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On the Fourth Order Grunsky Functionals for Bounded Univalent Functions

O funkcjonałach Grunsky'ego czwartego rzędu dla funkcji jednolistnych ograniczonych

О функционалах Грунского четвертого порядка для ограниченных однолистных функций

Introduction

Let $S(b_1)$ denote the class of univalent holomorphic functions

(1) $$f(z) = b_1(z + a_2 z^2 + \ldots),\ 0 < b_1 \leqslant 1,$$

which map the unit disc into itself. This class has been investigated since 1950 by several authors. Here we confine ourselves to refer to Z. Charzynski, Z. Nehari, O. Tammi, W. Janowski, M. Schiffer, and J. Ławrynowicz [1–10].

The present paper is concerned with functionals of the form

(2) $$B = |a_4 - pa_2a_3 + qa_2^3|.$$

The least upper bound of B is obtained (Section 1) for some intervals $0 < b^*(p, q) \leqslant b_1 \leqslant 1$, where p, q are complex. This is a generalization of the analogous result for p, q real, due to J. Ławrynowicz and O. Tammi [3].

The investigations are based on some generalization of a necessary and sufficient condition for a function of the form (1) to be in $S(b_1)$ due to Z. Nehari [5]. This generalization was obtained by M. Schiffer and O. Tammi [7] and can be formulated as follows.

Let A_{nm}, B_{nm} be defined by the relations

(3) $$\log\frac{f(z)-f(\zeta)}{z-\zeta} = \sum_{n,m=0}^{\infty} A_{nm} z^n \zeta^m,$$

$$-\log(1-f(z)f(\zeta)) = \sum_{n,m=0}^{\infty} B_{nm} z^n \zeta^m,\quad |z| < 1,\ |\zeta| < 1.$$

The generalized conditions in question, necessary and sufficient as well, read as follows:

$$\text{Re}\sum_{n,m=0}^{N}(A_{nm}x_n x_m + B_{nm}x_n\overline{x}_m) \leqslant \sum_{n=1}^{N}\frac{|x_n|^2}{n}, \quad N = 1, 2, \ldots, \tag{4}$$

where x_0 resp. $x_1, \ldots, x_n$ are arbitrary real resp. complex parameters. Equality holds if and only if

$$\text{Re}\{C_0\} = 0,$$

$$C_k = \begin{cases}(1/k)\overline{x}_k & \text{for } 0 < k \leqslant N,\\ 0 & \text{for } k > N,\end{cases} \tag{5}$$

where

$$C_k = x_0 A_{k0} + \sum_{n=1}^{N}(x_n A_{kn} + \overline{x}_n B_{kn}), \quad k \geqslant 0. \tag{6}$$

On putting $x_0 = 0$ in (4) one gets the original Nehari's result.

Recently M. Schiffer and O. Tammi [8] have found a further generalization of (4), which involves more complex parameters:

$$\text{Re}\Big\{t_0 y_0 + \sum_{n=1}^{N} n(y_{-n}y_n + t_{-n}t_n)\Big\} \leqslant |y_0|^2 + \sum_{n=1}^{N} n(|y_{-n}|^2 + |t_n|^2), \tag{7}$$

where

$$y_n = \sum_{k=-N}^{N} t_k c_{kn}, \quad n \geqslant -N, \tag{8}$$

$$[f(z)]^k = \sum_{n=k}^{\infty} c_{kn} z^n \quad \text{for } k \neq 0, \tag{9}$$

$$\log[f(z)]/z = \sum_{k=0}^{\infty} c_{0n} z^n \quad \text{for } k = 0,$$

and t_0 is assumed to be real.

Relations (4) follow from (7) by a special choice of parameters.

Aplication of inequalities (7) instead of (4) gives more freedom for choosing suitable parameters and, consequently, possibly better results. However, it is shown in this paper (Section 2) that in the case of B both kinds of inequalities lead to the same result.

In addition to the considerations concerned with B an analogous problem for

$$L = |a_3 - pa_2^2| \tag{10}$$

is solved (Section 3) with the same method. References concerning already known particular cases are given in [3].

Finally it is shown (Section 4) that, apart from a rotation, there is exactly one B that satisfies hypotheses of Theorem 1 and whose least upper bound is given by B^*, defined in (24) below, for the whole interval $0 < b_1 \leqslant 1$. Similarly, one can easily show that an analogous statement holds in the case of L and the only answer is $L = |a_3 - a_2^2|$.

Sharp estimates of B and L are useful when investigating the coefficient problem for f in $S(b_1)$. For instance J. Ławrynowicz and O. Tammi (oral information of the authors) are estimating $|a_6|$ with help of the following unpublished analogoue of the inequality (21), (given below):

$$\begin{aligned}
&\operatorname{Re}\Big\{a_2 x_1^2 + (a_3 - a_2^2)x_2^2 + \Big(a_4 - 2a_2 a_4 + \frac{13}{12}a_2^3\Big)x_3^2 + \Big(a_5 - 2a_2 a_4 - \\
&-\frac{2}{3}a_3^2 + 4a_2^2 a_3 - \frac{3}{2}a_2^4\Big)x_4^2 + \Big(a_6 - 2a_2 a_5 - 3a_3 a_4 + 4a_2^2 a_4 + \frac{21}{4}a_2 a_3^2 - \\
&-\frac{59}{8}a_2^3 a_3 + \frac{689}{320}a_2^5\Big) + \Big(2a_3 - \frac{3}{2}a_2^2\Big)x_1 x_3 + \Big(2a_4 - 3a_2 a_3 + \frac{5}{4}a_2^3\Big)x_1 + \\
&+\Big(2a_4 - 4a_2 a_3 + 2a_2^3\Big)x_2 x_4 + \Big(2a_5 - 4a_2 a_4 - \frac{5}{2}a_3^2 + \frac{29}{4}a_2^2 a_3 - \\
&-\frac{35}{32}a_2^4\Big)x_3\Big\} \leqslant 2\Big(1 - b_1\Big)|x_1|^2 + \Big(1 - b_1^2\Big)|x_2|^2 + \Big(\frac{2}{3} - \frac{1}{2}b_1|a_2|^2 - \\
&-\frac{2}{3}b_1^3\Big)|x_3|^2 + \Big(\frac{1}{2} - b_1^2|a_2|^2 - \frac{1}{2}b_1^4\Big)|x_4|^2 + \Big(\frac{2}{5} - \frac{1}{2}b_1\Big|a_3 - \frac{1}{4}a_2^2\Big|^2 - \\
&-\frac{3}{2}b_1^3|a_2|^2 - \frac{2}{5}b_1^5\Big) - 2\operatorname{Re}\Big[b_1\bar{a}_2 x_1\bar{x}_3 + b_1\Big(\bar{a}_3 - \frac{1}{4}\bar{a}_2^2\Big)x_1 + \\
&+b_1^2\bar{a}_2 x_2\bar{x}_4 + \frac{1}{2}b_1 a_2\Big(\bar{a}_3 - \frac{1}{2}\bar{a}_2^2\Big)x_3 + b_1^3\bar{a}_2 x_3\Big],
\end{aligned}$$

and it is easily seen that the knowledge of sharp estimates of functionals (2) and (10), and analogous five order functionals (cf. [4]) determines in the natural way the optimal choice of parameters.

1. Estimation of $|a_4 - pa_2a_3 + qa_2^3|$

In this section we prove

Theorem 1. *Suppose that p and q are constans (in general complex), $\operatorname{Re} p \leqslant \frac{5}{2}$, f belongs to $S(b_1)$, and one of the following six cases holds:*

(11) $$\frac{|2-p|^2+X}{|3-p|^2+X} \leqslant b_1 \leqslant 1 \quad \textit{for } X \geqslant 0,\ Y \geqslant 0,\ X \geqslant Y,$$

(12) $$\frac{|2-p|_2+Y}{|3-p|^2+Y} \leqslant b_1 \leqslant 1 \quad \textit{for } X \geqslant 0,\ Y \geqslant 0,\ X \leqslant Y,$$

(13) $$\frac{|2-p|^2+X-3Y}{|3-p|^2+X-3Y} \leqslant b_1 \leqslant 1 \quad \textit{for } X \geqslant 0,\ Y \leqslant 0,$$

(14) $$\frac{|2-p|^2-3X+Y}{|3-p|^2-3X+Y} \leqslant b_1 \leqslant 1 \quad \textit{for } X \leqslant 0,\ Y \geqslant 0,$$

(15) $$\frac{|2-p|^2-3X}{|3-p|^2-3X} \leqslant b_1 \leqslant 1 \quad \textit{for } X \leqslant 0,\ Y \leqslant 0,\ X \leqslant Y,$$

(16) $$\frac{|2-p|^2-3Y}{|3-p|^2-3Y} \leqslant b_1 \leqslant 1 \quad \textit{for } X \leqslant 0,\ Y \leqslant 0,\ X \geqslant Y,$$

where

(17) $$X = \operatorname{Re}\left(p^2-p-4q+\frac{7}{3}\right)$$

$$\textit{and } Y = (-\operatorname{sgn} v)\operatorname{Im}\left(p^2-p-4q+\frac{7}{3}\right),\quad v = \operatorname{Im} a_2.$$

Then

(18) $$|a_4 - pa_2a_3 + qa_2^3| \leqslant \frac{2}{3}(1-b_1^3).$$

The estimate is sharp for every p, q, and b_1. All the extremal functions are given by the formula

(19) $$f_c^*(z) = e^{-ic}P^{-1}\big(b_1P(e^{ic}z)\big),\ P(z) = \frac{z}{(1-z^3)^{2/3}},\ |z|<1,\ -\pi < c \leqslant \pi.$$

Proof. By (4) applied to

(20) $$\sqrt{f(z^2)} = b_1^{\frac{1}{2}}\left(z+\frac{a_2}{2}z^3+\left(\frac{a_3}{2}-\frac{a_2^2}{8}\right)z^5+\left(\frac{a_4}{2}-\frac{1}{4}a_2a_3+\frac{1}{16}a_2^3\right)z^7+\ldots\right),$$
$$|z|<1,$$

we get

$$A_{\mu\nu} = B_{\mu\nu} = 0 \text{ for } \mu+\nu \text{ odd},$$

$$A_{11} = \frac{1}{2}a_2, \quad A_{13} = A_{31} = \frac{1}{2}a_3 - \frac{3}{8}a_2^2,$$

$$A_{22} = \frac{1}{2}a_3 - \frac{1}{2}a_2^2, \quad A_{33} = \frac{1}{2}a_4 - a_2 a_3 + \frac{13}{24}a_2^3,$$

$$B_{11} = b_1, \quad B_{13} = \bar{B}_{31} = \frac{1}{2}b_1\bar{a}_2,$$

$$B_{22} = \frac{1}{2}b_1^2, \quad B_{33} = \frac{1}{4}b_1|a_2|^2 + \frac{1}{3}b_1^3.$$

Hence (4) with $N = 3$, $x_0 = 0$, and $x_3 = 1$ yields

$$\text{(21)} \qquad \operatorname{Re}\left\{a_4 - 2a_2a_3 + \frac{13}{12}a_2^3 + x_1^2 a_2 + x_2^2(a_3 - a_2^2) + 2x_1\left(a_3 - \frac{3}{4}a_2^2\right)\right\}$$
$$\leqslant \frac{2}{3}(1-b_1^3) - \frac{1}{2}b_1|a_2|^2 + 2|x_1|^2(1-b_1) + |x_2|^2(1-b_1^2) - 2b_1\operatorname{Re}(x_1\bar{a}_2).$$

Now we put $x_1 = \check{x}_1 a_2$ and $x_2^2 = \check{x}_2 a_2$, whence

$$\operatorname{Re}\left\{a_4 + a_2a_3(\check{x}_2 + 2\check{x}_1 - 2) + a_2^3\left(\check{x}_1^2 - \frac{3}{2}\check{x}_1 - \check{x}_2 + \frac{13}{12}\right)\right\} \leqslant \frac{2}{3}(1-b_1^3) +$$
$$\left[2|\check{x}_1|^2(1-b_1) - \frac{1}{2}b_1 - 2b_1\operatorname{Re}\check{x}_1\right]|a_2|^2 + |x_2|(1-b_1^2)|a_2|.$$

We choose $\check{x}_2$ so that, $\check{x}_2 + 2\check{x}_1 - 2 = -p$, i. e. $\check{x}_2 = 2 - p - 2\check{x}_1$. Therefore

$$\operatorname{Re}\{a_4 - pa_2a_3 + qa_2^3\} \leqslant \frac{2}{3}(1-b_1^3) + \left[2|\check{x}_1|^2(1-b_1) - \frac{1}{2}b_1 -\right.$$
$$\left.2b_1\operatorname{Re}\check{x}_1\right]|a_2|^2 + |2-p-2\check{x}_1|(1-b_1^2)|a^2|^2 - \operatorname{Re}\left\{\left(\check{x}_1^2 + \frac{1}{2}\check{x}_1 + p - q - \frac{11}{12}\right)a_2^3\right\}.$$

Since $(1-b_1^2)|a_2| \geqslant 0$ we choose $\check{x}_1$ so that $2 - p - 2\check{x}_1 = 0$, i. e. $\check{x}_1 = 1 - \frac{1}{2}p$.

Now we notice that there is not any loss of generality if we assume

$$\text{(22)} \qquad a_4 - pa_2a_3 + qa_2^3 > 0, \; a_2 = u + iv, \; u \leqslant 0, \; v \leqslant 0.$$

Consequently, by (2),

$$\text{(23)} \quad B - B^* \leqslant \frac{1}{2}u^2\left\{|2-p|^2 - b_1|3-p|^2 - \frac{1}{2}\operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right)u +\right.$$
$$\left.\frac{3}{2}\operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right)v\right\} + \frac{1}{2}v^2\left\{|2-p|^2 - b_1|3-p|^2 +\right.$$
$$\left.\frac{3}{2}\operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right)u - \frac{1}{2}\operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right)v\right\},$$

where

$$B^* = \frac{2}{3}(1-b_1^3). \tag{24}$$

According to (17) inequality (23) can be rewritten as

$$B-B^* \leqslant \frac{1}{2}u^2\left\{|2-p|^2-b_1|3-p|^2-\frac{1}{2}Xu+\frac{3}{2}Yv\right\}+ \tag{25}$$

$$\frac{1}{2}v^2\left\{|2-p|^2-b_1|3-p|^2+\frac{3}{2}Xu-\frac{1}{2}Yv\right\},$$

and hence it is natural to consider the following cases separately:

$$X \geqslant 0,\ Y \geqslant 0, \tag{26}$$

$$X \geqslant 0,\ Y \leqslant 0, \tag{27}$$

$$X \leqslant 0,\ Y \geqslant 0, \tag{28}$$

$$X \leqslant 0,\ Y \leqslant 0, \tag{29}$$

where in each case we assume that $p \neq 3$.

Here, by (22) and $|a_2| \leqslant 2(1-b_1)$ (cf. [6], p. 6), we have

$$-2(1-b_1) \leqslant u \leqslant 0,\ -2(1-b_1) \leqslant v \leqslant 0. \tag{30}$$

Consider first the case (26). By (25) and (30), in order to obtain (18) we have to assume

$$|2-p|^2-b_1|3-p|^2-\frac{1}{2}X\left[-2(1-b_1)\right]+\frac{3}{2}Y\cdot 0 \leqslant 0, \tag{31}$$

$$|2-p|^2-b_1|3-p|^2+\frac{3}{2}X\cdot 0-\frac{1}{2}Y\left[-2(1-b_1)\right] \leqslant 0 \tag{32}$$

or, equivalently,

$$b_1 \geqslant \frac{|2-p|^2+X}{|3-p|^2+X} = M_1,\quad b_1 \geqslant \frac{|2-p|^2+Y}{|3-p|^2+Y} = M_2.$$

It is easily verified that $M_1, M_2 \leqslant 1$ for $\operatorname{Re} p \leqslant \frac{5}{2}$. Hence for $\operatorname{Re} p \leqslant \frac{5}{2}$, inequalities (31) and (32), and, consequently, (18) are satisfied for $\max(M_1, M_2) \leqslant b_1 \leqslant 1$, where

$$\max(M_1, M_2) = \begin{cases} M_1 & \text{for } X \geqslant Y, \\ M_2 & \text{for } X \leqslant Y. \end{cases}$$

Hence, by (26) we conclude that the desired estimate $B \leqslant B^*$ holds for b_1, X, and Y satisfying (11) or (12).

Consider next the case (27). By (25) and (30) in order to obtain (17) we have to assume

$$(33)\quad |2-p|^2-b_1|3-p|^2-\frac{1}{2}X[-2(1-b_1)]+\frac{3}{2}Y[-2(1-b_1)]\leqslant 0,$$

$$(34)\quad |2-p|^2-b_1|3-p|^2+\frac{3}{2}X\cdot 0-\frac{1}{2}Y\cdot 0\leqslant 0,$$

or, equivalently,

$$b_1\geqslant\frac{|2-p|^2+X-3Y}{|3-p|^2+X-3Y},\quad b_1\geqslant\frac{|2-p|^2}{|3-p|^2}.$$

Since (27) implies

$$\frac{|2-p|^2}{|3-p|^2}\leqslant\frac{|2-p|^2+X-3Y}{|3-p|^2+X-3Y}$$

and

$$\frac{|2-p|^2+X-3Y}{|3-p|^2+X-3Y}\leqslant 1\quad\text{for } \operatorname{Re}p\leqslant\frac{5}{2},$$

then inequalities (33) and (34), and, consequently, (18) are satisfied for

$$\frac{|2-p|^2+X-3Y}{|3-p|^2+X-3Y}\leqslant b_1\leqslant 1.$$

Hence by (27) we conclude that the desired estimate $B\leqslant B^*$ holds for b_1, X, and Y satisfying (13).

The proof in the cases (28) and (29) and for $v\geqslant 0$ is quite similar, so it remains to prove that (18) is sharp and to find the corresponding extremal functions.

It remains to prove that if f^* belongs to $S(b_1)$ and is extremal for (18) then it is given by (19) with $c=0$.

Indeed, applying (25) for f^* we get $a_2=0$ and, according to (22), $a_4>0$. We utilize then for f^* the inequality (21) with $x_1=\tilde{x}_1\bar{a}_3$, $x_2^2=\tilde{x}_2\bar{a}_3$, where $\tilde{x}_1$ and $\tilde{x}_2$ are supposed to be nonnegative. Therefore

$$|a_4|+\tilde{x}_2|a_3|^2+2\tilde{x}_1|a_3|^2\leqslant\frac{2}{3}(1-b_1^3)+2\tilde{x}_1^2(1-b_1)|a_3|^2+\tilde{x}_2(1-b_1^2)|a_3|,$$

whence

$$B-B^*\leqslant\tilde{x}^2(1-b_1^2)|a_3|-2\left[\tilde{x}_1-\tilde{x}_1^2(1-b_1)-\frac{1}{2}\tilde{x}_2\right]|a_3|^2.$$

On putting $\tilde{x}_1=1$ and $\tilde{x}_2=0$ the above relation becomes

$$B-B^*\leqslant -2b_1|a_3|^2,$$

whence $a_3=0$. Consequently, f^* must satisfy

$$a_2=a_3=0,\ a_4>0.$$

Therefore we conclude, by (21), that f^* is extremal for (4) with $N = 3$, $x_0 = x_1 = x_2 = 0$, $x_3 = 1$. Thus, according to the result of M. Schiffer and O. Tammi quoted in Introduction, the corresponding coefficients C_k, $k = 0, 1, \ldots$, satisfy (5), i. e.

$$C_0 = C_1 = C_2 = 0,\ C_3 = \frac{1}{3},\ C_k = 0 \text{ for } k > 3. \tag{35}$$

Now we recall (cf. [7]) that if f is in $S(b_1)$, $D_r = f(\{z\colon |z| \leqslant r < 1\})$, $\Delta_r = \{w\colon |w| < 1 \setminus D_r\}$, x_0, resp. x_k, $k = 1, 2, \ldots$, are real resp. complex parameters, $F_k(t)$ are Faber polynomials defined by

$$-\log\big(1 - tf(z)\big) = \sum_{k=1}^{\infty} \frac{1}{k} F_k(t) z^k, \tag{36}$$

and, finally,

$$g(w) = x_0 \log w - \sum_{n=1}^{N} \left[\frac{x_n}{n} F_n\left(\frac{1}{w}\right) - \frac{\bar{x}_n}{n} \overline{F_n(\bar{w})}\right], \tag{37}$$

then $\iint\limits_{\Delta r} |g'(w)|^2 d\tau \geqslant 0$ implies (4).

On putting $w = f^*(z)$ in (38) we get

$$g\big(f^*(z)\big) = x_0 \log z - \sum_{m=1}^{N} \frac{x_m}{m} z^{-m} + \sum_{m=0}^{\infty} C_m z^m, \tag{38}$$

where C_m, $m = 0, 1, \ldots$, are given by (6).
Relations (38), (37) with $w = f^*(z)$, and (35) yield

$$\frac{f^*}{(1 - f^{*3})^{2/3}} = b_1 \frac{z}{(1 - z^3)^{2/3}},\ |z| < 1,$$

and this implies (19) with $c = 0$.

On the other hand one verifies directly that any function of the form (19) belongs to $S(b_1)$ and is extremal for (18). Thus the proof of Theorem 1 is completed.

2. Application of the Schiffer-Tammi inequalities

We proceed to prove that the generalized inequalities (7) due to M. Schiffer and O. Tammi lead to the same result as formulated in Theorem 1.

Indeed, we take $t_0 = 0$, and $N = 3$, and apply (7) to (20). Therefore

$$\operatorname{Re}\{y_{-1}y_1 + t_{-1}t_1 + 2(y_{-2}y_2 + t_{-2}t_2) + 3(y_{-3}y_3 + t_{-3}t_3)\} \leqslant |y_{-1}|^2 + |t_1|^2 + 2|y_{-2}|^2 + 2|t_2|^2 + 3|y_{-3}|^2 + 3|y_3|^2, \tag{39}$$

where, by (8) and (9),

$$y_{-1} = -\frac{3}{2}t_{-3}b_1^{-3/2}a_2 + t_{-1}b_1^{-1/2},\ y_1 = \frac{1}{2}t_{-1}b_1^{-1/2}a_2 + t_1 b_1^{1/2},$$

$$y_{-2} = t_{-2}b_1^{-1},\ y_2 = t_{-2}b_1^{-1}(a_2^2 - a_3) + t_2 b_1,$$

$$y_{-3} = t_{-3}b_1^{-3/2},$$

$$y_3 = t_{-3}b_1^{-3/2}\left(-\frac{3}{2}a_4 + \frac{15}{4}a_2 a_3 - \frac{35}{16}a_2^3\right) + t_{-1}b_1^{-1/2}\left(\frac{3}{8}a_2^2 - \frac{1}{2}a_3\right) + \\ + t_1 b_1^{1/2}a_2 + t_3 b_1^{3/2}.$$

Hence (39) yields

$$\mathrm{Re}\left\{\left(-\frac{9}{2}a_4 + \frac{27}{2}a_2 a_3 - \frac{75}{8}a_2^3\right)b_1^{-3}t_{-3}^2 + \left(\frac{15}{4}a_2^2 - 3a_3\right)b_1^{-2}t_{-3}t_{-1} + \right.$$

$$\left. 2(a_2^2 - a_3)b_1^{-2}t_{-2}^2 - \frac{1}{2}a_2 b_1^{-1}t_{-1}^2 + 6t_{-3}t_3 + 4t_{-2}t_2 + 2t_{-1}t_1\right\} \leqslant$$

$$\left|-\frac{3}{2}a_2 b_1^{-3/2}t_{-3} + b_1^{-1/2}t_{-1}\right|^2 + 2|b_1^{-1}t_{-2}|^2 + 3|b_1^{-3/2}t_{-3}|^2 +$$

$$3|t_3|^2 + 2|t_2|^2 + |t_1|^2.$$

Now we put $t_{-3} = t_3 = 1$ and $t_2 = t_{-2}\check{t}_2$, whence

$$\mathrm{Re}\left\{\left(-\frac{9}{2}a_4 + \frac{27}{2}a_2 a_3 - \frac{75}{8}a_2^3\right) + \left(\frac{15}{4}a_2^2 - 3a_3\right)b_1 t_{-1} + 2(a_2^2 - a_3)b_1 t_{-2}^2 - \right.$$

$$\left.\frac{1}{2}a_2 b_1^2 t_{-1}^2\right\} \leqslant 3(1 - b_1^3) + b_1^3\mathrm{Re}(-4|t_{-2}|^2\check{t}_2 - 2t_{-1}t_1) +$$

$$\left|t_{-1} - \frac{3}{2}a_2 b_1^{-1}\right|^2 b_1^2 + 2|t_{-2}|^2 b_1 + (2|t_{-2}|^2|\check{t}_2|^2 + |t_1|^2)b_1^3.$$

We choose $t_{-2}^2 = a_2\check{t}_{-2}$, $t_{-1} = a_2\check{t}_{-1}$, and $t_1 = \bar{a}_2\check{t}_1$. Therefore

$$\mathrm{Re}\left\{-\frac{9}{2}a_4 + \left(\frac{27}{2} - 3b_1\check{t}_{-1} - 2b_1\check{t}_{-2}\right)a_2 a_3 + \left(-\frac{75}{8} + \frac{15}{4}b_1\check{t}_{-1} + 2b_1\check{t}_{-2} - \right.\right.$$

$$\left.\left.-\frac{1}{2}b_1^2\check{t}_{-1}^2\right)a_1^3\right\} \leqslant 3(1 - b_1^3) + |a_2|^2\left(-2b_1^3\mathrm{Re}\check{t}_{-1}\check{t}_1 + \left|\check{t}_{-1} - \frac{3}{2}b_1^{-1}\right|^2 b_1^2 + \right.$$

$$\left.|\check{t}_1|^2 b_1^3\right) + |a_2|(-4b_1^3|\check{t}_{-2}|\mathrm{Re}\check{t}_2 + 2|\check{t}_{-2}||\check{t}_2|^2 b_1).$$

Next we choose t_{-2} so that $\frac{2}{3}b_1 t_{-1}+\frac{9}{4}b_1 t_{-2}-3 = -p$, i. e.

$$t_{-2} = \frac{9}{4}b_1^{-1}\left(3-p-\frac{2}{3}b_1 t_{-1}\right).$$

Consequently, we obtain

$$\operatorname{Re}\{a_4-pa_2a_3+qa_2^3\}+\frac{2}{3}(1-b_1^3) \geqslant -\frac{2}{9}|a_2|^2\Bigg(-2b_1^3\operatorname{Re}t_{-1}t_1+ \left|t_{-1}-\frac{3}{2}b_1^{-1}\right|^2 b_1^2+|t_1|^2b_1^3\Bigg)-|a_2|\left|3-p-\frac{2}{3}b_1t_{-1}\right|(|t_2|^2b_1^2-2b_1^2\operatorname{Re}t_2+1)- \operatorname{Re}\left\{\left(\frac{1}{9}b_1^2t_{-1}^2-\frac{1}{6}b_1t_{-1}-\frac{11}{12}+p-q\right)a_2^3\right\}.$$

Now we notice that there is not any loss of generality if we assume

$$a_4-pa_2a_3+qa_2^3<0,\ a_2 = u+iv,\ u\geqslant 0,\ v\geqslant 0. \tag{40}$$

Consequently, by (2) and (24),

$$B-B^* \leqslant \frac{2}{9}|a_2|^2\left(-2b_1^3\operatorname{Re}t_{-1}t_1+\left|t_{-1}-\frac{3}{2}b_1^{-1}\right|^2b_1^2+|t_1|^2b_1^3\right)+ |a_2|\left|3-p-\frac{2}{3}b_1t_{-1}\right|(|t_2|^2b_1^2\operatorname{Re}t_2+1)+\operatorname{Re}\left\{\left(\frac{1}{9}b_1^2t_{-1}^2- \frac{1}{6}b_1t_{-1}-\frac{11}{12}+p-q\right)a_2^3\right\}. \tag{41}$$

Since $|t_2|^2b_1^2-2\operatorname{Re}t_2b_1^2+1\geqslant 0$ for each t_2, the right-hand side of (41) is minimized by $t_{-1} = \frac{3}{2}b_1^{-1}(3-p)$, and it becomes

$$B-B^* \leqslant \frac{2}{9}|a_2|^2(|t_1|^2b_1^3-3b_1^2\operatorname{Re}(3-p)t_1+\frac{9}{4}|2-p|^2)+ \operatorname{Re}\left\{\left[\frac{1}{4}(3-p)^2-\frac{1}{4}(3-p)-\frac{11}{12}+p-q\right]a_2^3\right\}.$$

The right-hand side of the last inequality can be minimized by

$$\operatorname{Re}t_1 = \frac{3}{2}\operatorname{Re}(3-p)b_1^{-1}, \qquad \operatorname{Im}t_1 = \frac{3}{2}\operatorname{Im}(3-p)b_1^{-1},$$

and, although the choice of all parameters was optimal, we arrive to the same result as formulated in Theorem 1.

3. Estimation of $|a_3 - pa_2^2|$

In this section we prove

Theorem 2. *Suppose that p is a constant (in general complex),* $\operatorname{Re} p < 1$, *f belongs to $S(b_1)$, and*

$$\exp\frac{\operatorname{Re} p - 1}{(\operatorname{Re} p - 1)^2 + (\operatorname{Im} p)^2} \leqslant b_1 \leqslant 1. \tag{42}$$

Then

$$|a_3 - pa_2^2| \leqslant 1 - b_1^2. \tag{43}$$

The estimate is sharp for every p and b_1. All the extremal functions are given by the formula

$$f^{**}(z) = e^{-ic}\check{P}^{-1}\big(b_1 \check{P}(e^{ic}z)\big),\; P(z) = \frac{z}{1-z^2},\; -\pi < c \leqslant \pi,\; |z| < 1. \tag{44}$$

Proof. By (4) applied to the function in question we get

$$A_{00} = \log b_1,\; A_{01} = A_{10} = a_2,\; A_{11} = a_3 - a_2^2,\; B_{11} = b_1^2.$$

Hence (4) with $N = 1$ and $x_1 = 1$ yields

$$\operatorname{Re}(a_3 - a_2^2) \leqslant 1 - b_1^2 - x_0^2 \log b_1 - 2x_0 \operatorname{Re} a_2. \tag{45}$$

There is no loss of generality if we assume that

$$a_3 - pa_2^2 > 0. \tag{46}$$

Consequently, by (10),

$$L - L^* \leqslant -x_0^2 \log b_1 - 2x_0 \operatorname{Re} a_2 + \operatorname{Re}[(1-p)a_2^2], \tag{47}$$

where

$$L^* = 1 - b_1^2. \tag{48}$$

Since $\log b_1 \leqslant 0$ the right-hand side of (47) can be minimized by

$$x_0 = -\operatorname{Re} a_2 [\log b_1]^{-1}. \tag{49}$$

Therefore

$$L - L^* \leqslant u^2[(\log b_1)^{-1} + 1 - \operatorname{Re} p] + 2uv \operatorname{Im} p - v^2(1 - \operatorname{Re} p), \tag{50}$$

where $u = \operatorname{Re} a_2$, $v = \operatorname{Im} a_2$.

Direct calculation shows that a necessary condition for u to be optimal in (50) is

$$u[(1 - \operatorname{Re} p)(\log b_1)^{-1} + (1 - \operatorname{Re} p)^2 + (\operatorname{Im} p)^2] = 0,$$

where the expression in the square brackets can only vanish for $p<1$. Since the latter case is well known (cf. [9], p. 10) we confine ourselves to the case where $u=0$. Now we easily check that the analogous necessary condition for v is $v=0$, and that the sufficient condition for $u=0$, $v=0$ to realize the maximum in the right-hand side of (50)

$$-[(\log b_1)^{-1}+1-\operatorname{Re}p](1-\operatorname{Re}p)-(\operatorname{Im}p)^2>0$$

is satisfied provided that

$$\operatorname{Re}p<1,\ \exp\{[\operatorname{Re}p-1]/[(\operatorname{Re}p-1)^2+(\operatorname{Im}p)^2]\}<b_1\leqslant 1.$$

Finally, we verify directly that our choice is still optimal for

$$b_1=\exp\{[\operatorname{Re}p-1]/[(\operatorname{Re}p-1)^2+(\operatorname{Im}p)^2]\},\ \operatorname{Re}p<1.$$

Consequently, (50) yields $L-L^*\leqslant 0$ for $\operatorname{Re}p<1$ and (42) as desired.

It remains to prove that if f^{**} belongs to $S(b_1)$ and is extremal for (43) then it is given by (44) with $c=0$.

Indeed, applying (50) for f^{**} we get $a_2=0$ and, according to (46), $a_3>0$. Therefore we conclude, by (45), that f^{**} is extremal for (4) with $N=1$, $x_0=0$, $x_1=1$. Thus, according to the result of M. Schiffer and O. Tammi quoted in Introduction, the corresponding coefficients C_k, $k=0,1,\ldots$, satisfy (5), i. e.

(51) $$C_0=0,\ C_1=1,\ C_k=0,\ \text{for}\ k>1.$$

Now, arguing as in the analogous part of proof of Theorem 1, we obtain that f^{**} satisfies the equation

$$\frac{f^{**}}{1-f^{**2}}=b_1\frac{z}{1-z^2},\quad |z|<1$$

and this implies (44) with $c=0$.

On the other hand one verifies directly that any function of the form (44) belongs to $S(b_1)$ and is extremal for (43). Thus the proof of Theorem 2 is completed.

4. A uniqueness theorem

In this section we prove

Theorem 3. *If f belongs to $S(b_1)$ and* $\operatorname{Re}p<\frac{5}{2}$ *then, apart from a rotation, there is exactly one functional* (2) *for which the alternative of* (11)–(16) *implies* $0<b_1\leqslant 1$:

(52) $$B=\left|a_4-2a_2a_3+\frac{13}{12}a_2^3\right|.$$

Proof. We consider (21) with $x_1 = x_2 = 0$:

$$\operatorname{Re}\left\{a_4 - 2a_2a_3 + \frac{13}{12}a_2^3\right\} - \frac{2}{3}(1 - b_1^3) \leqslant -\frac{1}{2}b_1|a_2|^2.$$

After a suitable rotation we can assume $a_4 - 2a_2a_3 + \frac{13}{12}a_2^3 > 0$, whence $B - B^* \leqslant -\frac{1}{2}b_1|a_2|^2$ and, consequently, $B - B^* \leqslant 0$ for all b_1, $0 < b_1 \leqslant 0$. We have to show that, apart from rotation, the problem has no solutions other than (46). To this end we consider, separately six cases:

$$\text{(53)}\quad \begin{cases} |2-p|^2 + \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant 0, \ \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \geqslant 0, \\ \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right) \geqslant 0, \ \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \geqslant \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right), \end{cases}$$

$$\text{(54)}\quad \begin{cases} |2-p|^2 + \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant 0, \ \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \geqslant 0, \\ \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right) \geqslant 0, \ \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right), \end{cases}$$

$$\text{(55)}\quad \begin{cases} |2-p|^2 + \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) - 3\operatorname{Im}\left(p - p - 4q + \frac{7}{3}\right) \leqslant 0, \\ \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \geqslant 0, \ \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant 0, \end{cases}$$

$$\text{(56)}\quad \begin{cases} |2-p|^2 - 3\operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) + \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant 0, \\ \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant 0, \ \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right) \geqslant 0, \end{cases}$$

$$\text{(57)}\quad \begin{cases} |2-p|^2 - 3\operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant 0, \ \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant 0, \\ \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant 0, \ \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right), \end{cases}$$

$$\text{(58)}\quad \begin{cases} |2-p|^2 - 3\operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant 0, \ \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant 0, \\ \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right) \leqslant 0, \ \operatorname{Re}\left(p^2 - p - 4q + \frac{7}{3}\right) \geqslant \operatorname{Im}\left(p^2 - p - 4q + \frac{7}{3}\right). \end{cases}$$

Since considerations in all cases are analogous to each other we confine ourselves to describe (53) only. The first two inequalities in (53) imply $|2-p|^2 \leqslant -\operatorname{Re}\left(p^2-p-4q+\frac{7}{3}\right) \leqslant 0$, whence $p = 2$ and $\operatorname{Re} q = \frac{13}{12}$. Now applying this result to the remaining two inequalities in (53) we conclude that $\operatorname{Im} q = 0$, i. e. the answer required. Cases (54)–(58) lead to the same answer, and this completes the proof.

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STRESZCZENIE

W niniejszej pracy rozpatrywane są funkcjonały (2) i (10) dla p i b zespolonych w klasie funkcji postaci (1), odwzorowujących holomorficznie i jednolistnie koło jednostkowe w siebie. W oparciu o uogólnione nierówności Grunsky'ego-Nehariego (4) otrzymano kresy górne równe odpowiednio B^* i L^* dla b_1 odpowiednio z przedziałów $0 < b^*(p, q) \leqslant b_1 \leqslant 1$ oraz $0 < b^{**}(p) \leqslant b_1 \leqslant 1$ i znaleziono wszystkie funkcje ekstremalne. Ponadto udowodniono, że zastosowanie nierówności (7), będących dalszym uogólnieniem nierówności Grunsky'ego-Nehariego nie poprawia wyniku w przypadku funkcjonału (2) oraz że dla omawianej metody istnieje dokładnie

jeden funkcjonał postaci (2), o którym można dowieść, że jego kres górny wynosi B^* dla każdego $0 < b_1 \leqslant 1$.

РЕЗЮМЕ

В этой работе рассматриваются функционалы (2) и (10) для комплексных p и q в классе функций вида (1), отображающих голоморфно и однолистно-единичный круг в себя. Опираясь на обобщенных неравенствах Грунского-Нехари (4), получено верхние грани, соответственно равные B^* и L^* для b_1 соответственно из интервалов $0 < b^*(p, q) \leqslant b_1 \leqslant 1$ и $0 < b^{**}(p) \leqslant b_1 \leqslant 1$, а также найдено все экстремальные функции. Кроме того доказано, что применение неравенств (7), являющихся дальнейшим обобщением неравенств Грунского-Нехари, не улучшает результата в случае функционала (2) и что для обсуждаемого метода существует ровно один функционал вида (2), о котором можно доказать, что его верхняя грань равна B^* для каждого b_1 из $0 < b_1 \leqslant 1$.