values $\sigma^2(B_N|a)$).

We prove thus the following

Theorem 5.2. *The variance of conditional expectation is given by the formula*

(5.12) $$\sigma^2[E(B_N|A_n)] = a^2N^2\sigma^2[E(P|A_n)],$$

and expected value of conditional variance by the formula

(5.13) $$E[\sigma^2(B_N|A_n)] = \sigma^2B_N - a^2N^2\sigma^2[E(P|A_n)].$$

Proof. The formula (5.12) easily follows from (5.1). Since $E[\sigma^2(B_N|A_n)] = \sigma^2(B_N)-\sigma^2[E(B_N|A_n)]$ therefore taking into account (5.12) we easily get (5.13).

It is easy to see (by the formulas (5.12), (5.13) and (4.6)), that the variance of the conditional expectation of the random variable B_N and the expected value of the conditional variance of B_N tend to infinity when the number N of experiments approaches to infinity and n is constant. Speaking precisely they are of $O(N^2)$ order.

By virtue of (5.12) and (4.6), we obtain

(5.14) $$\lim_{N\to\infty}\frac{\sigma^2[E(B_N(A_n)]}{\sigma^2B_N} = \frac{a\sigma^2[E(P|A_n)]}{EP^2-aE^2P}.$$

From the formulas (5.13) and (4.6), we have

$$(5.15)\qquad \lim_{N\to\infty}\frac{E[\sigma^2(B_N|A_n)]}{\sigma^2 B_N}=1-\frac{a\sigma^2[E(P|A_n)]}{EP^2-aE^2P}$$

Now, let us suppose that the random variables λ and P are uniformly distributed. Then by (5.8) and (1.17), we obtain

$$(5.16)\qquad \sigma^2[E(P|A_n)] = \frac{(n-3)(n+2)(n+1)+12(k_0+1)[(n+1)^2+k_0^2-nk_0]}{24(n+1)(n+2)^2}.$$

Taking into account that in this case $EP=\frac{1}{2}$, $EP^2=\frac{1}{3}$, $a=\frac{1}{2}$ and using (5.14), we have

$$(5.17)\qquad \lim_{N\to\infty}\frac{\sigma^2[E(B_N|A_n)]}{\sigma^2 B_N} = \frac{(n-3)(n+2)(n+1)+12(k_0+1)[(n+1)^2+k_0^2-nk_0]}{10(n+1)(n+2)^2}.$$

Let us notice, that in this case

$$(5.18)\qquad \lim_{\substack{N\to\infty\\ n\to\infty}}\frac{\sigma^2[E(B_N|A_n)]}{\sigma^2 B_N}=\frac{1}{10},$$

$$(5.19)\qquad \lim_{\substack{N\to\infty\\ n\to\infty}}\frac{E[\sigma^2(B_N|A_n)]}{\sigma^2 B_N}=\frac{9}{10}.$$

In the case when the random variable λ is as in above, and the random variable P has the beta distribution, by virtue of (5.10) and (1.27), we have

$$(5.20)\qquad \sigma^2[E(P|A_n)] = \frac{n(r+1)(s-r+1)}{2(s+2)^2(s+3)(n+s+2)} - \left(\frac{k_0+r+1}{n+s+2}\right)^2\binom{n}{k_0}\frac{B(k_0+r+1,\,n-k_0+s-r+1)}{2B(r+1,\,s-r+1)} + \left[\frac{B(r+2,\,s-r+1)+\binom{n}{k_0}B(k_0+r+2,\,n-k_0+s-r+1)}{B(r+1,\,s-r+1)+\binom{n}{k_0}B(k_0+r+1,\,n-k_0+s-r+1)}\right]^2 \times \left[\frac{1}{2}+\frac{1}{2}\binom{n}{k_0}\frac{B(k_0+r+1,\,n-k_0+s-r+1)}{B(r+1,\,s-r+1)}\right].$$

Further, since in this case

$$EP = \frac{r+1}{s+2}, \ EP^2 = \frac{(r+1)(r+2)}{(s+2)(s+3)},$$

therefore by (5.14) and (5.15), we have

(5.21) $$\lim_{N\to\infty} \frac{\sigma^2[E(B_N|A_n)]}{\sigma^2 B_N} = \frac{n(s-r+1)}{2(n+s+2)(rs+r+3s+5)} -$$

$$-\binom{n}{k_0}\frac{B(k_0+r+1, n-k_0+s-r+1)(s+3)}{2B(r+1, s-r+1)(rs+r+3s+5)} \times \frac{(s+2)^2}{(r+1)}\left(\frac{k_0+r+1}{n+s+2}\right)^2 +$$

$$+\frac{(s+2)^2(s+3)}{(r+1)(rs+r+3s+5)}\left[\frac{1}{2}+\frac{B(k_0+r+1, n-k_0+s-r+1)}{2B(r+1, s-r+1)}\right] \times$$

$$\times \left[\frac{B(r+2, s-r+1)+\binom{n}{k_0}B(k_0+r+2, n-k_0+s-r+1)}{B(r+1, s-r+1)+\binom{n}{k_0}B(k_0+r+1, n-k_0+s-r+1)}\right]^2.$$

Let us notice finally, that in case from the formula (5.21), we have

(5.22) $$\lim_{\substack{N\to\infty \\ n\to\infty}} \frac{\sigma^2[E(B_N|A_n)]}{\sigma^2 B_N} = \frac{(r+2)(s+2)}{2(rs+r+3s+5)}.$$

Hence, and from (5.15), we have

(5.23) $$\lim_{\substack{N\to\infty \\ n\to\infty}} \frac{E[\sigma^2(B_N|A_n)]}{\sigma^2 B_N} = \frac{rs+4s+6}{2(rs+r+3s+5)}.$$

6. The correlation coefficient between random variables A_n and B_N.

Since by the assumption $P(A_n, B_N|P) = P(A_n|P)\cdot P(B_N|P)$, therefore

(6.1) $$E(B_N A_n) = E[E(A_n|P)E(B_N|P)].$$

Now, taking advantage of (4.9), we obtain

(6.2) $$E(A_n B_N) = nNa^2 EP^2 + bk_0(N+n)EP + b^2 k_0^2,$$

hence and by (4.1) and (4.2)

(6.3) $$\mathrm{Cov}(A_n, B_N) = EA_nB_N - EA_n\cdot EB_N = na^2N\sigma^2(P).$$

If the random variables λ and P are uniformly distributed, then

(6.4) $$\mathrm{Cov}(A_n, B_N) = \frac{1}{48}Nn.$$

In the case when the random variable λ is as in above, and the random variable P has the beta distribution, we have

(6.5) $$\mathrm{Cov}(A_n, B_N) = \frac{Nn(r+1)(s-r+1)}{4(s+2)^2(s+3)}.$$

On the basis of (6.3), (4.5) and (4.6), we obtain

(6.6) $$\varrho A_n B_N = \frac{\mathrm{Cov}(A_n, B_N)}{\sigma A_n \sigma B_N} = \frac{naN\sigma^2(P)}{\sqrt{[b(k_0-nEP)^2+n^2\sigma^2P+n(EP-EP^2)]}} \times \frac{1}{\sqrt{[b(k_0-NEP)^2+N^2\sigma^2P+N(EP-EP^2)]}}$$

Hence

(6.7) $$\lim_{\substack{N\to\infty\\ n\to\infty}} \varrho_{A_nB_N} = \frac{E\lambda\sigma^2P}{EP^2-E\lambda EP^2}.$$

It is easy to observe that if the random variables λ and P are uniformly distributed, then

(6.8) $$\varrho_{A_nB_N} = \frac{nN}{\sqrt{[5n^2+12k_0(k_0-n)+4n][5N^2+12k_0(k_0-N)+4N]}}$$

and

(6.9) $$\lim_{\substack{n\to\infty\\ N\to\infty}} \varrho_{A_nB_N} = \frac{1}{5}.$$

In the case when the random variable λ is as in above, and the random variable P has the beta distribution, we have

(6.10) $$\varrho_{A_nB_N} = \frac{n(r+1)(s-r+1)}{\sqrt{k_0^2(s+2)^2+(k_0s+2k_0-nr-n)^2+2n(r+1)(s-r+1)}} \times$$
$$\times \frac{N}{(s+3)\sqrt{k_0^2(s+2)^2+(k_0s+2k_0-Nr-N)^2+2N(r+1)(s-r+1)}},$$

and

(6.11) $$\lim_{\substack{n\to\infty\\ N\to\infty}} \varrho_{A_nB_N} = \frac{s-r+1}{s(r+3)+r+5}.$$

So we have proved the following

Theorem 6.1. *The covariance between the random variables A_n and B_N is expressed by the formula* (6.3) *and it is directly proportional to n and N as well, with the coefficient of proportionality equal to $E^2\lambda\sigma^2 P$. The correlation coefficient of these random variables is expressed by the formula* (6.6). *When n and N tends to infinity at the same time $\varrho_{A_n B_N}$ tends to* $\dfrac{E\lambda\sigma^2 P}{EP^2 - E\lambda E^2 P}$.

7. Remarks about applications

In a quality control of mass production the probability of getting k bad pieces equal in a sample of n, is given by the Bernoulli formula

$$P(k) = \binom{n}{k} p^k q^{n-k}, \; k = 0, 1, 2, \ldots, n \text{ (see [2]).}$$

Such an approach refers to the majority of practical applications. However situations exist where an article is subjected for studing not immediately after producing but after a certain period of time, for example glass, tinned meat, eggs and many alike.

At that time the binomial distribution does not precisely reflect the probability of number of bad pieces, especially when $k = 0$. With the regard to this fact, applying the inflated binomial distribution

$$P(k) = \begin{cases} 1 - s + sq^n & \text{for } k = 0, \\ s\dbinom{n}{k} p^k q^{n-k} & \text{for } k = 1, 2, \ldots, n \end{cases}$$

seems be more substantiated in such cases.

This observation has also to be taken into consideration during a priori construction of a distribution function.

In the paper [1] the possibilities of applying distributions $p(a)$, $p(\beta|a)$ to the quality control were indicated. The applying $p(a) = \binom{n}{a}\int_0^1 p^a q^{n-a} \times f(p)\,dp$ is being suggested instead of $P[A_n = a | P = p] = \binom{n}{a} p^a q^{n-a}$. The reasons are given that the $p(a)$ distribution is a theoretical model for results of really collected samples from whole extension of production process. On the contrary $p(a|p)$ is a model for results of thought out but practically not carried out sample replications from the same isolated parcel.

The formulas from [1] can be applied in case when the control was accomplished directly after the production cycle (or when the article was not subjected to deterioration). The formulas from [1] will not precisely reflect the investigated reality in case when the control is carried out after a certain period of time, during which the number of pieces can increase. The number of bad pieces in sample, in such case, with given p is described by the inflated binomial distribution with greater accuraty and in case, when p and s are values of random variables P and λ respectively, by the distribution

$$P(k) = \begin{cases} 1-E\lambda+E\lambda\int_0^1 q^n f(p)\,dp & \text{for } k = 0, \\ E\lambda\binom{n}{k}\int_0^1 p^k q^{n-k} f(p)\,dp & \text{for } k = 1, 2, \ldots, n. \end{cases}$$

Hence, the results given in this paper, in particular the formulas determining $p(a)$ and $p(\beta|a)$ can have a similar application to the quality control as the corresponding formulas from [1].

REFERENCES

[1] Krysicki, W. Olekiewicz, M. *O uogólnionym połączeniu zagadnień Bayesa i Bernoulliego*, Zastosowania matematyki, VII (1963).

[2] Oderfeld, J. *Statystyczny odbiór towarów klasycznych według alternatywy*, Studia i prace statystyczne, 2 (1950), 100-104.

[3] Singh, M. P. *Inflated Binomial Distribution*, J. Scient. Res. Banares Hindu Univ., (1965/66), 16, 1, 87-90.

[4] Singh, S. N. *Probability Models for the Variation in the Number of Births per Couple*, J. Amer. Statist. Ass., 58 (1963), 721-727.

STRESZCZENIE

Niech P i λ będą niezależnymi zmiennymi losowymi przyjmującymi odpowiednio wartości p i s z przedziałów $(0, 1)$ i $(0, 1)$. Niech dalej A_n i B_N będą dyskretnymi zmiennymi losowymi takimi, że

$$P[A_n = a|P = p;\ \lambda = s] = \begin{cases} 1-s+s\binom{n}{k_0}p^{k_0}q^{n-k_0}, & a = k_0, \\ s\binom{n}{a}p^a q^{n-a}, & a = 0, 1, 2, \ldots, k_0-1, k_0+1, \ldots, n; \end{cases}$$

$$P[B_N = \beta\,|P = p;\ \lambda = s] = \begin{cases} 1-s+s\binom{N}{k_0}p^{k_0}q^{N-k_0}, & \beta = k_0, \\ s\binom{N}{\beta}p^\beta q^{N-\beta}, & \beta = 0, 1, 2, \ldots, k_0-1, k_0+1, \ldots, N. \end{cases}$$

W pracy rozpatruje się uogólnione połączenie zagadnienia Bernoulliego i Bayesa, które łączy się z poszukiwaniem rozkładów łącznych, warunkowych i brzegowych wyżej wprowadzonych zmiennych losowych. Ponadto zbadano asymptotyczne własności znalezionych rozkładów oraz wyznaczono ich charakterystyki. Rozważono dwa szczególne przypadki rozkładów zmiennej losowej P (jednostajny i beta). Podane wyniki mogą znaleźć zastosowanie w statystycznej kontroli jakości.

W przypadku szczególnym $\lambda = \text{const.} = 1$, otrzymuje się wyniki podane w [1].

РЕЗЮМЕ

Пусть P и λ — независимые случайные величины принимающие соответственно величины p и s из пределов $(0, 1)$ и $(0, 1\rangle$. Пусть A_n и B_N — дискретные случайные величины такие, что

$$P[A_n = a|P = p; \lambda = s] = \begin{cases} 1-s+s\binom{n}{k_0}p^{k_0}q^{n-k_0}, & a = k_0 \\ s\binom{n}{a}p^a q^{n-a}, & a = 0, 1, 2, \ldots, k_0-1, k_0+1, \ldots n; \end{cases}$$

$$P[B_N = \beta|P = p; \lambda = s] = \begin{cases} 1-s+s\binom{N}{k_0}p^{k_0}q^{N-k_0}, & \beta = k_0 \\ s\binom{N}{\beta}p^\beta q^{N-\beta}, & \beta = 0, 1, 2, \ldots, k_0-1, k_0+1, \ldots N. \end{cases}$$

В работе рассматривается обобщенно-объединенная проблема Бернули и Бейеса, связанная с поиском совместных, маргинальных и условных распределений приведенных выше случайных величин. Кроме того, исследуются асимптомические свойства найденных распределений, а также определяются их характеристики. Рассматриваются два частных случая распределений случайной величины P (прямоугольное и β-распределение). Полученные результаты могут найти применение в статистическом контроле качества.

В частном случае $\lambda = \text{const} = 1$ получены результаты работы [1].