ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA
VOL. XXVI, 1 SECTIO A 1972

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Geometric Interpretation of the π-Geodesics

Interpretacja geometryczna π-geodetyk

Геометрическая интерпретация π-геодезических

In [3] K. Radziszewski has given the definition of the π-geodesic in the n-dimensional space with the affine connection and with a given tensor π. This paper deals with π-geodesics on a surface $S \subset E_3$ determined by tensors associated in a natural way with the surface and it gives their interpretation by means of a parallel displacement.

Analogously to the definition of the projective surface [2] we introduce the definition of the π-projective surface and deal with mappings that map the π_1-geodesics on the surface S_1 into the π_2-geodesics on the surface S_2. We start with some definitions and notations.

Let S be the surface in the Euclidean space E_3 given in the local chart U:

$$\bar{x}\colon (u^1, u^2) \to \bar{x}(u^1, u^2), \; u = (u^1, u^2) \in D$$

where $\bar{x}(u^1, u^2)$ is the radius vector of a point $X(u^1, u^2) \in E_3$ and D is the domain in $R \times R$ (R — the set of real numbers). Let g_{ij} denote components of the metric tensor g of the surface S in the local chart U or more precisely:

$$g_{X(u)}\colon (\bar{v}_{X(u)}, \bar{w}_{X(u)}) \to \bar{v}_{X(u)} \bar{w}_{X(u)}$$
$$= g_{ij}(u) v^i_{X(u)} w^j_{X(u)}, \; \bar{v}_{X(u)} = v^i_{X(u)} \bar{x}_i(u) \in T_{X(u)}$$
$$\bar{w}_{X(u)} = w^i_{X(u)} \bar{x}_i(u) \in T_{X(u)}$$

where $T_{X(u)}$ is tangent vector space to S at the point $X(u)$.

$$\bar{x}_i(u) = \frac{\partial \bar{x}(u)}{\partial u^i}, \; g_{ij}(u) = \bar{x}_i(u) \bar{x}_j(u),$$

$$g\colon X(u) \to g_{X(u)} = g(u), \; g_{ij}\colon X(u) \to g_{ij}(u)$$

Let $\pi\colon X\to\pi_x$, $\pi_{X(u)}\colon (\bar v_{X(u)}, \bar w_{X(u)})\to\pi_{ij}(u)v^i_{Xu}\, w^j_{X(u)}$ be a tensor of the type (0, 2) on S and let $\pi_{ij}\colon X(u)\to\pi_{ij}(u)$ be components of π in U. A covariant derivative of the function π_{ij} with respect to g is denoted by $\nabla_r\pi_{ij}$.

If $\bar v\colon X\to\bar v_X\in T_X$, $X\in S$ is a vector field on S ($\bar v = v^i\bar x_i$ in U), then the functions $\pi_i^v = \pi_{ij}v^j$ are the components of the covector π^v in U. The symbol $\nabla_v\pi_i^w = \nabla_r\pi_i^w v^r$ denotes the value of the covariant differential $D\pi_i^w$ of the components π_i^w of the tensor π^w on the field $\bar v$.

The tensor π is called non-singular if $\det(\pi_{ij})\neq 0$ at each local chart U.

Definition 1 [3]. A vector field $\bar w$ on the surface S:

$$\bar x\colon (u^1, u^2)\to\bar x(u^1, u^2)$$

is said to be π-geodesic, if:

$$\nabla_{\bar w}\pi_i^w = \lambda\pi_i^w, \tag{1}$$

where $\lambda\in F(S)$ and π is non-singular. ($F(S)$ denotes a set of differentiable functions defined on S).

The integral curves of π-geodesic vector field on S are called π-geodesic lines.

This definition is equivalent (in the 2-dimensional case) to the following:

Definition 1′. A vector field $\bar w$ on S is said to be π-geodesic, if there exists such vector field $\bar v$ ($\bar w\neq\bar v$) on S that:

$$\pi_i^w v^i = 0 \text{ and } \nabla_{\bar w}\pi_i^w v^i = 0,\ i = 1, 2, \bar v\neq 0. \tag{2}$$

Let's write the equation (1) in the extended form. If we get rid of λ, then we obtain:

$$\pi_k^w\nabla_w\pi_i^w - \pi_i^w\nabla_w\pi_k^w = 0 \text{ or if } w^j = \frac{du^j}{dt} \tag{3}$$

$$\pi_{ij}\frac{d^2u^j}{dt^2} + (\nabla_k\pi_{is} + \pi_{ij}\Gamma^j_{ks})\frac{du^k}{dt}\frac{du^s}{dt} = \lambda\pi_{ij}\frac{du^j}{dt} \tag{4}$$

The equation (1) can be expressed:

$$\nabla_{\bar w}(\pi_{ij}w^j) = \lambda\pi_{ij}w^j, \text{ where } \bar w = w^j\bar x_j.$$

Multiplying both sides of this equation by g^{ik} (an inverse tensor to the metric tensor g_{ik}) and setting $w^i = \dfrac{du^i}{dt}$ we obtain the equivalent equation:

$$\nabla_{\bar w}\left(\pi_{ij}g^{ik}\frac{du^j}{dt}\right) = \lambda\pi_{ij}g^{ik}\frac{du^j}{dt} \tag{5}$$

which constitutes the necessary and sufficient condition for the existence of a vector $\bar{a}(t)$ having the direction of the vector:

(6) $$\bar{v} = \pi_{ij} g^{ik} \frac{du^j}{dt} \bar{x}_k$$

and simultaneously for the vector $\bar{a}(t)$ to be displaced parallel along the curve

$$\bar{x}: t \to \bar{x}(u^1(t), u^2(t))$$

We shall now deal with π-geodesic that are determined by the tensors associated in a natural way with a surface. Consider now the tensor of the form:

(7) $$h_{ij} = \alpha b_{ij} + \beta g_{ij} \ (\alpha, \beta - \text{scalar functions})$$

Then the vector (6) takes the form:

$$\bar{h} = (\alpha b_{ij} + \beta g_{ij}) g^{ik} \frac{du^j}{dt} \bar{x}_k$$

$$= \alpha b_{ij} g^{ik} \bar{x}_k \frac{du^j}{dt} + \beta g_{ij} g^{ik} \bar{x}_k \frac{du^j}{dt}$$

$$= -\alpha N_i \frac{du^i}{dt} + \beta \delta_j^k \bar{x}_k \frac{du^j}{dt}$$

(8) $$\bar{h} = \beta \frac{d\bar{x}}{dt} - \alpha \frac{d\bar{N}}{dt} \text{ where } \bar{N} = \frac{\bar{x}_1 \times \bar{x}_2}{|\bar{x}_1 \times \bar{x}_2|}$$

$$\frac{d\bar{x}}{dt} = \bar{x}_i \frac{du^i}{dt}, \frac{d\bar{N}}{dt} = \bar{N}_i \frac{du^i}{dt}$$

$$\bar{N}_i = -b_{ik} g^{kp} \bar{x}_p$$

and we get the following:

Theorem 1. *The necessary and sufficient condition for the curve Γ: $\bar{x}: t \to \bar{x}(u^1(t), u^2(t))$ on the surface S: $\bar{x}: (u^1, u^2) \to \bar{x}(u^1, u^2)$ to be h-geodesic (i.e. the integral curve of h-geodesic field, $h_{ij} = \alpha b_{ij} + \beta g_{ij}$) is the existence of a vector having the direction of the vector* (8) *and which is displaced parallel along this curve.*

Using the Bonnet – Kowalewski formulas:

(9) $$\frac{d\bar{t}}{ds} = k_g \bar{B} + k_n \bar{N}$$
$$\frac{d\bar{B}}{ds} = -k_g \bar{t} + \tau_g \bar{N}$$
$$\frac{d\bar{N}}{ds} = -k_n \bar{t} - \tau_g \bar{B}$$

where $\bar{t} = \frac{d\bar{x}}{ds}$, $\bar{B} = \bar{N} \times \bar{t}$

k_g — the geodesic curvature

τ_g — the geodesic torsion

k_n — the normal curvature,

the vector $\bar{h}$ can be expressed in the following form:

$$\bar{h} = \beta \frac{d\bar{x}}{dt} - a \frac{d\bar{N}}{dt} = \left(\beta \frac{d\bar{x}}{ds} - a \frac{d\bar{N}}{ds}\right) \frac{ds}{dt} = (\beta \bar{t} + a k_n \bar{t} + a \tau_g \bar{B}) \frac{ds}{dt}$$

$$= \frac{ds}{dt}\left((\beta + a k_n)\bar{t} + a \tau_g \bar{B}\right)$$

Let $\bar{h}' = \frac{\bar{h}}{|\bar{h}|}$, then

$$\bar{h}' = \frac{(\beta + a k_n)\bar{t} + a \tau_g \bar{B}}{\sqrt{(\beta + a k_n)^2 + (a \tau_g)^2}} \tag{10}$$

Now we can state:

Theorem 1′. *The necessary and sufficient condition for a curve Γ on a surface S to be h-geodesic (determined by the tensor $h_{ij} = ab_{ij} + \beta g_{ij}$) is that, the vector:*

$$\bar{h}' = \frac{(\beta + a k_n)\bar{t} + a \tau_g \bar{B}}{\sqrt{(\beta + a k_n)^2 + (a \tau_g)^2}}$$

be displaced parallel along Γ.

From the equation (3) of the π-geodesic line it follows, that if $\hat{\pi} = \lambda \pi$, then π-geodesics and $\hat{\pi}$-geodesics are the same curves, where $0 \neq \lambda \in F(S)$ Put in (7) $a = 0$, $\hat{g}_{ij} = \beta g_{ij}$. Then the vector (10) takes the form: $\bar{v} = \bar{t}$, or g-geodesic is a geodesic in the usual sense; in particular we can state:

Theorem 2. *The Ricci tensor $R = Kg$ $(K \neq 0)$ determines the R-geodesic being the geodesic in the usual sense.*

Let's put $a = 2H$ and $\beta = -K$ in (7), then the tensor (7) becomes the third fundamental tensor of the surface S:

$$\gamma_{ij} = 2Hb_{ij} - Kg_{ij},$$

where H is the mean curvature and K is the Gaussian curvature of the surface S. The vector (10) takes the form:

$$\bar{w} = \frac{(2Hk_n - K)\bar{t} + 2H\tau_g \bar{B}}{\sqrt{(2Hk_n - K)^2 + 4H^2\tau_g^2}}$$

hence, we get:

Theorem 3. *The necessary and sufficient condition for a curve Γ to be γ-geodesic line ($\gamma_{ij} = 2Hb_{ij} - Kg_{ij}$) provided that γ is non-singular, is that, the vector:*

$$\overline{w} = \frac{(2Hk_n - K)\bar{t} + 2H\tau_g\overline{B}}{\sqrt{(2Hk_n - K)^2 + 4H^2\tau_g^2}}$$

be displaced parallel along Γ.

If we put $\beta = 0$ in (7), we'll get $\hat{b}_{ij} = ab_{ij}$ and then the vector (8) is given by:

$$\overline{v} = -a\frac{d\overline{N}}{dt}$$

hence, we get:

Theorem 4. *The necessary and sufficient condition for a curve Γ on a surface S to be b-geodesic (i.e. determined by the tensor $\hat{b}_{ij} = ab_{ij}$ and provided that b is non-singular) is that, there exists a vector $\overline{u}$ having a direction of the vector $\frac{d\overline{N}}{dt}$ and displaced parallel along Γ;*

or equivalent:

Theorem 4′. *The necessary and sufficient condition for a curve Γ on S to be b-geodesic is that, the vector:*

$$\overline{v} = \frac{k_n\bar{t} + \tau_g\overline{B}}{\sqrt{k_n^2 + \tau_g^2}} \text{ (in (10) we put } \beta = 0)$$

be displaced parallel along Γ.

Definition 2. A curve Γ on a surface S is said to be a line of shadow if there exists a vector field $\overline{v} \neq \overline{w} = \frac{du^j}{dt}\bar{x}_j$ defined on Γ such that: $d_{\overline{w}}\overline{v} = 0$ and $\nabla_{\overline{w}}\overline{v} = 0$, where $d_{\overline{w}}\overline{v}$ denotes $\partial_i\overline{v}w^i$.

This definition means that the line of shadow Γ on S is such a curve that there exists a vector field $\overline{v}$ defined on Γ which is tangent to S, but is not tangent to this line and is displaced parallel along Γ and simultaneously in E_3, what means that $\overline{v}$ is constant in E_3. If $\overline{v}$ is displaced parallel in E_3, then it defines generating lines of a cylindrical surface W which is tangent to S along a line of shadow. This property allows us to define a line of shadow as a curve Γ on S such, that there exists some cylindrical surface which is tangent to S along Γ what justifies the name for these lines. Observe that, if we neglected the condition $\overline{v} \neq \overline{w}$, then every straight line on S would be a line of shadow (of course, if there exists a straight line on S). It is easy to see that a vector field $\overline{w}$ satisfying the following conditions:

(2′) a) $\pi^w_i v^i = 0$ ($\overline{w}$ and $\overline{v}$ are π-conjugate) and

b) $\nabla_{\overline{w}} v^i = \lambda v^i \quad \lambda \in F(S)$

is π-geodesic vector field.

In particular, if π is the second fundamental tensor b of a surface S, then the conditions (2′) are equivalent to the condition (2), so we can state:

Definition 2′. A curve Γ on a surface $S \subset E_3$ is a line of shadow if there exists a vector field $\overline{v}$ defined on Γ which is conjugate to tangent vector to Γ and is displaced parallel along Γ.

From the definition 2′ it follows immediately:

Theorem 5 [3]. *An b-geodesic on a surface S, b-being the second fundamental tensor of S with* $\det b \neq 0$, *is its line of shadow and conversely.*

Now we shall express a vector field $\overline{v}$ defined on a line of shadow Γ: $\overline{x}\colon t \to \overline{x}(u^1(t), u^2(t))$ in an invariant form. Vectors of the field $\overline{v}$ satisfy at each point of Γ following conditions:

$$\overline{v}(u^1(t), u^2(t))\overline{N}(u^1(t), u^2(t)) = 0 \text{ and}$$

$$\overline{v}(u^1(t), u^2(t))\frac{d\overline{N}(u^1(t), u^2(t))}{dt} = 0,$$

hence, we get

$$\overline{v}(u^1(t), u^2(t)) = \lambda\left[\overline{N}(u^1(t), u^2(t))\frac{d\overline{N}(u^1(t), u^2(t))}{dt}\right]$$

Using formulas (9), we have:

$$\overline{v}(u^1(t), u^2(t)) = \frac{\tau_g\overline{t} - k_n\overline{B}}{\sqrt{\tau_g^2 + k_n^2}} \tag{11}$$

We get:

Theorem 6. *If Γ on S is a line of shadow, then the vector:*

$$\overline{v} = \frac{\tau_g\overline{t} - k_n\overline{B}}{\sqrt{\tau_g^2 + k_n^2}}$$

is constant vector in E_3 and conversely, if the vector (11) *is displaced parallel along Γ (in Levi-Civita sense), then Γ is a line of shadow and the vector* (11) *is constant vector in E_3.*

The second part of the theorem 6 one can obtain in the following way: If the vector (11) is displaced parallel along a curve Γ then the vector:

$$\hat{\bar{v}} = \frac{k_n \bar{t} + \tau_g \bar{B}}{\sqrt{k_n^2 + \tau_g^2}}$$

(according to the theorem 4′) is also displaced parallel along ($|\bar{v}| = |\hat{\bar{v}}|$, $\hat{\bar{v}} \perp \bar{v}$) and from this it follows that Γ is a line of shadow.

Corollaries

From the shape of the vector (11) it is easy to observe that, if the curve Γ is a curvature line (respectively an asymptotic line) then Γ is a line of shadow if and only if it is simultaneously a geodesic line. As the vector (11) is constant vector in E_3 and as the cases when $\tau_g = 0$ or $k_n = 0$ were considered, we can assume now that, $\tau_g \neq 0$ and $k_n \neq 0$, and then we get:

$$\frac{d\bar{v}}{ds} = 0, \text{ or}$$

$$\frac{d}{ds}\left(\frac{\tau_g \bar{t} - k_n \bar{B}}{\sqrt{\tau_g^2 + k_n^2}}\right) = 0$$

Denoting $\frac{d\tau_g}{ds} = \tau_g'$ and $\frac{dk_n}{ds} = k_n'$, we get $\left(\tau_g' \bar{t} + \tau_g \frac{d\bar{t}}{ds} - k_n' \bar{B} - k_n \frac{d\bar{B}}{ds}\right)$ $\sqrt{\tau_g^2 + k_n^2} - \frac{\tau_g \tau_g' + k_n k_n'}{\sqrt{\tau_g^2 + k_n^2}} (\tau_g \bar{t} - k_n \bar{B}) = 0$. Using the formulas (9), we have:

$$(k_g k_n^2 + k_g \tau_g^2 - k_n' \tau_g + \tau_g' k_n)(k_n \bar{t} + \tau_g \bar{B}) = 0$$

From this we get the following equation of a line of shadow:

$$k_g(k_n^2 + \tau_g^2) - k_n' \tau_g + \tau_g' k_n = 0 \text{ or} \tag{12}$$

$$\left[\left(\frac{k_n}{\tau_g}\right)^2 + 1\right] k_g = \left(\frac{k_n}{\tau_g}\right)' \tag{12′}$$

If a line of shadow is a geodesic in usual sense (i.e. $k_g = 0$) then from (12′) we have:

$$\frac{k_n}{\tau_g} = \text{const}.$$

and conversely, if $\frac{k_n}{\tau_g} = \text{const}$, then $k_g = 0$, so we can state:

Theorem 7. *The necessary and sufficient condition for a line of shadow to be a geodesic line, is that it be a so called cylindrical (or general) helix* (i.e. $\frac{k_n}{\tau_g} = \text{const}$).

We shall prove the following:

Theorem 8. *Let $S \subset E_3$ be a surface and $K \neq 0$, being its Gaussian curvature. A family of lines of shadow of the surface S coincides with a family of geodesic lines of this surface if and only if $K = \text{const}$ and $H = \text{const}$.*

Proof. Let the equations of the lines of shadows (b-geodesics [3]) and the geodesic lines (g-geodesic) on the surface S be given respectively:

$$\frac{d^2u^i}{dt^2} + (\nabla_s b_{jk} b^{ki} + G^i_{sj}) \frac{du^j}{dt}\frac{du^s}{dt} = \lambda \frac{du^i}{dt}$$

$$\frac{d^2u^i}{dt^2} + G^i_{sj}\frac{du^j}{dt}\frac{du^s}{dt} = \mu \frac{du^i}{dt}$$

Substracting the second equation from the first one, we have:

$$\nabla_s b_{jk} b^{ki} \frac{du^j}{dt}\frac{du^s}{dt} = (\lambda - \mu)\frac{du^i}{dt}$$

Having got rid of $(\lambda - \mu)$ and symmetrizing over lower indices, we get:

$$\nabla_{(s} b_{j|k|} b^{ki} \delta^q_{r)} - \nabla_{(s} b_{j|k|} b^{kq} \delta^i_{r)} = 0$$

Putting $r = q$ and summing over q, we have:

(13) $$\nabla_s b_{jk} b^{ki} = p_s \delta^i_j + p_j \delta^i_s, \text{ where}$$

$$p_r = \frac{1}{3}\nabla_s b_{rk} b^{rk} = \frac{1}{3}\frac{K_r}{K} \text{ ([2])}, \; K_r = \frac{\partial K}{\partial u^r}$$

As the spherical image of the line of shadow is the geodesic line [3], the spherical mapping of the surface S is the geodesic mapping [2], but the only surfaces which can be geodesically mapped upon the surface of the constant Gaussian curvature are those of constant curvature [1], hence, we get:

$$K = \text{const},$$

which means, that: $K_r = 0$. The condition (13) can be expressed now:

$$\nabla_s b_{jk} b^{ki} = 0$$

from this it follows that:

$$\nabla_s b_{jk} = 0$$

and it is equivalent to:

$$K = \text{const and } H = \text{const ([2])}.$$

Now, conversely, let $K = \text{const}$ and $H = \text{const}$. These conditions are equivalent to the condition $\nabla_s b_{jk} = 0$, so it follows from this that the lines of shadow and the geodesic lines coincide.

According to the theorem 7 and 8 we can state the following:

Theorem 9. *The only surfaces of the Gaussian curvature $K \neq 0$ on which geodesic lines are the so called cylindrical (or general) helices are those of the constant Gaussian curvature and the constant mean curvature.*

In [2] Kagan has given the definition of the projective surface. Analogously we give the following:

Definition 3. A surface S is said to be a local π-projective if there exists such coordinate system on S, that π-geodesics are expressed by linear equations.

The equation of the π-geodesic, provided that $\det \pi_{ij} \neq 0$, has the form [3]:

$$\frac{d^2u^k}{dt^2} + (\nabla_r \pi_{ij}\pi^{ik} + G^k_{rj})\frac{du^r}{dt}\frac{du^j}{dt} = \lambda\frac{du^k}{dt}$$

Let: $P^k_{rj} = \nabla_r \pi_{ij}\pi^{ik} + G^k_{rj}$ Let us assume, that: $u^i = a^i t + b^i$ are the equations of the π-geodesic. By replacing u^i in the equation of the π-geodesic with these u^i, we get:

$$P^k_{rj}a^r a^j = \lambda a^k$$

Removing λ, we have:

$$P^{[i\ \ j]}_{(\alpha\beta\delta\gamma)} = 0,$$

where (...) denotes the symmetrization and [...] the alternation, hence, putting:

$$P^+_\beta = \tfrac{1}{3}P^j_{j\beta},\ P^-_\beta = \tfrac{1}{3}P^j_{\beta j}$$

we get:

$$P^i_{(\alpha\beta)} = \frac{P^+_\beta + P^-_\beta}{2}\delta^i_\alpha + \frac{P^+_\alpha + P^-_\alpha}{2}\delta^i_\beta \tag{14}$$

Now, let the equation (14) be satisfied. Writting the equation of the π-geodesic two times:

$$\frac{d^2u^k}{dt^2} + (\nabla_r \pi_{ij}\pi^{ik} + G^k_{rj})\frac{du^r}{dt}\frac{du^j}{dt} = \lambda\frac{du^k}{dt}$$

$$\frac{d^2u^k}{dt^2} + (\nabla_j \pi_{ir}\pi^{ik} + G^k_{jr})\frac{du^j}{dt}\frac{du^r}{dt} = \lambda\frac{du^k}{dt}$$

and adding them and dividing them by 2, we have:

$$\frac{d^2u^k}{dt^2} + P^k_{(rj)}\frac{du^r}{dt}\frac{du^j}{dt} = \lambda\frac{du^k}{dt}$$

Putting suitable values $P^k_{(rj)}$ from (14) in this equation, we get:

$$\frac{d^2u^k}{dt^2}+\left(A_\beta\frac{du^\beta}{dt}+A_\alpha\frac{du^\alpha}{dt}\right)\frac{du^k}{dt}=\lambda\frac{du^k}{dt}$$

or

$$\frac{d^2u^k}{dt^2}+2A_s\frac{du^s}{dt}\frac{du^k}{dt}=\lambda\frac{du^k}{dt}$$

where $$A_s=\frac{P^+_s+P^-_s}{2}$$

Let $\Phi=2A_s\dfrac{du^s}{dt}-\lambda$.

So we have:

$$\frac{d^2u^k}{dt^2}+\Phi\frac{du^k}{dt}=0$$

Removing Φ, we have:

$$\frac{d^2u^1}{dt^2}\frac{du^2}{dt}-\frac{d^2u^2}{dt^2}\frac{du^1}{dt}=0$$

And from this, it follows that:

$\dfrac{du^1}{dt}$ and $\dfrac{du^2}{dt}$ are linear dependent i.e.

$A_i\dfrac{du^i}{dt}=0$, where A_1, A_2 – const.

hence, the equation of the π-geodesic, if the condition (14) is satisfied, has the form:

$$A_1u^1+A_2u^2=A_3$$

so it means that the surface is π-projective. We can state now:

Theorem 10. *The condition* (14) *is necessary and sufficient for the surface S to be π-projective, provided that* $\det\pi_{ij}\neq 0$.

When the surface S is a b-projective surface ($K\neq 0$), the condition (14) can be written like this:

$$P^i_{\alpha\beta}=P_\beta\delta^i_\alpha+P_\alpha\delta^i_\beta$$

Because

$$P^i_{\alpha\beta}=\nabla_\alpha b_{\beta\gamma}b^{\gamma i}+G^i_{\alpha\beta}$$

and

$$\nabla_\alpha b_{\beta\gamma}b^{\gamma i}=\Gamma^i_{\alpha\beta}-G^i_{\alpha\beta},$$

where $\Gamma^i_{\alpha\beta}$ are the Christoffel symbols of the spherical image [3] of the surface S, we have:

$$\Gamma^i_{\alpha\beta}=P_\beta\delta^i_\alpha+P_\alpha\delta^i_\beta$$

and this is a necessary and sufficient condition for geodesic lines of the spherical image to be expressed in a linear form [2]. We get:

Theorem 11. *Each surface S of The Gaussian curvature $K \neq 0$ is a locally b-projective surface, that is the lines of shadow can be expressed by means of linear equations on each surface S of $K \neq 0$. (locally)*

Theorem 12. *Given two surfaces S_1 and $S_2 \subset E_3$. Suppose, that the Gaussian curvature of S_1 is different from zero, and there exists a mapping $\varphi: S_1 \to S_2$ which maps b-geodesics on the surface S_1 into g-geodesics on the surface S_2. Then S_2 is the surface of the constant Gaussian curvature.*

Proof. As the spherical image of the b-geodesic is the g-geodesic, there exists the geodesical mapping of the spherical image of S_1 into S_2 induced by φ. The Gaussian curvature of the spherical image is constant, and the only surfaces which can be geodesically maped upon the surface of the constant curvature are those of constant Gaussian curvature [3], hence S_2 must have constant Gaussian curvature.

Q.E.D

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STRESZCZENIE

K. Radziszewski w pracy [1] podał definicję linii π-geodezyjnych w przestrzeniach o koneksji aficznej. W pracy tej zajmujemy się badaniem tych linii w przypadku powierzchni $S \subset E_3$ i określonych przez tensory związane w naturalny sposób z powierzchnią. Podajemy ich interpretację geometryczną za pomocą przeniesienia równoległego wektorów. Następnie, analogicznie do definicji powierzchni rzutowych wprowadzonych przez W. F. Kagana w [2] podajemy definicję powierzchni π-rzutowych.

Na koniec rozpatrujemy odwzorowania dwóch powierzchni na siebie przeprowadzające π_1-geodezyjne w π_2-geodezyjne.

РЕЗЮМЕ

К. Радишевски в работе [3] определил понятие π-геодезических в пространстве афинной связности. Авторы настоящей работы изучают π-геодезические на поверхности $S \subset E_3$ определимые тенсорами, которые натуральным образом связаны с поверхностью. S. Дается их геометрическая интерпретация при помощи параллельного переноса векторов. Затем, аналогично дефиниции проективных поверхностей [2] дается дефиниция π-проективных поверхностей. В заключение авторы изучают отображения поверхностей, переводящие π_1-геодезические в π_2-геодезические.