ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XXII/XXIII/XXIV, 16 SECTIO A 1968/1969/1970

Instytut Matematyczny Polskiej Akademii Nauk, Pracownia w Łodzi

JULIAN ŁAWRYNOWICZ and WŁODZIMIERZ WALISZEWSKI

Conformality and Pseudo-riemannian Manifolds

Konforemność i rozmaitości pseudoriemannowskie

Конформность и псевдоримановые многообразия

Introduction

Conformal mappings of Riemannian manifolds were investigated by several authors in the local and global formulation as well (cf. e.g. [3], [8], and [10]), and some results were also obtained in the case of pseudo-riemannian manifolds, however, in the local formulation only (cf. e.g. [3], [9], and [7]). Quasiconformal mappings of Riemannian manifolds were introduced and investigated in [13].

In the present paper we are concerned with conformal mappings of pseudo-riemannian manifolds in the global formulation.

We begin our study with preliminaries. We introduce first some notation and terminology, in particular the notion of an essentially pseudo-riemannian manifold, develop measurability and integration (Theorems 1 and 2), introduce the notion of an angle, and define its inner measure. We deal then with curves, especially we distinguish some kinds of curves: space-like, time-like, regular, and rectifiable, define the length of a regular curve, introduce some kinds of mappings: type-preserving and type-reversing, and give a basic theorem on these mappings (Theorem 3). Next we introduce the notion of the p-modulus of a family of regular curves and study basic properties of these moduli (Theorems 4-8).

In the second part of the paper (Section 6) we are concerned with conformal mappings of essentially pseudo-riemannian manifolds. We introduce the notion of conformality that, roughly speaking, means that the isotropic cone is preserved at each point of the manifold in question. We give then a necessary and sufficient condition for conformality in terms of quadratic forms determined by the metrics of the manifolds in question (Theorem 9). Now we give a characterization of conformal mappings in terms of angles and their inner measure (Theorems 10 and 11), and, finally, in terms of families of regular curves and their moduli (Theorems 12 and 13).

In the last section we define regular quasiconformal mappings and conclude the paper with the result that in the case of essentially pseudo-riemannian manifolds there is no analogue of regular quasiconformal mappings other than conformal. Here we mention that the problem of the existence of some irregular quasiconformal mappings remains open. We also pose some other natural problems, some of them being planned to be discussed in a subsequent paper.

The authors are deeply indebted to Dr Kalevi Suominen for helpful discussions and suggestions during their research in the subject in question.

The detailed version of this paper will appear in the Mathematica Scandinavica 28 (1971).

1. Notation and terminology

Throughout this paper the set of all points (resp. vectors) of a manifold (resp. vector space) X is denoted by $\operatorname{supp} X$. If f is a mapping from a set (resp. manifold or a vector space) X into a set (resp. manifold or vector space) Y, we write $f: X \to Y$, and denote the image of any subset E of X (resp. $\operatorname{supp} X$) by $f[E]$. If, in particular, f is a homeomorphism, $\twoheadrightarrow$ means "onto", i.e. $Y = f[X]$ (resp. $\operatorname{supp} Y = f[\operatorname{supp} X]$).

The n-dimensional Euclidean space is denoted by R^n, and its subspace that consists of points with the last component positive — by R^n_+. In the case where $n = 1$ we drop out the index n.

Under a *pseudo-riemannian* manifold we mean a C^∞-differentiable paracompact connected manifold endowed with a *pseudo-riemannian metric*, i.e. a symmetric C^∞ tensor field of type $(0, 2)$ which is nondegenerate and has the same index at each point. Let g be the metric in question. Denote by n and p its dimension and index, respectively. Clearly, there is no loss of generality if we assume that $p \leqslant \frac{1}{2}n$, i.e. if we replace, if convenient, g by $-g$. We say that a pseudo-riemannian manifold is *essentially pseudo-riemannian* if $1 \leqslant p \leqslant \frac{1}{2}n$. For the definition and properties of C^∞-differentiable manifolds as well as tensor fields we refer to [1].

Given a pseudo-riemannian manifold M and an $x \in \operatorname{supp} M$, $T_x M$ denotes the tangent space to M at x, while

$$I^0_x M = \{v \in \operatorname{supp} T_x M:\ g(v, v) = 0\},$$

$$I^+_x M = \{v \in \operatorname{supp} T_x M:\ g(v, v) > 0\},$$

$$I^-_x M = \{v \in \operatorname{supp} T_x M:\ g(v, v) < 0\}.$$

In other words $I_x^0 M$ is the collection of vectors of all isotropy subspaces of $T_x M$, while $I_x^+ M$ and $I_x^- M$ are the collections of vectors of all positive and negative definite subspaces of $T_x M$, respectively. Further, TM denotes the tangent bundle of M. Finally, if N is another pseudo-riemannian manifold and $f\colon M \to N$ a diffeomorphism, then $Df\colon TM \to TN$ denotes the derivative of f.

2. Measurability and integration

Suppose that X and Y are C^∞-differentiable paracompact connected manifolds, while M and N are pseudo-riemannian manifolds with metrics g and g', respectively. Under a *Borel measure* on X we mean a measure which is defined on the collection of Borel subsets of $\operatorname{supp} X$. A mapping $f\colon X \to Y$ is said to be a *Borel function* if the preimage $f^{-1}[E]$ of each open set $E \subset \operatorname{supp} Y$ is a Borel set.

As in [13], p. 8, we say that a set $E \subset \operatorname{supp} X$ is a *null set* if for each coordinate neighbourhood $U \subset \operatorname{supp} X$ and each coordinate C^∞-mapping $\mu\colon U \to \operatorname{supp} R^n$ the set $\mu[E \cap U]$ has Lebesgue measure zero. A condition is said to hold for *almost every* $x \in \operatorname{supp} X$, or *almost everywhere* on X, if it holds everywhere except perhaps for a null set. In our considerations as derivatives of functions are differentiable almost everywhere we shall meet functions which are not defined on a Borel null set. If such a function is Borel on its set of definition, then its extension by a constant value will also be Borel. We will carry out always such an extension by the value 0. Hence we may regard all functions as defined everywhere.

Theorem 1. *Suppose that $f\colon M \to N$ is continuous and differentiable almost everywhere. For any $x \in \operatorname{supp} M$ consider arbitrary coordinate C^∞-mappings $\mu = (\mu^i)$ on M at x and $\nu = (\nu^i)$ on N at $f(x)$ whose dimensions are equal to the dimensions of the corresponding manifolds. Then*

(i) *the quantities*

$$\|Df\|(x) = \sup\left|g'\big(Df(x)(v), Df(x)(v)\big)\right|^{1/2}, \quad x \in \operatorname{supp} M,$$

where the supremum is taken over all $v \in T_x M$ such that $|g(v, v)| \leqslant 1$, and

$$(\det Df)(x) = \det(\nu^j \circ f \circ \mu^{-1})_{|i} \circ \mu(x) \left|\frac{\det g'_{ij} \circ \nu \circ f(x)}{\det g_{ij} \circ \mu(x)}\right|^{1/2}, \quad x \in \operatorname{supp} M,$$

where $_{|i}$ denotes partial differentiation with respect to μ^i, do not depend on the choice of μ and ν,

(ii) *the functions $\|Df\|\colon M \to R$ and $\det Df\colon M \to R$ are Borel.*

We introduce then the notion of jacobian. If $f\colon M \to N$ is a C^1-diffeomorphism, then, by Theorem 1, $\det Df$ is a real-valued Borel function of $x \in \operatorname{supp} M$, i.e. $\det Df\colon M \to R$. Moreover, as it is easily seen, it is continuous. The function

$$J_f = |\det Df|$$

is called the *jacobian* of f.

The following theorem enables us to define the Lebesgue measure on a pseudo-riemannian manifold.

Theorem 2. *With each M we can associate a unique Borel measure $\tau(M)$ so that the following conditions are satisfied:*

(i) *if N is an open pseudo-riemannian submanifold of M, then $\tau(M)(E) = \tau(N)(E)$ for all Borel sets $E \subset \operatorname{supp} N$,*

(ii) *if $f\colon M \to N$ is a C^1-diffeomorphism, then*

$$\tau(N)(f[E]) = \int_E J_f d\tau(M)$$

for all Borel sets $E \subset \operatorname{supp} M$,

(iii) *if $M = R^m$ or R^m_+, $m = 1, 2, \ldots$, then $\tau(M)$ is the Lebesgue measure.*

Now we define the *Lebesgue measure* on a pseudo-riemannian manifold M as the measure $\tau(M)$ determined in Theorem 2.

We conclude this section by a corollary.

Corollary 1. *If $f\colon M \to N$ is a C^1-diffeomorphism, then a Borel function $\varrho\colon N \to R$ is $\tau(N)$-integrable if and only if $(\varrho \circ f)J_f$ is $\tau(M)$-integrable and*

$$\int_N \varrho d\tau(N) = \int_M (\varrho \circ f) J_f d\tau(M).$$

The proofs are analogous to that given in [13], p. 9-12, in the case of Riemannian manifolds.

3. Angles and their inner measure

Let M be an essentially pseudo-riemannian manifold with metric g. For a real number a, $a \neq 0$, let

$$I_x^a M = \{v \in \operatorname{supp} T_x M : g(v, v) = a\}.$$

We say that a set E *forms an ordinary angle* $\arg(x, E)$ at a point x of M if E is a Borel subset of some $I_x^a M$, $a \neq 0$. We say that a set E *forms a topological angle* $\arg(x, E)$ at a point x of M if E is a Borel subset of either $I_x^+ M$ or $I_x^- M$.

Let $x \in \operatorname{supp} M$. Given a set E that forms a topological angle at x, let

$$I_x E = \{bv\colon v \in E, 0 < b < 1/|g(v, v)|^{1/2}\}.$$

It is easily seen that $I_x E$ is Lebesgue-measurable on $T_x M$. According to Section 2, we denote its Lebesgue measure by $\tau(T_x M)(I_x E)$ and the volume element by $d\tau(T_x M)$. We then define the *inner measure* $A(x, E)$ of $\arg(x, E)$ by

$$A(x, E) = \tau(T_x M)(I_x E).$$

4. Curves and arc length

Let M be an essentially pseudo-riemannian manifold with metric g. Under a *curve* on M we understand a continuous mapping c from a closed interval $[a; b]$, $a \leqslant b$, to M. If c is differentiable, we identify the derivative $Dc(t)$, $t \in [a; b]$, with a tangent vector to M at $c(t)$. This determines a curve Dc in the tangent bundle TM.

A curve c is called *space-like* (resp. *time-like*) if it is absolutely continuous and $Dc(t)$ is a vector of a positive (resp. negative) definite subspace of $T_{c(t)} M$ at every point of differentiability. If c is either space-like orl time-like, it is called *regular*.

The length of a regular curve c is defined by

$$l(c) = \int_{[a;b]} |g(Dc(t), Dc(t)|^{1/2} dt.$$

If $l(c)$ is finite, c is said to be *rectifiable*. Now let $\varrho\colon M \to R$ be a Bore function, c_0 – the parametrization of c by arc length, and ds – the arc length element. The integral of ϱ along c is defined by

$$\int_c ds = \int_0^{l(c)} \varrho \circ c_0 \, ds,$$

provided that the latter integral exists. Otherwise the integral of ϱ along c is undefined.

Finally, suppose that N is an essentially pseudo-riemannian manifold and $f\colon M \to N$ a C^1-diffeomorphism. Then f is said to be *type-preserving* (resp. *type-reversing*) if it transforms space-like curves onto space-like (resp. space-like) curves. Here we confine ourselves to one theorem needed later on:

Theorem 3. *Suppose that $f\colon M \to N$ is either type-preserving or type--reversing, $c\colon [a; b] \to M$ is rectifiable, while $\varrho\colon N \to R$ is Borel and non-negative. Then $f(c)$ is rectifiable and*

$$\int_{f(c)} \varrho \, ds \leqslant \int_c (\varrho \circ f) \|Df\| \, ds.$$

The proof is analogous to that given in [13], p. 14, in the case of Riemannian manifolds.

5. Moduli

Here we give an analogue of the p-moduli discussed in [13], p. 15-20. Our composition and proofs, however, follow rather [4] or [11]. Throughout the whole section M is an essentially pseudo-riemannian manifold, while $C, C_0, C_1, C_2, \ldots$ are families of regular curves on M.

Denote by $\operatorname{adm} C$ the class of all nonnegative Borel functions ϱ on M which satisfy

$$\int_c \varrho\, ds \geqslant 1$$

for all rectifiable $c \in C$. Here we do not assume that the integrals in question are finite. If $\varrho \in \operatorname{adm} C$, ϱ is said to be an ***admissible metric*** for C. For each positive number p we define the ***p-modulus*** $\operatorname{mod}_p C$ of C by

$$\operatorname{mod}_p C = \inf \int_M \varrho^p\, d\tau ,$$

where the infimum is taken over all $\varrho \in \operatorname{adm} C$. If $\operatorname{adm} C$ is empty, we put $\operatorname{mod}_p C = \infty$. The quantity $1/\operatorname{mod}_p C$ is called the ***p-extremal length*** of C.

If in $\operatorname{adm} C$ there is a metric ϱ_0 such that

$$\operatorname{mod}_p C = \int_M \varrho_0^p\, d\tau ,$$

then ϱ_0 is called ***p-extremal.*** It has the following important property:

Theorem 4. (uniqueness of an extremal metric). *If, for some positive integer p, $\operatorname{mod}_p C$ is finite and ϱ_0, ϱ_0^* are p-extremal, then $\varrho_0^* = \varrho_0$ almost everywhere on M.*

Now we formulate other basic properties of p-moduli. Thereafter p is a positive number and Σ, $\cup$ denote summation over all positive integers k.

Theorem 5 (monotoneity of moduli). *If $C_1 \subset C_2$ or, more generally, each c_1 of C_1 contains a c_2 of C_2, then*

$$\operatorname{mod}_p C_1 \leqslant \operatorname{mod}_p C_2 .$$

Theorem 6 (the principle of composition for extremal lengths). *Suppose that C_k, $k = 1, 2, \ldots$, consist of curves lying in disjoint Borel subsets E_k of* supp *M, respectively, and that any c of C contains some curve of C_k for each k. Then*

$$\sum 1/\operatorname{mod}_p^{\frac{1}{p-1}} C_k \leqslant 1/\operatorname{mod}_p^{\frac{1}{p-1}} C, \; p > 1,$$

$$\sum 1/\operatorname{mod}_p C_k \leqslant 1/\operatorname{mod}_p C, \; p \geqslant 2 .$$

Theorem 7 (subadditivity of moduli). *If* $C = \bigcup C_k$, *then*

$$\operatorname{mod}_p C \leq \sum \operatorname{mod}_p C_k .$$

Theorem 8 (superadditivity of moduli). (i) *Suppose that* $\bigcup C_k \subset C$ *and that all* C_k *consist of curves lying in disjoint Borel subsets* E_k *of* supp M, *respectively. Then*

(1) $$\sum \operatorname{mod}_p C_k \leq \operatorname{mod}_p C .$$

(ii) *Estimate* (1) *remains valid if the condition* $\bigcup C_k \subset C$ *is replaced by the requirement for each* c_k *of* C_k, $k = 1, 2, \ldots$, *to contain some curve of* C.

Theorems 4-8 are valid also in the case where M is pseudo-riemannian but not essentially. If the index p of M equals 1, i.e. in the riemannian case, $C, C_0, C_1, C_2, \ldots$ denote just families of curves on M (cf. [13], p. 15-20). If $\frac{1}{2}n < p < n$, we can establish the same results as those given above on replacing the metric g of M by $-g$.

6. Conformality

Thereafter we always assume that M and N are essentially pseudo-riemannian manifolds with metrics g and g', repsectively, while $f\colon M \to N$ is a C^1-diffeomorphism.

A C^1-diffeomorphism $f\colon M \to N$ is said to be *conformal* if

(2) $$Df(x)[I_x^0 M] = I_{f(x)}^0 N, \; x \in \operatorname{supp} M,$$

in the case where the index of g is less than $\frac{1}{2}n$, while

(3) $$Df(x)[I_x^+ M] = I_{f(x)}^+ N, \; x \in \operatorname{supp} M,$$

in the case where the index of g equals $\frac{1}{2}n$, n being the dimension of g.

We begin with a theorem that gives a necessary and sufficient condition for conformality. This condition agrees with the usual definition applied in the case of Riemannian manifolds (cf. [8], p. 106, [10], vol. I, p. 309, and [13], p. 16) as well as in the local formulation in the case of pseudo-riemannian manifolds (cf. [3], p. 89, and [9], p. 5).

Theorem 9. *A* C^1*-diffeomorphism* $f\colon M \to N$ *is conformal if and only if*

$$g'\big(Df(x)(v), Df(x)(v)\big) = a(x) g(v, v),$$

$$a(x) > 0, \; x \in \operatorname{supp} M, \; v \in \operatorname{supp} T_x M,$$

where a *does not depend on* v.

Theorem 9 implies:

Corollary 2. *Conditions* (3) *and*

$$Df(x)[I_x^- M] = I_f^- N, \; x \in \operatorname{supp} M,$$

are both necessary and sufficient for a C^1-diffeomorphism $f\colon M \to N$ to be conformal.

Next we give a characterization of conformal mappings in terms of the inner measure of angles.

Theorem 10. *If $f\colon M \to N$ is conformal and E forms a topological angle at $x \in \operatorname{supp} M$, then*

(i) *$Df(x)[E]$ forms a topological angle at $f(x)$ and*

$$A\big(f(x), Df(x)[E]\big) = A(x, E),$$

(ii) *the relation $E \subset I_x^+ M$ implies $Df(x)[E] \subset I_{f(x)}^+ N$, while $E \subset I_x^- M$ imples $Df(x)[E] \subset I_{f(x)}^- N$.*

If, in particular, E forms an ordinary angle at x, then $Df(x)[E]$ forms an ordinary angle at $f(x)$.

Theorem 11. *Suppose that $f\colon M \to N$ is a C^1-diffeomorphism and that if E forms an ordinary angle at $x \in \operatorname{supp} M$, then*

(i) *$Df(x)[E]$ forms a topological angle at $f(x)$,*

(ii) *the relation $E \subset I_x^+ M$ implies $Df(x)[E] \subset I_{f(x)}^+ N$, while $E \subset I_x^- M$ implies $Df(x)[E] \subset I_{f(x)}^- N$.*

Then f is conformal.

Finally we give a characterization of conformal mappings in terms of moduli.

Theorem 12. *If f is conformal, then it is type-preserving. Furthermore, if C is a family of regular curves on M, then*

$$\operatorname{mod}_n f(C) = \operatorname{mod}_n C. \tag{4}$$

If, in particular,

$$0 < k \leqslant \|Df(x)\| \leqslant K < \infty, \; x \in \operatorname{supp} M,$$

then

$$K^{n-p} \operatorname{mod}_p C \leqslant \operatorname{mod}_p f(C) \leqslant k^{n-p} \operatorname{mod}_p C \quad \text{for} \quad p \geqslant n$$

and

$$k^{n-p} \operatorname{mod}_p C \leqslant \operatorname{mod}_p f(C) \leqslant K^{n-p} \operatorname{mod}_p C \text{ for } p \leqslant n.$$

Theorem 13. *If $f\colon M \to N$ is type-preserving, then it is conformal.*

7. Conclusions

Suppose that $f\colon M \to N$ is type-preserving and that there is a constant Q, $1 \leqslant Q < \infty$, such that

$$(1/Q) \operatorname{mod}_n C \leqslant \operatorname{mod}_n f(C) \leqslant Q \operatorname{mod}_n C \tag{5}$$

for some family C of regular curves. Then, by Theorem 13, f is conformal and consequently, by Theorem 12, we get (4). Hence we conclude that in the case of essentially pseudo-riemannian manifolds there is no analogue of regular quasiconformal mappings other than conformal (cf. [12], p. 18, 179, and 222 (Theorems 3.2 and 4.2), for the plane case; [14], p. 18-19, for the euclidean case; and [13], p. 24-25, for the riemannian case). Nevertheless, it is quite possible that if we properly weaken the hypotheses of Theorem 13 in the sense that we allow some less smooth mappings and assume, in addition, that f preserves the n-moduli, we will still be able to prove that f is conformal (cf. [5], p. 388-390). Then it will be natural to consider also the case where the preservation of the n-moduli is replaced by a quasi-preservation in the sense of (5) with some fixed Q, where C ranges over the class of all families of regular curves on M.

Other important problems that seem to be very natural are the convergence properties of sequences of conformal mappings, in particular, the problem of finding some conditions under which the limit mapping is conformal. These questions, including the problem of obtaining some analogue of the Carathéodory convergence theorem (cf. [2] and [6]), are essential for physical applications. They were not solved even in the riemannian case.

The authors plan to discuss at least some of these problems in a subsequent paper.

REFERENCES

[1] Bishop, R. L., and Goldberg, S. I., *Tensor analysis on manifolds*, New York — London 1968.

[2] Carathéodory, C., *Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten*, Math. Ann. 72 (1912), p. 107–144.

[3] Eisenhart, L. P., *Riemannian geometry*, Princeton 1949.

[4] Fuglede, B., *Extremal length and functional complection*, Acta Math. 98 (1957) p. 171–219.

[5] Gehring, F. W., *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. 103 (1962), p. 353–393.

[6] — *The Carathéodory convergence theorem for quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. Ser. AI 336/11 (1963), 21 p.

[7] — and Haahti, H., *The transformations which preserve the harmonic functions*, ibid. 293 (1960), 12 p.

[8] Goldberg, S. I., *Curvature and homology*, New York — London 1962.

[9] Haahti, H., *Über konforme Abbildungen eines euklidischen Raumes in eine Riemannsche Mannigfaltigkeit*, Ann. Acad. Sci. Fenn. Ser. AI 287 (1960), 54 p.

[10] Kobayashi, S., and Nomizu, K., *Foundations of differential geometry* I–II, New York — London — Sydney 1963–1969.

[11] Krzyż, J., and Ławrynowicz, J., *Metoda długości ekstremalnej i jej zastosowania* [*The method of extremal length and its applications*], Materiały Konferencji Szkoleniowej Funkcji Analitycznych w Uniejowie, Łódź 1969, p. 19–83.

[12] Lehto, O., und Virtanen, K. I., *Quasikonforme Abbildungen*, Berlin — Heidelberg — New York 1965.

[13] Suominen, K., *Quasiconformal maps in manifolds*, Ann. Acad. Sci. Fenn. Ser. AI 393 (1966), 39 p.

[14] Väisälä, J., *On quasiconformal mappings in space*, ibid. 298 (1961), 36 p.

STRESZCZENIE

Praca dotyczy odwzorowań konforemnych rozmaitości pseudoriemannowskich rozpatrywanych globalnie.

W części przygotowawczej wprowadzamy pewne oznaczenia i pojęcia, w szczególności pojęcie rozmaitości pseudoriemannowskiej, dyskutujemy zagadnienia mierzalności i całkowalności, wprowadzamy pojęcie kąta i definiujemy jego miarę wewnętrzną. Z kolei zajmujemy się krzywymi, w szczególności wyróżniamy pewne ich rodziny: przestrzenne, czasowe, regularne i prostowalne, definiujemy długość krzywej regularnej, wprowadzamy pewne klasy odwzorowań: zgodne i niezgodne oraz uzyskujemy podstawowe twierdzenie o tych odwzorowaniach, które daje pewną nierówność istotną dla dalszych badań. Następnie wprowadzamy pojęcie modułu rzędu p rodziny krzywych regularnych oraz badamy podstawowe własności tych modułów.

W drugiej części pracy zajmujemy się odwzorowaniami konforemnymi rozmaitości istotnie pseudoriemannowskich. Wprowadzone pojęcie konforemności oznacza, z grubsza biorąc, zachowanie stożka izotropowego w każdym punkcie rozpatrywanej rozmaitości. Z kolei uzyskujemy warunek konieczny i dostateczny konforemności w terminach form kwadratowych określonych przez metryki rozpatrywanych rozmaitości. Podajemy charakteryzację odwzorowań konforemnych w terminach kątów i ich miary wewnętrznej, a w końcu, w terminach krzywych regularnych i ich modułów.

Wreszcie, definiujemy odwzorowania quasi-konforemne regularne i podsumowujemy pracę wynikiem orzekającym, iż w przypadku rozmaitości istotnie pseudoriemannowskich nie ma odpowiednika odwzorowań quasi-konforemnych regularnych i niekonforemnych jednocześnie. Pragniemy tu zaznaczyć, że problem istnienia stosownie określonych odwzorowań quasi-konforemnych nieregularnych pozostaje otwarty. W zakończeniu stawiamy także pewne inne naturalne problemy.

РЕЗЮМЕ

Работа касается конформных отображений псевдоримановых многообразий, рассматриваемых в целом.

В предварительной части введены некоторые обозначения и терминология, в частности понятие псевдориманового многообразия, рассмотрены вопросы измеримости и интегрирования, введено понятие угла и определена его внутренняя мера. Далее рассмотрены кривые с особенным выделением нескольких семейств кривых: пространственных, временных, регулярных и спрямляемых; определена длина регулярной кривой, введено несколько видов отображений: согласные

и несогласные и получена основная теорема об этих отображениях, дающая некоторое неравенство существенное для дальнейших исследований. Введено понятие p-модуля семейства регулярных кривых и изучаются основные свойства этих модулей.

Во второй части работы рассмотрены конформные отображения, по существу псевдоримановых многообразий. Введено понятие конформности, которое обозначает, приблизительно, сохранение изотропного конуса в каждой точке рассматриваемого многообразия. Получено необходимое и достаточное условие для конформности в терминах квадратных форм, которые определены метриками рассматриваемых многообразий. Дана характеристика конформных отображений в терминах углов и их внешней меры, а также в терминах регулярных кривых и их модулей.

В конце работы определены регулярные квазиконформные отображения и сделан вывод, что в случае по существу псевдоримановых многообразий нет совместного аналога для квазиконформных и регулярных отображений. Следует отметить, что проблема существования соответственно определенных нерегулярных квазиконформных отображений остается открытой. Представлены также и другие естественные проблемы.