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Some Inequalities For Bounded Univalent Functions

O pewnych nierównościach dla funkcji jednolistnych ograniczonych

Некоторые неравенства для однолистных ограниченных функций

1. Introduction

If $f(z)$ is regular in the open unit disk Δ, $\Delta = \{z: |z| < 1\}$, then $f(z)$ is univalent in Δ if and only if

$$\log \frac{f(z)-f(\zeta)}{z-\zeta} = \sum_{k,h=0}^{\infty} A_{kh} z^k \zeta^h \tag{1.1}$$

is defined and convergent in $\Delta \times \Delta$. This observation has been useful to investigators of Geometric Function Theory, particularly for deriving "Grunsky-type" inequalities for functions which are univalent or carry some similar restriction (see Hummel, [2]); the techniques used and the results so obtained have often been unusually difficult and complicated. Jenkins [3], and other authors [1], [2], simplified and enhanced much of this work by the application of generalized area principles.

In a subsequent paper [4], Jenkins used the area method to derive inequalities for functions which are of the Bieberbach-Eilenberg or similar classes. The purpose of this note is to illustrate applications of these techniques to pairs of univalent functions which are bounded and have non-overlapping domains and to bounded univalent functions; the novelty of these applications stems in part from using the unit circle as a boundary component for the region of integration. Some bounds for functions and their derivatives are obtained as corollaries. These appear to be new; some have meaning for functions considered earlier by Nehari [6].

2. One Method

The following notation is used. For a suitable set S, $h[S]$ denotes the image of S under the function $h(z)$. Δ_r is the open disk centered at the origin with radius r and γ_r is its boundary; $\Delta = \Delta_1$ and $\gamma = \gamma_1$.

Theorem 1. *If $f(z)$ and $g(z)$ are regular and univalent in Δ; $g(z) \neq 0$, $z \in \Delta$; $f[\Delta]$ and $g[\Delta]$ are disjoint and both contained in Δ;* (1.1) *holds along with*

$$\log\left\{\frac{g(z)-g(\zeta)}{z-\zeta}\right\} = \sum_{k,h=0}^{\infty} B_{kh} z^k \zeta^h, \quad z \in \Delta, \ \zeta \in \Delta, \tag{2.1}$$

and

$$\log\left\{1-\frac{f(z)}{g(\zeta)}\right\} = \sum_{k,h=0}^{\infty} \lambda_{kh} z^k \zeta^h, \quad z \in \Delta, \ \zeta \in \Delta; \tag{2.2}$$

and if for $k = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, z_k and ζ_j are arbitrary numbers in Δ whereas a_k and β_j are any complex numbers such that $\sum_{k=1}^{n} a_k = \sum_{j=1}^{m} \beta_j = 0$; *then*

$$\begin{aligned}\sum_{k=1}^{\infty}\Big|\sum_{j=1}^{n} a_j \sum_{h=1}^{\infty} A_{kh} z_j^h + \sum_{q=1}^{m} \beta_q \sum_{h=1}^{\infty} \lambda_{kh} \zeta_q^h\Big|^2 + \\ + \sum_{h=1}^{\infty} h \Big|\sum_{j=1}^{n} a_j \sum_{k=1}^{\infty} \lambda_{kh} z_j^k + \sum_{q=1}^{m} \beta_q \sum_{k=1}^{\infty} B_{kh} \zeta_q^k\Big|^2 + \\ + \sum_{s=1}^{\infty} \frac{1}{s}\Big|\sum_{j=1}^{n} a_j f(z_j)^s + \sum_{q=1}^{m} \beta_q g(\zeta_q)^s\Big|^2 \\ \leqslant \sum_{j,k=1}^{n} a_j \overline{a_k} \log\left(\frac{1}{1-z_j\overline{z_k}}\right) + \sum_{j,k=1}^{m} \beta_j \overline{\beta_k} \log\left(\frac{1}{1-\zeta_j\overline{\zeta_k}}\right).\end{aligned} \tag{2.3}$$

To simplify the notation of the proof we let $\Gamma_r = f[\gamma_r]$, $\Omega_\varrho = g[\gamma_\varrho]$ and $D_{r,\varrho}$ be the region in Δ whose boundary is $\Gamma_r \cup \Omega_\varrho \cup \gamma$, i.e., $D_{r,\varrho}$ is Δ without $f[\overline{\Delta_r}]$ and $g[\overline{\Delta_\varrho}]$.

Let

$$\Phi(w) = \sum_{j=1}^{n} a_j (w-w_j)^{-1} + \sum_{q=1}^{m} \beta_q (w-\omega_q)^{-1}, \tag{2.4}$$

where $w_j = f(z_j)$, $j = 1, 2, \ldots, n$ and $\omega_q = g(\zeta_q)$, $q = 1, 2, \ldots, m$ and r and ϱ are taken sufficiently large to make $\Phi(w)$ regular in $D_{r,\varrho}$. It follows that

$$A_{r,\varrho} = \iint_{D_{r,\varrho}} |\Phi(w)|^2 dA \tag{2.5}$$

is strictly positive for all r and ϱ sufficiently close to and smaller than 1. Because of the restrictions $\Sigma a_j = \Sigma \beta_k = 0$ we can define

$$\phi(w) = \sum_{j=1}^{n} a_j \log(w-w_j) + \sum_{q=1}^{m} \beta_q \log(w-\omega_q)$$

to be regular in $D_{r,\varrho}$; and, by an application of Green's Theorem [6] or "integration by parts in the complex plane" [1] we obtain

$$A_{r,\varrho} = \frac{1}{2i}\left\{\int_{\gamma} \overline{\Phi(w)}\, d\phi(w) - \int_{\Gamma_r} \overline{\phi(w)}\, d\phi(w) - \int_{\Omega_\varrho} \overline{\phi(w)}\, d\phi(w)\right\}, \tag{2.6}$$

the integrals being taken in the positive direction (with respect to their own interiors) in each case.

Again making use of the restrictions on the $a_{j's}$ and $\beta_{k's}$ to eliminate terms containing $\log w$, we have on γ, with $w = e^{i\theta}$, that

$$\begin{aligned}
\phi(w) &= \sum_{j=1}^{n} a_j \log\big(w - f(z_j)\big) + \sum_{q=1}^{m} \beta_q \log\big(w - g(\zeta_q)\big) \\
&= \sum_{j=1}^{n} a_j \log\left(1 - \frac{f(z_j)}{w}\right) + \sum_{q=1}^{m} \beta_q \log\left(1 - \frac{g(\zeta_q)}{w}\right) \\
&= -\sum_{j=1}^{n} a_j \sum_{s=1}^{\infty} \frac{1}{s}\left(\frac{f(z_j)}{w}\right)^s - \sum_{q=1}^{m} \beta_q \sum_{s=1}^{\infty} \frac{1}{s}\left(\frac{g(\zeta_q)}{w}\right)^s \\
&= -\sum_{s=1}^{\infty} \frac{1}{s}\left[\sum_{j=1}^{n} a_j f(z_j)^s + \sum_{q=1}^{m} \beta_q g(\zeta_q)^s\right] w^{-s}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{2i}\int_{\gamma} \overline{\phi(w)}\, d\phi(w) &= \frac{1}{2i}\int_{0}^{2\pi} \overline{\phi(e^{i\theta})}\, d\phi(e^{i\theta}) \\
&= \frac{1}{2i}\int \overline{\sum_{s=1}^{\infty} \frac{1}{s}\left|\sum_{j=1}^{n} a_j f(z_j)^s + \sum_{q=1}^{m} \beta_q g(\zeta_q)^s\right| e^{-is\theta}}\, .
\end{aligned}$$

$$\sum_{s=1}^{\infty}\left[\sum_{j=1}^{n} a_j f(z_j)^s + \sum_{q=1}^{m} \beta_q g(\zeta_q)^s\right](-ie^{-is\theta})\, d\theta \tag{2.8}$$

$$= -\pi \sum_{s=1}^{\infty} \frac{1}{s}\left|\sum_{j=1}^{n} a_j f(z_j)^s + \sum_{q=1}^{m} \beta_q g(\zeta_q)^s\right|^2.$$

Then, if w is on Γ_r, $w = f(z)$ for $z = re^{i\theta}$ and

$$\phi(w) = \sum_{j=1}^{n} a_j \log(f(z) - f(z_j)) + \sum_{q=1}^{m} \beta_q \log(f(z) - g(\zeta_q))$$

$$= \sum_{j=1}^{n} a_j \log\left(\frac{f(z)-f(z_j)}{z-z_j}\right) + \sum_{j=1}^{n} a_j \log(z - z_j) +$$

$$+ \sum_{q=1}^{m} \beta_q \log\left(1 - \frac{f(z)}{g(\zeta_q)}\right) + \sum_{q=1}^{m} \beta_q \log(-g(\zeta_q))$$

$$= \sum_{j=1}^{n} a_j \sum_{k,h=0}^{\infty} A_{kh} z^k z_j^h - \sum_{j=1}^{n} a_j \sum_{p=1}^{\infty} \frac{1}{p}\left(\frac{z_j}{z}\right)^p +$$

$$+ \sum_{q=1}^{m} \beta_q \sum_{k,h=0}^{\infty} \lambda_{kh} z^k \zeta_q^h + K.$$

K is a constant which depends on the way in which logarithms are chosen, it disappears in the subsequent operation.

$$\frac{1}{2i} \int_{\Gamma_r} \overline{\phi(w)}\, d\phi(w) = \frac{1}{2i} \int_0^{2\pi} \overline{\phi(f(re^{i\theta}))}\, d\phi(f(re^{i\theta}))$$

$$= \pi \left\{ \sum_{k=1}^{\infty} k \left| \sum_{j=1}^{n} a_j \sum_{h=0}^{\infty} A_{kh} z_j^h + \sum_{q=1}^{m} \beta_q \sum_{h=0}^{\infty} \lambda_{kh} \zeta_q^h \right|^2 r^{2k} - \right. \tag{2.9}$$

$$\left. - \sum_{p=1}^{\infty} \frac{1}{p} \left| \sum_{j=1}^{n} a_j z_j^p \right|^2 r^{2p} \right\}.$$

Finally, on Ω_ϱ, $w = g(z)$ for $z = \varrho e^{i\theta}$, therefore

$$\phi(w) = \sum_{j=1}^{n} a_j \log(g(z) - f(z_j)) + \sum_{q=1}^{m} \beta_q \log(g(z) - g(\zeta_q))$$

$$= \sum_{j=1}^{\infty} a_j \log\left(1 - \frac{f(z_j)}{g(z)}\right) + \sum_{q=1}^{m} \beta_q \log\left(\frac{g(z)-g(\zeta_q)}{z - \zeta_q}\right) + \sum_{q=1}^{m} \beta_q \log(z - \zeta_q) + K$$

$$= \sum_{h=1}^{\infty} \left\{ \sum_{j=1}^{n} a_j \sum_{k=1}^{\infty} \lambda_{kh} z_j^k + \sum_{q=1}^{m} \beta_q \sum_{k=1}^{\infty} B_{hk} \zeta_q^k \right\} - \sum_{p=1}^{\infty} \left\{ \frac{1}{p} \sum_{q=1}^{m} \beta_q \zeta_q^p \right\} z^{-p} + K;$$

and consequently

$$\frac{1}{2i}\int_{\Omega_\varrho}\overline{\phi(w)}\,d\phi(w)=\frac{1}{2i}\int_0^{2\pi}\overline{\phi(g(\varrho e^{i\theta}))}\,d\phi(g(\varrho e^{i}))$$

$$(2.10)\qquad =\pi\Big\{\sum_{h=1}^{n} h\Big|\sum_{j=1}^{\infty}a_j\sum_{k=1}^{\infty}\lambda_{kh}z_j^k+\sum_{q=1}^{m}\beta_q\sum_{k=1}^{\infty}B_{hk}\zeta_q^k\Big|^2\varrho^{2h}$$
$$-\sum_{p=1}^{\infty}\frac{1}{p}\Big|\sum_{q=1}^{m}\beta_q\zeta_q^p\Big|^2\varrho^{-2p}\Big\}.$$

Now, combining (2.5), (2.7), (2.8), (2.9) and (2.10) gives

$$A_{r,\varrho}=\pi\Big\{-\sum_{k=1}^{\infty}k\Big|\sum_{j=1}^{n}a_j\sum_{h=1}^{\infty}A_{kh}z_j^k+\sum_{q=1}^{m}\beta_q\sum_{h=1}^{\infty}\lambda_{kh}\zeta_q^h\Big|^2r^{2k}$$
$$-\sum_{h=1}^{\infty}h\Big|\sum_{j=1}^{n}a_j\sum_{k=1}^{\infty}\lambda_{kh}z_j^k+\sum_{q=1}^{m}\beta_q\sum_{k=1}^{\infty}B_{hk}\zeta_q^k\Big|^2\varrho^{2h}$$
$$(2.11)\qquad +\sum_{p=1}^{\infty}\frac{1}{p}\Big|\sum_{j=1}^{n}a_jz_j^p\Big|^2r^{-2p}+\sum_{p=1}^{\infty}\frac{1}{p}\Big|\sum_{q=1}^{m}\beta_q\zeta_q^p\Big|^2\varrho^{-2p}$$
$$-\sum_{s=1}^{\infty}\frac{1}{s}\Big|\sum_{j=1}^{n}a_jf(z_j)^s+\sum_{q=1}^{m}\beta_qg(\zeta_q)^s\Big|^2\Big\}>0.$$

Then letting $r\to1$ and $\varrho\to1$ and observing that

$$(2.12)\qquad \sum_{p=1}^{\infty}\frac{1}{p}\Big|\sum_{j=1}^{n}a_jz_j^p\Big|^2=-\sum_{j=1}^{n}\sum_{h=1}^{n}a_j\overline{a_h}\log(1-z_j\overline{z_h}),$$

we see that (2.11) implies the conclusion of the theorem, (2.3).

Corollary 1.1. *If along with the conditions of Theorem 1 we suppose that* x_k *and* y_j *are generic points in* Δ, *for* $k=1,2,\ldots,N$ *and* $j=1,2,\ldots,M$, *and* $\sum_{j=1}^{M}\delta_j=\sum_{j=1}^{M}\varepsilon_k=0$ *for arbitrary complex numbers* δ_k *and* ε_j, *then*

$$\Big|\sum_{j=1}^{n}\sum_{h=1}^{N}a_j\delta_h\log\Big\{\frac{f(x_h)-f(\zeta_j)}{x_h-\zeta_j}\Big\}+\sum_{q=1}^{m}\sum_{h=1}^{N}\beta_q\delta_h\log\Big\{1-\frac{f(x_h)}{g(\zeta_q)}\Big\}\Big|^2$$
$$-\sum_{h,l=1}^{M}\delta_h\overline{\delta_l}\log(1-x_h\overline{x_l})$$

$$+\frac{\left|\sum_{j=1}^{n}\sum_{h=1}^{M}a_j\varepsilon_h\log\left\{1-\frac{f(z_j)}{g(y_k)}\right\}+\sum_{q=1}^{m}\sum_{h=1}^{M}\beta_q\varepsilon_h\log\left\{\frac{g(y_k)-g(\zeta_q)}{y_h-\zeta_q}\right\}\right|^2}{-\sum_{h,l=1}^{M}\varepsilon_h\bar{\varepsilon}_l\log(1-y_h\bar{y}_l)}$$

(2.13)

$$+\sum_{s=1}^{\infty}\frac{1}{s}\left|\sum_{j=1}^{n}a_jf(z_j)^s+\sum_{q=1}^{m}\beta_qg(\zeta_q)^s\right|^2$$

$$\leqslant-\sum_{j,h=1}^{n}a_j\bar{a}_h\log(1-z_j\bar{z}_h)-\sum_{j,h=1}^{m}\beta_j\bar{\beta}_h\log(1-\zeta_j\bar{\zeta}_h).$$

This is obtained by showing that the first two terms in (2.13) do not exceed those in (2.3). Making use of the Cauchy-Schwarz inequality we see that

$$\left|\sum_{j=1}^{n}\sum_{h=1}^{N}a_j\delta_h\log\left\{\frac{f(x_h)-f(\zeta_j)}{x_h-\zeta_j}\right\}+\sum_{q=1}^{m}\sum_{h=1}^{N}\beta_q\delta_h\log\left\{1-\frac{f(x_h)}{g(\zeta_q)}\right\}\right|^2$$

$$=\left|\sum_{j=1}^{n}\sum_{h=1}^{N}a_j\delta_h\sum_{k,s=1}^{\infty}A_{ks}x_h^kz_j^s+\sum_{q=1}^{m}\sum_{h=1}^{N}\beta_q\delta_h\sum_{k,s=1}^{\infty}\lambda_{ks}x_h^k\zeta_q^s\right|^2$$

(2.14)

$$=\left|\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{1/2}\left(\sum_{h=1}^{N}\delta_hx_h^k\right)\cdot k^{1/2}\left\{\sum_{j=1}^{n}a_j\sum_{s=1}^{\infty}A_{ks}z_j^s+\sum_{q=1}^{m}\beta_q\sum_{s=1}^{\infty}\lambda_{ks}\zeta_q^s\right\}\right|^2$$

$$\leqslant\sum_{k=1}^{\infty}\frac{1}{k}\left|\sum_{k=1}^{N}\delta_hx_h^k\right|^2\cdot\sum_{k=1}^{\infty}k\left|\sum_{j=1}^{n}a_j\sum_{s=1}^{\infty}A_{ks}z_j^s+\sum_{q=1}^{m}\beta_q\sum_{s=1}^{\infty}\lambda_{ks}\zeta_q^s\right|^2$$

$$=-\sum_{h,l=1}^{N}\delta_h\bar{\delta}_l\log(1-x_h\bar{x}_l)\cdot\sum_{k=1}^{\infty}k\left|\sum_{j=1}^{n}a_j\sum_{s=1}^{\infty}A_{ks}z_j^s+\sum_{q=1}^{m}\beta_q\sum_{s=1}^{\infty}\lambda_{ks}\zeta_q^s\right|^2.$$

A similar calculation enables us to compare the second terms of (2.3) and (2.13), respectively.

For specific choices of the parameters appearing in (2.13) we may obtain inequalities which are more easily interpreted. For example, choosing $m=n=N=M=2$, $a_1=\delta_1=\varepsilon_1=\beta_1=1$, $a_2=\beta_2=\delta_2=\varepsilon_2=-1$, $x_2=\zeta_2=z_2=y_2=0$, $x_1=\zeta_1=z$ and $z_1=y_1=\zeta$, gives

$$\frac{\left|\log\left\{\frac{z^2f'(z)f'(0)}{(f(z)-f(0))^2}\right\}+\log\left\{\frac{(g(z)-f(z))(g(0)-f(0))}{(g(z)-f(0))(g(0)-f(z))}\right\}\right|^2}{-\log(1-|z|^2)}$$

$$+\frac{\left|\log\left\{\frac{(g(\zeta)-f(\zeta))(g(0)-f(0))}{(g(\zeta)-f(0))(g(0)-f(\zeta))}\right\}+\log\left\{\frac{(g(z)-g(\zeta))}{(g(z)-g(0))}\left(\frac{\zeta z}{z-\zeta}\right)\frac{g'(0)}{(g(\zeta)-g(0))}\right\}\right|^2}{-\log(1-|\zeta|^2)}$$

(2.15)

$$+\sum_{s=1}^{\infty}\frac{1}{s}\left|\left(f(\zeta)^s-f(0)^s\right)+\left(g(z)^s-g(0)^s\right)\right|^2\leqslant-\log(1-|\zeta|^2)-\log(1-|z|^2).$$

Now if the first term in (2.15) is dropped and we let $f(0)=0$ and $z=0$, then we can conclude that

$$\left|\log\left\{\frac{g(\zeta)-f(\zeta)}{g(0)-f(\zeta)}\cdot\frac{g(0)}{g(\zeta)}\right\}\right|^2\leqslant\log\left(\frac{1-|f(\zeta)|^2}{1-|\zeta|^2}\right)\cdot\log\left(\frac{1}{1-|\zeta|^2}\right) \tag{2.16}$$

for any ζ in Δ. Taking the square root of both sides in (2.16) and making use of the relation $2ab\leqslant a^2+b^2$ which holds for any real numbers a and b, we obtain

$$\left|\log\left\{\frac{g(\zeta)-f(\zeta)}{g(0)-f(\zeta)}\cdot\frac{g(0)}{g(\zeta)}\right\}\right|\leqslant\log\left\{\frac{\sqrt{1-|f(\zeta)|^2}}{(1-|\zeta|^2)}\right\}. \tag{2.17}$$

In summary we have the following.

Corollary 1.2. *If $f(z)$ and $g(z)$ are univalent in Δ, $f[\Delta]$ and $g[\Delta]$ are non-intersecting and both contained in Δ, and $f(0)=0$, then for any ζ in Δ*

$$\frac{1-|\zeta|^2}{\sqrt{1-|f(\zeta)|^2}}\leqslant\left|\frac{g(\zeta)-f(\zeta)}{g(0)-f(\zeta)}\cdot\frac{g(0)}{g(\zeta)}\right|\leqslant\frac{\sqrt{1-|f(\zeta)|^2}}{1-|\zeta|^2} \tag{2.18}$$

and

$$\log\frac{1-|\zeta|^2}{\sqrt{1-|f(\zeta)|^2}}\leqslant\arg\left\{\frac{g(\zeta)-f(\zeta)}{g(0)-f(\zeta)}\cdot\frac{g(0)}{g(\zeta)}\right\}\leqslant\log\frac{\sqrt{1-|f(\zeta)|^2}}{1-|\zeta|^2}. \tag{2.19}$$

Different choices of terms and variables in (2.15) and of the parameters in (2.13) will yield other bounds on $f(z)$ and $g(z)$, however no further illustrations will be given here. It should be observed that some of the preceding methods and results may be extended to the case when there are more than two bounded univalent functions with non-overlapping images. The case where we have a single function is sufficiently important to be considered separately.

Theorem 2. *If $f(z)$ is regular and univalent in Δ and $f[\Delta]\subset\Delta$, then*

$$\sum_{k=1}^{\infty} k\left|\sum_{j=1}^{n} a_j \sum_{h=1}^{\infty} A_{kh} z_j^h\right|^2 \leqslant \sum_{j,h=1}^{n} a_j\overline{a_h}\log\left\{\frac{1-f(z_j)\overline{f(z_h)}}{1-z_j\overline{z_h}}\right\}, \tag{2.20}$$

where the A_{kh} are defined by (1.1), $z_j\in\Delta$ *for all j and the numbers a_j are restricted only by $\sum_{j=1}^{n} a_j = 0$.*

Using the notation defined above we let D_r be the region bounded by $\gamma\cup\Gamma_r$ with r chosen sufficiently large to insure that $w_j\notin D_r$, and $w_j = f(t_j)$ $j = 1, 2, \ldots, n$. Let

$$\psi'(w) = \sum_{j=1}^{n} a_j(w-w_j)^{-1}, \tag{2.21}$$

then

$$\begin{aligned}\iint_{D_r} |\psi'(w)|^2 dA &= \frac{1}{2i}\left[\int_{\gamma}\overline{\psi(w)}\,d\psi(w) - \int_{\Gamma_r}\overline{\psi(w)}\,d\psi(w)\right]\\ &= -\pi\left(\sum_{s=1}^{\infty}\frac{1}{s}\left|\sum_{j=1}^{n} a_j f(z_j)^s\right|^2\right) - \pi\left(\sum_{k=0}^{\infty} k\left|\sum_{j=1}^{\infty} a_j \sum_{h=0}^{\infty} A_{kh} z_j^h\right|^2 r^{2k}\right.\\ &\qquad \left. - \sum_{k=1}^{\infty}\frac{1}{k}\left|\sum_{j=1}^{n} a_j z_j^k\right|^2 r^{-2k}\right),\end{aligned} \tag{2.22}$$

which is non-negative. Letting $r\to 1$ and making use of (2.12) gives (2.20).

Choosing δ_h and x_h as in Corollary 1.1 we may write

$$\begin{aligned}&\left|\sum_{j=1}^{n}\sum_{h=1}^{N} a_j\delta_h \log\left\{\frac{f(x_h)-f(z_j)}{x_h-z_j}\right\}\right|^2\\ &= \left|\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{1/2}\left(\sum_{h=1}^{N}\delta_h x_h^k\right) k^{1/2}\left(\sum_{j=1}^{n} a_j \sum_{p=1}^{\infty} A_{kp} z_j^p\right)\right|^2\\ &\leqslant \sum_{k=1}^{\infty}\frac{1}{k}\left|\sum_{h=1}^{N}\delta_h x_h^k\right|^2 \cdot \sum_{k=1}^{\infty} k\left|\sum_{j=1}^{n} a_j \sum_{k=1}^{\infty} A_{kh} z_j^h\right|^2,\end{aligned} \tag{2.23}$$

having made use of the Cauchy-Schwarz inequality. Using (2.12), (2.20) nad (2.23) give the following results.

Corollary 2.1. *If in addition to the conditions of Theorem 2, numbers x_j and δ_j, $j = 1, 2, \ldots, N$ are chosen so that $x_j\in\Delta$ for each j and $\sum_{j=1}^{N}\delta_j = 0$,* *then*

$$\left|\sum_{j=1}^{n}\sum_{h=1}^{N} a_j\,\delta_h \log\left(\frac{f(x_h)-f(z_j)}{x_h-z_j}\right)\right|^2$$

$$(2.24)\qquad \leq \left\{\sum_{j=h=1}^{N}\delta_j\,\overline{\delta_h}\log\left(\frac{1}{1-x_j\overline{x_h}}\right)\right\}\left\{\sum_{j=h=1}^{n} a_j\,\overline{a_h}\log\left(\frac{1-f(z_j)\overline{f(z_h)}}{1-z_j\overline{z_h}}\right)\right\}.$$

Choosing $n = N = 2$, $x_1 = z_1 = z$, $x_2 = z_2 = 0$, $a_1 = \delta_1 = 1$ and $\delta_2 = a_2 = -1$ in (2.24) yields

$$\left|\log\left\{\frac{z^2 f'(z)f'(0)}{[f(z)-f(0)]^2}\right\}\right|^2 \leq \log\left(\frac{1}{1-|z|^2}\right)\log\left\{\frac{1-|f(z)|^2}{1-|z|^2}\cdot\frac{1-|f(0)|^2}{|1-f(z)\overline{f(0)}|^2}\right\}.$$

If for simplicity, we let $f(0) = 0$ and again make use of the relation $2ab \leq a^2+b^2$, then

$$(2.25)\qquad \left|\log\left(\frac{z^2 f'(z)f'(0)}{f(z)^2}\right)\right| \leq \log\frac{\sqrt{1-|f(z)|^2}}{1-|z|^2}.$$

Separating real and imaginary parts on the left of (2.25) gives the following interesting bounds.

Corollary 2.2. *If $f(z)$ is univalent in Δ, $f(0) = 0$ and $f[\Delta] \subset \Delta$, then for any z in Δ*

$$(2.26)\qquad \log\frac{1-|z|^2}{\sqrt{1-|f(z)|^2}} \leq \arg\left(\frac{z^2 f'(z)f'(0)}{f(z)^2}\right) \leq \log\frac{\sqrt{1-|f(z)|^2}}{1-|z|^2},$$

and

$$(2.27)\qquad \left|\frac{f(z)}{z}\right|^2\frac{1-|z|^2}{\sqrt{1-|f(z)|^2}} \leq |f'(z)f'(0)| \leq \left|\frac{f(z)}{z}\right|^2\frac{\sqrt{1-|f(z)|^2}}{1-|z|^2}.$$

These results can be generalized easily by choosing $x_2 = z_2 = \zeta$ in the substitutions following Corollary 2.1 and in this case (2.25) is replaced by an inequality which relates the values of $f(z)$ and its derivative at two distinct points of the disk Δ. Choosing values of n and N exceeding 2 or non-real values for the a_j's and δ_k's in (2.24) will of course give new inequalities, they are, however, extremely cumbersome and difficult to interpret.

3. A Second Method

In this section we apply the method of Section 2 replacing γ as a component of the boundary of the region of integration with the image of γ_r under $1/f(z)$, when $f(0) = 0$. The calculations are similar to those above.

Theorem 3. *If $f(z)$ and $g(z)$ are regular and univalent in Δ, $f[\Delta]\subset\Delta$; $g[\Delta]\subset\Delta$; $f[\Delta]\cap g[\Delta]=\Phi$; $f(0)=0$; (1.1) and (2.1) hold;*

$$\log(1-f(z)g(\zeta)) = \sum_{\substack{k=1\\h=0}}^{\infty} u_{kh} z^k \zeta^h; \tag{3.1}$$

and, for $k=1,2,\ldots,n$ and $j=1,2,\ldots,m$, z_k and ζ_j are in Δ and a_k and β_j are complex numbers such that $\sum_{j=1}^{m}\beta_j=0$; then

$$\sum_{k=1}^{\infty} k\left|\sum_{j=1}^{n} a_j \sum_{h=1}^{\infty} A_{kh} z_j^h + \sum_{q=1}^{m}\beta_q \sum_{h=1}^{\infty} u_{kh}\zeta_q^h\right|^2 + \tag{3.2}$$
$$+\sum_{k=1}^{\infty} k\left|\sum_{j=1}^{n} a_j \sum_{h=1}^{\infty} u_{hk} z_j^h + \sum_{q=1}^{m}\beta_q \sum_{h=1}^{\infty} b_{kh}\zeta_q^{h}\right|^2$$
$$\leqslant \sum_{j,h=1}^{n} a_j\bar{a}_h \log\left(\frac{1}{1-z_j\bar{z}_h}\right) + \sum_{,h=1}^{m}\beta_j\bar{\beta}_h\log\left(\frac{1}{1-\zeta_j\bar{\zeta}_h}\right).$$

The method of proof is similar to that of Theorem 1. Let Γ_r^* be the closed Jordan curve defined by the set of points $1/f(re^{i\theta})$, $0\leqslant\theta\leqslant 2\pi$, which has positive direction with respect to its own interior; and let A_r be the annular region bounded by $\Gamma_r^*\cup\Omega_r$, $0<r<1$, $\Omega_r=g[\gamma_r]$. Choose r sufficiently large so that

$$\chi'(w) = \sum_{j=1}^{n}\frac{-a_j w_j}{(1-ww_j)} + \sum_{q=1}^{m}\frac{\beta_q}{(w-\omega_q)},$$

$w_j=f(z_j)$, $j=1,2,\ldots,n$ and $\omega_q=g(\zeta_q)$, $q=1,2,\ldots,m$ is regular in A_r; this can be done since $\chi'(w)$ has only simple poles at the points $1/f(z_j)$ and $g(\zeta_q)$. Letting

$$\chi(w) = \sum_{j=1}^{n} a_j\log(1-ww_j) + \sum_{q=1}^{M}\beta_q\log(w-\omega_q),$$

we have

$$\iint_{A_r} |\chi'(w)|^2 dA = \frac{1}{2i}\left\{\int_{\Gamma_r^*}\overline{\chi(w)}\,d\chi(w) - \int_{\Omega_r}\overline{\chi(w)}\,d\chi(w)\right\}.$$

Carrying out these calculations as was done in Section 2 gives (3.2).

Comparison of (2.3) and (3.2) suggests that (2.3) may be stronger due to the presence of the additional term on the left side of (2.3). However, the significant feature of (3.2) is that the numbers a_j may be chosen without restriction, whereas in (2.3) their sum must be zero. This leads, for example, to bounds on the Grunsky-type coefficients defined in (1.1) and (3.1). Choosing $\zeta_q = 0$ all q, $a_1 = 1$, $z_1 = z$ and $n = 1$ in (3.2) yields the following.

Corollary 3.1. *If the conditions of Theorem 3 are fulfilled, then for z in Δ*

$$\sum_{k=1}^{\infty} k\left(\left|\sum_{h=1}^{\infty} A_{kh} z^h\right|^2 + \left|\sum_{h=1}^{\infty} u_{kh} z^h\right|^2\right) \leqslant \log\left(\frac{1}{1-|z|^2}\right). \tag{3.3}$$

Again choosing $\zeta_q = 0$ in (3.1) and using the methods of Section 2, we can derive inequalities like those in (2.13).

Corollary 3.2. *If along with the conditions of Theorem 3 we have $\sum_{j=1}^{n} a_j = \sum_{h=1}^{N} \delta_h = 0$, for complex numbers a_j and δ_h, and x_h is in Δ, for $h = 1, 2, \ldots N$, then*

$$\left|\sum_{j=1}^{n}\sum_{h=1}^{N} a_j \delta_h \log\left\{\frac{f(x_h)-f(z_j)}{x_h - z_j}\right\}\right|^2 + \left|\sum_{j=1}^{n}\sum_{h=1}^{N} a_j \delta_h \log\left(1-f(x_h)g(z_j)\right)\right|^2 \leqslant \sum_{h,l=1}^{N} \delta_h \bar{\delta}_l \log\left(\frac{1}{1-x_h\bar{x}_l}\right) \cdot \sum_{h,j=1}^{n} a_j \bar{a}_h \log\left(\frac{1}{1-z_j\bar{z}_h}\right). \tag{3.4}$$

Now, choosing $a_1 = \delta_1 = 1$, $a_2 = \delta_2 = -1$, $z_1 = x_1 = z$, $z_2 = x_2 = 0$ and $n = N = 2$ in (3.4) gives

$$\left|\log\left(\frac{z^2 f'(0) f'(z)}{f(z)^2}\right)\right|^2 + \left|\log\left(\frac{1-f(z)g(z)}{1-f(z)g(0)}\right)\right|^2 \leqslant \left(\log(1-|z|^2)\right)^2 \tag{3.5}$$

This can be reduced further to give bounds like those in (2.18) and (2.19).

In conclusion we will apply the method of the last theorem to the case of a single univalent function.

Theorem 4. *If $f(z)$ is univalent in Δ, $f(0) = 0$, $f[\Delta] \subset \Delta$, and (1.1) holds along with*

$$\log\left(1-f(z)f(\zeta)\right) = \sum_{k,h=0}^{\infty} v_{kh} z^k \zeta^h, \tag{3.6}$$

then

$$\sum_{k=1}^{\infty} k\left|\sum_{j=1}^{n} a_j \sum_{h=1}^{\infty} v_{kh} z_j^h + \sum_{q=1}^{m} \beta_q \sum_{h=1}^{\infty} A_{kh} \zeta_q^h\right|^2 + \sum_{k=1}^{\infty} k\left|\sum_{j=1}^{n} a_j \sum_{h=1}^{\infty} A_{kh} z_j^h + \sum_{q=1}^{m} \beta_q \sum_{h=1}^{\infty} v_{kh} \zeta_q^h\right|^2 \leqslant \sum_{j,k=1}^{n} a_j \bar{a}_k \log\left(\frac{1}{1-z_j\bar{z}_k}\right) + \sum_{j,k=1}^{m} \beta_j \bar{\beta}_k \log\left(\frac{1}{1-\zeta_j\bar{\zeta}_k}\right), \tag{3.7}$$

for complex numbers a_j *and* β_j *such that* $\sum_{j=1}^{n} a_j = 0$ *and* z_j *and* ζ_q *are in* Δ.

Using the notation defined above we let B_r be the annular region bounded by $\Gamma_r \cup \Gamma_r^*$ for $0 < r < 1$. And, let $w_j = f(z_j)$, $j = 1, 2, \ldots, n$ and $W_j = f(\zeta_j)$, $j = 1, 2, \ldots, m$. Then for r sufficiently large

$$\Lambda'(w) = \sum_{j=1}^{n} a_j (w - w_j)^{-1} + \sum_{q=1}^{m} \beta_q (1 - wW_q)^{-1} \text{ and}$$

$$\Lambda(w) = \sum_{j=1}^{n} a_j \log(w - w_j) + \sum_{q=1}^{m} \beta_q \log(1 - wW_q)$$

are regular in B_r and

$$\int\int_{\Delta_r} |\Lambda'(w)|^2 dA = \frac{1}{2i}\Big[\int_{\Gamma_r^*} \overline{\Lambda(w)}\, d\Lambda(w) - \int_{\Gamma_r} \overline{\Lambda(w)}\, d\Lambda(w)\Big]$$

is non-negative. Carrying through the calculations as above gives (3.7).

To illustrate an application of (3.7) we first choose $\zeta_q = 0$ for all q in (3.7); this gives

$$\sum_{k=1}^{\infty} k \Big|\sum_{j=1}^{n} a_j \sum_{q=1}^{\infty} v_{kq} z_j^q\Big|^2 + \sum_{k=1}^{\infty} k \Big|\sum_{j=1}^{n} a_j \sum_{q=1}^{\infty} A_{kq} z_j^q\Big|^2 \leqslant \sum_{j,k=1}^{n} a_j \bar{a}_k \log\left(\frac{1}{1 - z_j \bar{z}_k}\right). \tag{3.8}$$

Making calculations similar to (2.12) and (2.14) we can obtain the following from (3.8).

Corollary 4.1. *If along with the conditions of Theorem* 4 *we assume that* $x_k \in \Delta$, $k = 1, 2, \ldots, N$ *and* $\sum_{k=1}^{N} \delta_k = 0$ *for generic complex numbers* δ_k, *then*

$$\Big|\sum_{j=1}^{n} \sum_{k=1}^{N} a_j \delta_k \log\Big\{\frac{f(x_k) - f(z_j)}{x_k - z_j}\Big\}\Big|^2 + \Big|\sum_{j=1}^{n} \sum_{k=1}^{N} a_j \delta_k \log\{1 - f(x_k) f(z_j)\}\Big|^2 \leqslant \sum_{k,j=1}^{N} \delta_k \bar{\delta}_j \log(1 - x_k \bar{x}_j) \cdot \sum_{k,j=1}^{n} a_k \bar{a}_j \log(1 - z_k \bar{z}_j). \tag{3.9}$$

As an example, we choose $n = N = 2$, $a_1 = \delta_1 = 1$, $x_1 = z_1 = z$ and $x_2 = z_2 = 0$. The result can be rephrased as follows.

Corollary 4.2. *If $f(z)$ is regular and univalent in Δ, $f(0) = 0$ and $|f(z)| < 1$, $z \in \Delta$, then, for z in Δ,*

$$\left|\log\left\{\frac{z^2 f'(0) f'(z)}{f(z)^2}\right\}\right|^2 + |\log\{1 - f(z)^2\}|^2 \leqslant \{\log(1-|z|^2)\}^2. \tag{3.10}$$

Dropping the first term on the left of (3.10) reduces it to the Schwarz Lemma, whereas dropping the next term gives

$$\left|\frac{z^2 f'(0) f'(z)}{f(z)^2}\right| \leqslant \frac{1}{1-|z|^2},$$

or

$$\left|\frac{zf'(z)}{f(z)}\right| \leqslant \frac{|f(z)|}{|zf'(0)(1-|z|^2)|}.$$

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STRESZCZENIE

W pracy tej stosuje się metodę nierówności Grunsky'ego do par funkcji jednolistnych i ograniczonych, mających rozłączne zbiory przyjmowanych wartości. Otrzymano w ten sposób pewne nowe oszacowania dla funkcji jednolistnych i ograniczonych.

РЕЗЮМЕ

В настоящей работе применяется метод неравенств Грунского для пар однолистных и ограниченных, которые принимают значения из непересекающихся областей.

Получены некоторые новые оценки для однолистных и ограниченных функций.