ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA
VOL. XXI, 7 SECTIO A 1967

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Univalent Polynomials of Small Degree

Wielomiany jednolistne małego stopnia

Однолистные многочлены небольшой степени

1. Introduction and statement of results

Let S denote the class of functions $f(z) = z + a_2 z^2 + \ldots$ regular and univalent in the unit disk $K = \{z: |z| < 1\}$. The subclass of S consisting of polynomials of degree n will be denoted by $\mathscr{P}_n$. It was believed (see for example [1]) that the transformation

$$f(z) \to F(z) = \int_0^z \zeta^{-1} f(\zeta)\, d\zeta$$

preserves the class S. But in 1963 it was observed by Krzyż and Lewandowski [5] that the function

$$f(z) = z\exp\{(i-1)\,\mathrm{Log}(1-iz)\} = \sum_{k=1}^{\infty} a_k z^k,$$

where Log denotes the principal branch of the logarithm, belongs to S but the corresponding $F(z)$ does not. In fact, $F(z)$ assumes the same value at the points

$$z_1 = i(e^{2\pi}-1)(e^{2\pi}+1)^{-1}, \quad z_2 = -z_1,$$

i.e. the function

$$\int_0^z \exp\{(i-1)\,\mathrm{Log}(1-i\zeta)\}\, d\zeta = \sum_{k=1}^{\infty} k^{-1} a_k z^k$$

is at least 2-valent in $|z| < e^{2\pi}(e^{2\pi}+1)^{-1}$.

A theorem of Montel ([8], p. 8) states that if the sequence of functions

$$f_1(z), f_2(z), \ldots, f_n(z), \ldots$$

each regular and p-valent in a domain D converges uniformly to a non-constant function $f(z)$, then $f(z)$ is at most p-valent in D. It follows that the n-th partial sums $S_n(z) = \sum_{k=1}^{n} k^{-1} a_k z^k$ of the function

$$F(z) = \int_0^z \zeta^{-1} f(\zeta)\, d\zeta = \sum_{k=1}^{\infty} k^{-1} a_k z^k$$

are at least 2-valent in $|z| < e^{2\pi}(e^{2\pi}+1)^{-1}$ if n is sufficiently large, say $n \geqslant N_1$. According to another theorem of Montel ([8], p. 9) if a sequence of functions

$$f_1(z), f_2(z), \ldots, f_n(z), \ldots$$

each regular in a domain D converges uniformly to a function $f(z)$ at most p-valent in D then on any given compact subset D' of D the functions $f_n(z)$ are at most p-valent if n is sufficiently large. Thus the n-th partial sums $s_n(z) = \sum_{k=1}^{n} a_k z^k$ of the function

$$f(z) = z \exp\{(i-1)\operatorname{Log}(1-i\zeta)\} = \sum_{k=1}^{\infty} a_k z^k$$

are univalent in

$$|z| < 2^{-1}(2e^{2\pi}+1)(e^{2\pi}+1)^{-1} = \lambda \text{ say},$$

if $n \geqslant N_2$. We see that if $n \geqslant \max(N_1, N_2)$ then $p_n(z) = \lambda^{-1} s_n(z\lambda)$ is univalent in $|z| < 1$ but $P_n(z) = \int_0^z \zeta^{-1} p_n(\zeta)\, d\zeta$ is not. Hence the transformation

$$p_n(z) \to P_n(z) = \int_0^z \xi^{-1} p_n(\zeta)\, d\zeta$$

does not preserve the class $\mathscr{P}_n$ if n is large enough.

We prove

Theorem 1. *If the polynomial* $p_n(z) = z + \sum_{k=2}^{n} a_k z^k$ *is univalent in* $|z| < 1$ *then the polynomial* $P_n(z) = \int_0^z \zeta^{-1} p_n(\zeta)\, d\zeta$ *is univalent in* $|z| < 2 \sin \pi/n$. *Hence the transformation* $p_n(z) \to \int_0^z \zeta^{-1} p_n(\zeta)\, d\zeta$ *preserves the class* $\mathscr{P}_n$ *if* $n \leqslant 6$.

We show that for polynomials of degree 2, 3 the hypothesis can be slightly weakened. In fact, it is enough to assume that $p'(z) \neq 0$ in $|z| < 1$.

With reference to Theorem 1 we may ask the converse question:

If $p_n(z) \in \mathscr{P}_n$ what is the radius ϱ_n of the largest disk centred at the origin in which $zp_n'(z)$ is necessarily univalent? While trying to answer this question we restrict ourselves to polynomials of degree 3.

The polynomial $p_3(z) = z - \frac{2\sqrt{2}}{3} z^2 + \frac{1}{3} z^3$ which is univalent in $|z| < 1$ and for which $[zp_3(z)]'$ vanishes at $z = \frac{1}{9}(4\sqrt{2} - \sqrt{5})$ shows that $\varrho_3 \leqslant \frac{1}{9}(4\sqrt{2} - \sqrt{5}) = 0.380087$ approximately.

Theorem 2. *If $p(z) = z + a_2 z^2 + a_3 z^3$ is univalent in $|z| < 1$ then the polynomial $zp'(z)$ is univalent in $|z| < 1/\sqrt{7}$. Hence $\varrho_3 \geqslant 1/\sqrt{7}$.*

Since $1/\sqrt{7}$ is approximately equal to 0.377964 we have determined ϱ_3 with an error of at most 0.56 per cent.

In analogy with Theorem 1 we prove

Theorem 3. *If the polynomial $p_n(z) = z + \sum_{k=2}^{n} a_k z^k$ is univalent in $|z| < 1$ then the polynomial $\check{p}_n(z) = 2z^{-1} \int_0^z p_n(\zeta) d\zeta$ is univalent in $|z| < 2\sin\pi/(n+1)$. Hence the transformation $p_n(z) \to \check{p}_n(z) = 2z^{-1} \int_0^z p(\zeta) d\zeta$ preserves the class $\mathscr{P}_n$ if $n \leqslant 5$.*

It has been shown by Libera [6] that if $f(z) = z + a_2 z^2 + \dots$ is close-to-convex with respect to $g(z)$ then $\check{f}(z) = 2z^{-1} \int_0^z f(\zeta) d\zeta$ is close-to-convex with respect to $\check{g}(z) = 2z^{-1} \int_0^z g(\zeta) d\zeta$. The radius of close-to-convexity for functions belonging to S was determined [4] to be r_0 where $0.80 < r_0 < 0.81$. Hence if $f(z) = z + \sum_{k=2}^{n} a_k z^k$ is univalent in $|z| < 1$ then $\check{f}(z) = 2z^{-1} \int_0^z f(\zeta) d\zeta$ is univalent in $|z| < r_0$. It is still an open question whether $\check{f}(z)$ is univalent in $|z| < 1$.

2. Lemmas

Lemma 1. (Dieudonné Criterion). *The polynomial $p_n(z) = z + \sum_{k=2}^{n} a_k z^k \in \mathscr{P}_n$ if and only if the associated polynomial*

$$1 + a_2 \frac{\sin 2\theta}{\sin\theta} z + \dots + a_n \frac{\sin n\theta}{\sin\theta} z^{n-1}$$

does not vanish in $|z| < 1$ for $0 \leqslant \theta \leqslant \pi/2$.

Lemma 1 is proved in ([3], p. 310).

Lemma 2. *If all the zeros of the polynomial*

$$f_n(z) = \sum_{k=0}^{n} \binom{n}{k} A_k z^k,$$

lie in the circle $|z| \leqslant r_f$ *and if all the zeros of*

$$g_n(z) = \sum_{k=0}^{n} \binom{n}{k} B_k z^k$$

lie in the circle $|z| \leqslant r_g$, *then all the zeros of*

$$h_n(z) = \sum_{k=0}^{n} \binom{n}{k} A_k B_k z^k$$

lie in the circle $|z| \leqslant r_f \cdot r_g$.

For a proof of Lemma 2 see [7], pp. 65–66.

The following result is due to D. A. Brannan [2].

Lemma 3. *Suppose* $p(z) = z + a_2 z^2 + tz^3$ *where* t *is real and positive, and* $a_2 = \alpha + i\beta$. *Then*

a) *For* $0 \leqslant t \leqslant 1/5$, $p(z) \in \mathscr{P}_3$ *iff:*

$$\left(\frac{2\alpha}{1+3t}\right)^2 + \left(\frac{2\beta}{1-3t}\right)^2 \leqslant 1.$$

b) *If* $1/5 \leqslant t \leqslant 1/3$, $p(z) \in \mathscr{P}_3$ *iff:*

$$\alpha + i\beta \in \bigcap_{(1-2t)t^{-1} \leqslant d \leqslant 3} E_d$$

where

$$E_d = \left\{\alpha + i\beta : \frac{\alpha^2}{\left(\frac{1+td}{\sqrt{1+d}}\right)^2} + \frac{\beta^2}{\left(\frac{1-td}{\sqrt{1+d}}\right)^2} \leqslant 1\right\}.$$

Suppose t is a fixed nonnegative number $\leqslant \frac{1}{3}$. Let $H(t)$ be the region of possible values of a_2 in order that $p(z) = z + a_2 z^2 + tz^3 \in \mathscr{P}_3$. From Lemma 3 it follows that $H(t)$ is a closed and bounded convex set containing the origin and symmetric with respect to the coordinate axes. For $0 \leqslant \varphi \leqslant 2\pi$ let $K(\varphi) = \max_{\zeta \in H(t)} \operatorname{Re}(\zeta e^{-i\varphi})$ be the supporting function of $H(t)$. We shall need the following estimates for $K(\varphi)$.

Lemma 4. *If* $0 \leqslant t \leqslant 1/5$ *then*

$$K(\varphi) = \tfrac{1}{2}(1 + 9t^2 + 6t\cos 2\varphi)^{1/2}. \tag{2.1}$$

If $1/5 < t < 1/3$ *and* φ_0 *is the unique root of the equation*

$$\cos 2\varphi = (2t)^{-1}(1 - 15t^2), \tag{2.2}$$

contained in $(0, \pi/2)$, *then*

$$K(\varphi) \leqslant \{2t(1 + t^2 - 2t\cos 2\varphi)^{1/2} + 2t\cos 2\varphi - 2t^2\}^{1/2} \tag{2.3}$$

for $|\varphi|\leqslant\varphi_0$, $|\varphi-\pi|\leqslant\varphi_0$, *whereas for* $\left|\varphi-\frac{\pi}{2}\right|<\frac{\pi}{2}-\varphi_0$, $\left|\varphi-\frac{3\pi}{2}\right|<\frac{\pi}{2}-\varphi_0$,

$$K(\varphi)\leqslant\tfrac{1}{2}(1+9t^2+6t\cos 2\varphi)^{1/2}. \tag{2.4}$$

If $t=1/3$, *then*

$$K(\varphi)=\frac{2\sqrt{2}}{3}|\cos\varphi|. \tag{2.5}$$

Proof.

(i) The case $0\leqslant t\leqslant\frac{1}{5}$.

Since $H(0)=\{z\colon |z|\leqslant\frac{1}{2}\}$, the formula (2.1) obviously holds for $t=0$. If $0<t\leqslant\frac{1}{5}$, then $H(t)$ is the ellipse $\left\{x+iy\colon \frac{x^2}{\left(\frac{1+3t}{2}\right)^2}+\frac{y^2}{\left(\frac{1-3t}{2}\right)^2}\leqslant 1\right\}$.

It is easy to verify that the supporting function of the ellipse $E=\left\{x+iy\colon \frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\right\}$ has the form

$$K(\varphi)=(a^2\cos^2\varphi+b^2\sin^2\varphi)^{1/2}. \tag{2.6}$$

Thus we see that the supporting function of $H(t)$ is given by (2.1).

(ii) The case $\frac{1}{5}<t<\frac{1}{3}$.

Because of obvious symmetry we may restrict ourselves to φ in $\left[0,\frac{\pi}{2}\right]$. Let $K(d,\varphi)$ be the supporting function of the ellipse E_d mentioned above. According to (2.6)

$$K^2(d,\varphi)=(1+d)^{-1}(1+t^2d^2+2td\cos 2\varphi). \tag{2.7}$$

Lemma 3 says that for $\frac{1}{5}<t<\frac{1}{3}$ the set $H(t)$ is the intersection of the family of ellipses E_d where d varies over the interval $I_t=[t^{-1}-2, 3]$.

Hence

$$K(\varphi)\leqslant\inf_{d\in I_t}K(d,\varphi). \tag{2.8}$$

For $t>0$, the equation $\frac{\partial}{\partial d}K^2(d,\varphi)=0$ has a unique, positive root d_1 where

$$d_1=-1+t^{-1}(1+t^2-2t\cos 2\varphi)^{1/2}. \tag{2.9}$$

Moreover, $\frac{\partial}{\partial d}K^2(d,\varphi)<0$ for $0<d<d_1$, $\frac{\partial}{\partial d}K^2(d,\varphi)>0$ for $d>d_1$. If $d_1\in I_t$, then for a fixed φ, $K(d,\varphi)$ attains its absolute minimum at $d=d_1$, whereas, if $d_1\notin I_t$ then the absolute minimum is attained at one of the end points of I_t. In view of (2.9), $d_1 < t^{-1}-2$ is not possible. For a fixed t, $d_1 \geqslant 3$ if and only if

$$\cos 2\varphi \leqslant (2t)^{-1}(1-15t^2). \tag{2.10}$$

If $1/5 < t < 1/3$, then $-1 < (2t)^{-1}(1-15t^2) < 1$ and equation (2.2) has a unique solution φ_0 in $(0, \pi/2)$. If $\varphi_0 \leqslant \varphi \leqslant \pi/2$ then $\cos 2\varphi \leqslant (2t)^{-1}(1-15t^2)$ and hence $d \geqslant 3$. This implies that $K(d, \varphi)$ is a strictly decreasing function of d in I_t and

$$\inf_{d \in I_t} K(d, \varphi) = K(3, \varphi) = \tfrac{1}{2}(1+9t^2+6t\cos 2\varphi)^{1/2}.$$

On the other hand, if $0 \leqslant \varphi \leqslant \varphi_0$, then

$$\cos 2\varphi \geqslant (2t)^{-1}(1-15t^2), \tag{2.11}$$

and this implies that $d_1 \in I_t$. Hence

$$K(\varphi) \leqslant \inf_{d \in I_t} K(d, \varphi) = K(d_1, \varphi) = \{2t(1+t^2-2t\cos 2\varphi)^{1/2}+2t\cos 2\varphi-2t^2\}^{1/2}.$$

(iii) The case $t = \frac{1}{3}$.

It follows from Lemma 3 that $H\left(\dfrac{1}{3}\right) = \left[-\dfrac{2\sqrt{2}}{3}, \dfrac{2\sqrt{2}}{3}\right]$ and formula (2.5) can be easily verified.

Lemma 5. *Let H_1 and H_2 be two convex sets containing the origin. If K_1 and K_2 are respectively their supporting functions then $H_1 \subset H_2$ if and only if $K_1(\varphi) \leqslant K_2(\varphi)$ for $0 \leqslant \varphi < 2\pi$.*

Proof. If $H_1 \subset H_2$ then clearly

$$K_1(\varphi) = \max_{\zeta \in H_1} \operatorname{Re}(\zeta e^{-i\varphi}) \leqslant \max_{\zeta \in H_2} \operatorname{Re}(\zeta e^{-i\varphi}) = K_2(\varphi).$$

On the other hand,

$$H_j = \bigcap_{0 \leqslant \varphi < 2\pi} \{x+iy\colon x\cos\varphi + y\sin\varphi \leqslant K_j(\varphi)\}, \quad j = 1, 2.$$

Hence $H_1 \subset H_2$ if $K_1(\varphi) \leqslant K_2(\varphi)$ for $0 \leqslant \varphi < 2\pi$.

3. Proofs of the theorems

Proof of Theorem 1. Set $2\sin\dfrac{\pi}{n} = \mu$. It is clearly enough to prove that

$$Q_n(z) = \mu^{-1}P_n(\mu z) = z + \frac{1}{2}\mu a_2 z^2 + \frac{1}{3}\mu^2 a_3 z^3 + \ldots + \frac{1}{n}\mu^{n-1}a_n z^n$$

is univalent in $|z| < 1$. According to Dieudonné Criterion the polynomial $Q_n(z)$ is univalent in $|z| < 1$ if and only if the polynomial

$$H_{n-1}(z) = 1+\frac{1}{2}\mu a_2\frac{\sin 2\theta}{\sin\theta}z+\frac{1}{3}\mu^2 a_3\frac{\sin 3\theta}{\sin\theta}z^2+\ldots+\frac{1}{n}\mu^{n-1}a_n\frac{\sin n\theta}{\sin\theta}z^{n-1}$$

does not vanish in $|z|<1$ for $0\leqslant\theta\leqslant\pi/2$. Thus we need to show that the zeros of

$$\begin{aligned}h_{n-1}(z) &= z^{n-1}H_{n-1}(z^{-1})\\ &= z^{n-1}+\frac{1}{2}\mu a_2\frac{\sin 2\theta}{\sin\theta}z^{n-2}+\frac{1}{3}\mu^2 a_3\frac{\sin 3\theta}{\sin\theta}z^{n-3}+\ldots+\frac{1}{n}\mu^{n-1}a_n\frac{\sin n\theta}{\sin\theta}\\ &= z^{n-1}+\binom{n-1}{1}\frac{1}{2}\frac{\mu}{\binom{n-1}{1}}a_2\frac{\sin 2\theta}{\sin\theta}z^{n-2}+\ldots+\binom{n-1}{k-1}\frac{1}{k}\frac{\mu^{k-1}}{\binom{n-1}{k-1}}a_k\frac{\sin k\theta}{\sin\theta}z^{n-k}+\\ &\qquad+\ldots+\binom{n-1}{n-1}\frac{1}{n}\mu^{n-1}a_n\frac{\sin n\theta}{\sin\theta}\end{aligned}$$

lie in $|z|\leqslant 1$ for $0\leqslant\theta\leqslant\pi/2$.

Since $p_n(z)$ is univalent in $|z|<1$, the polynomial

$$1+a_2\frac{\sin 2\theta}{\sin\theta}z+a_3\frac{\sin 3\theta}{\sin\theta}z^2+\ldots+a_n\frac{\sin n\theta}{\sin\theta}z^{n-1}$$

does not vanish in $|z|<1$ for $0\leqslant\theta\leqslant\pi/2$. Hence the zeros of the polynomial

$$\begin{aligned}f_{n-1}(z) &= z^{n-1}+\binom{n-1}{1}\frac{1}{\binom{n-1}{1}}a_2\frac{\sin 2\theta}{\sin\theta}z^{n-2}+\\ &\qquad\ldots+\binom{n-1}{k-1}\frac{1}{\binom{n-1}{k-1}}a_k\frac{\sin k\theta}{\sin\theta}z^{n-k}+\ldots+\binom{n-1}{n-1}a_n\frac{\sin n\theta}{\sin\theta}\end{aligned}$$

lie in $|z|\leqslant 1$. Now let us set

$$\begin{aligned}g_{n-1}(z) &= z^{n-1}+\binom{n-1}{1}\frac{1}{2}\mu z^{n-2}+\binom{n-1}{2}\frac{1}{3}\mu^2 z^{n-3}+\\ &\qquad+\ldots+\binom{n-1}{k-1}\frac{1}{k}\mu^{k-1}z^{n-k}+\ldots+\frac{1}{n}\mu^{n-1}\\ &= \frac{1}{\mu n}\left\{\left[z^n+n\mu z^{n-1}+n\binom{n-1}{1}\frac{1}{2}\mu^2 z^{n-2}+\ldots+n\binom{n-1}{k-1}\frac{1}{k}\mu^k z^{n-k}+\right.\right.\\ &\qquad\left.\left.\ldots+\mu^n\right]-z^n\right\} = \frac{1}{n\mu}\left[(z+\mu)^n-z^n\right]\end{aligned}$$

which vanishes at the points $z_k=\mu/(e^{i2k\pi/n}-1)$ where $k=1,2,\ldots,n-1$.

Hence the zeros of $g_{n-1}(z)$ lie in $|z| \leqslant 1$. From Lemma 2 it follows that the zeros of $h_{n-1}(z)$ lie in $|z| \leqslant 1$ for $0 \leqslant \theta \leqslant \pi/2$. This completes the proof of Theorem 1.

If $p(z) = z + a_2 z^2 + \frac{1}{3} z^3$ then for $-\frac{2\sqrt{2}}{3} \leqslant a_2 \leqslant \frac{2\sqrt{2}}{3}$ the polynomial $p(z)$ is univalent in $|z| < 1$. It is easy to verify that the derivative of $\int_0^z \zeta^{-1} p(\zeta) d\zeta$ vanishes on $|z| = \sqrt{3}$. This shows that Theorem 1 is sharp for polynomials of degree 3.

We wish to show now that for polynomials of degree 2, 3 the conclusion remains unchanged if instead of univalence we assume $p'(z) \neq 0$ in $|z| < 1$. For $n = 2$ it is trivial. So let $p(z) = z + a_2 z^2 + a_3 z^3$ be a polynomial of degree 3 such that $p'(z) = 3a_3 z^2 + 2a_2 z + 1$ does not vanish in $|z| < 1$, i.e. the polynomial

$$f_2(z) = z^2 + \binom{2}{1} a_2 z + 3a_3$$

has both its zeros in $|z| \leqslant 1$. We wish to show that the polynomial

$$P(z) = \int_0^z \zeta^{-1} p(\zeta) d\zeta = z + \tfrac{1}{2} a_2 z^2 + \tfrac{1}{3} a_3 z^3$$

is univalent in $|z| < \sqrt{3}$ or equivalently the polynomial

$$H_2(z) = 1 + \tfrac{1}{2} a_2 \frac{\sin 2\theta}{\sin\theta} z + \tfrac{1}{3} a_3 \frac{\sin 3\theta}{\sin\theta} z^2$$

does not vanish in $|z| < \sqrt{3}$ for $0 \leqslant \theta \leqslant \pi/2$. The latter holds if the reciprocal polynomial

$$h_2(z) = z^2 + \frac{1}{2} a_2 \frac{\sin 2\theta}{\sin\theta} z + \frac{1}{3} a_3 \frac{\sin 3\theta}{\sin\theta} = z^2 + \binom{2}{1} \frac{1}{2^2} a_2 \frac{\sin 2\theta}{\sin\theta} z + \frac{1}{3} a_3 \frac{\sin 3\theta}{\sin\theta}$$

has both its zeros in $|z| \leqslant 1/\sqrt{3}$ for $0 \leqslant \theta \leqslant \pi/2$. Now let

$$\begin{aligned} g_2(z) &= z^2 + \binom{2}{1} \frac{1}{2^2} \frac{\sin 2\theta}{\sin\theta} z + \frac{1}{3^2} \frac{\sin 3\theta}{\sin\theta} \\ &= \left(z - \frac{-3\cos\theta + \sqrt{-3 + 7\sin^2\theta}}{6}\right)\left(z - \frac{-3\cos\theta - \sqrt{-3 + 7\sin^2\theta}}{6}\right) \end{aligned}$$

It is easy to verify that for $0 \leqslant \theta \leqslant \pi/2$

$$|(-3\cos\theta \pm \sqrt{-3 + 7\sin^2\theta})/6| \leqslant 1/\sqrt{3}.$$

Hence for $0 \leqslant \theta \leqslant \pi/2$ the zeros of $g_2(z)$ lie in $|z| \leqslant 1/\sqrt{3}$, and so do the zeros of $h_2(z)$ by Lemma 2.

In the same way we can prove that if $p(z) = z + \sum_{k=2}^{n} a_k z^k$ is a polynomial of degree n such that $p'(z) \neq 0$ in $|z| < 1$ then the polynomial $\int_0^z \zeta^{-1} p(\zeta) d\zeta$ is univalent in $|z| < \varrho_n$ where ϱ_n is the minimum modulus of the zeros of the polynomial

$$H^*_{n-1}(z) = 1 + \binom{n-1}{1} \frac{1}{2^2} \frac{\sin 2\theta}{\sin\theta} z + \ldots + \binom{n-1}{k-1} \frac{1}{k^2} \frac{\sin k\theta}{\sin\theta} z^{n-1} + \ldots + \frac{1}{n^2} \frac{\sin n\theta}{\sin\theta} z^{n-1}.$$

It is clear that

$$F^*_{n-1}(z) = 1 + \binom{n-1}{1} \frac{1}{2} \frac{\sin 2\theta}{\sin\theta} z + \ldots + \binom{n-1}{k-1} \frac{1}{k} \frac{\sin k\theta}{\sin\theta} z^{k-1} + \ldots + \frac{1}{n} \frac{\sin n\theta}{\sin\theta} z^{n-1} = \frac{(1+ze^{i\theta})^n - (1+ze^{-i\theta})^n}{2inz}$$

and hence its zeros are

$$z_k = -e^{i\theta} \frac{1-\exp(2\pi ik/n)}{\exp(2i\theta) - \exp(2\pi ik/n)}, \; k = 1, 2, \ldots, n-1.$$

Since $\min_{1 \leqslant k \leqslant n-1} |z_k| = \sin(\pi/n)$ it follows that the zeros of the polynomial

$$f^*_{n-1}(z) = z^{n-1} F^*_{n-1}(z^{-1}) = z^{n-1} + \binom{n-1}{1} \frac{1}{2} \frac{\sin 2\theta}{\sin\theta} z^{n-2} + \ldots + \binom{n-1}{k-1} \frac{1}{k} \frac{\sin k\theta}{\sin\theta} z^{n-k} + \ldots + \frac{1}{n} \frac{\sin n\theta}{\sin\theta}$$

lie in $|z| \leqslant \operatorname{cosec}(\pi/n)$. On the other hand,

$$g^*_{n-1}(z) = z^{n-1} + \binom{n-1}{1} \frac{1}{2} z^{n-2} + \ldots + \binom{n-1}{k-1} \frac{1}{k} z^{n-k} + \ldots + \frac{1}{n} = \frac{1}{n} [(z+1)^n - z^n]$$

and hence its zeros lie in $|z| \leqslant \frac{1}{2} \operatorname{cosec}(\pi/n)$.

By Lemma 2 the zeros of the polynomial

$$h^*_{n-1}(z) = z^{n-1} + \binom{n-1}{1} \frac{1}{2^2} \frac{\sin 2\theta}{\sin\theta} z^{n-2} + \ldots + \binom{n-1}{k-1} \frac{1}{k^2} \frac{\sin k\theta}{\sin\theta} z^{n-k} + \ldots + \frac{1}{n^2} \frac{\sin n\theta}{\sin\theta},$$

lie in $|z| \leqslant \frac{1}{2}\operatorname{cosec}^2(\pi/n)$ and those of the reciprocal polynomial $H_{n-1}^*(z)$ lie in $|z| \geqslant 2\sin^2(\pi/n)$. It follows that if $p(z) = z + \sum_{k=2}^{n} a_k z^k$ is a polynomial of degree n such that $p'(z) \neq 0$ in $|z| < 1$ then the polynomial $\int_0^z \zeta^{-1} p(\zeta)d\zeta$ is univalent in $|z| < 2\sin^2(\pi/n)$. This result is by no means sharp but it shows in particular that if $p(z) = z + \sum_{k=2}^{4} a_k z^k$ is a polynomial of degree 4 such that $p'(z) \neq 0$ in $|z| < 1$ then $\int_0^z \zeta^{-1} p(\zeta)d\zeta \in \mathscr{P}_4$.

Proof of Theorem 2. Without loss of generality we may suppose that $p(z) = z + a_2 z^2 + tz^3$ where $0 \leqslant t \leqslant 1/3$. Set $1/\sqrt{7} = \gamma$. The polynomial $zp'(z) = z + 2a_2 z^2 + 3tz^3$ is univalent in $|z| < \gamma$ if and only if the polynomial $zp'(\gamma z) = z + 2a_2\gamma z^2 + 3t\gamma^2 z^3$ is univalent in $|z| < 1$. We note that $3t\gamma^2 < 1/5$. Hence by Lemma 3 the region of possible values of $2a_2\gamma$ in order that $zp'(\gamma z) \in \mathscr{P}_3$ is

$$H_1 = \left\{x + iy : \left(\frac{2x}{1+9t\gamma^2}\right) + \left(\frac{2y}{1-9t\gamma^2}\right)^2 \leqslant 1\right\}.$$

Since $p(z) = z + a_2 z^2 + tz^3 \in \mathscr{P}_3$ by hypothesis, a_2 lies in a convex domain $H(t)$ whose supporting function $K(\varphi)$ has been estimated in Lemma 4. It follows that $2a_2\gamma$ lies in a convex domain H_2 whose supporting function is $K_2(\varphi) = 2\gamma K(\varphi)$. In order to prove that $zp'(\gamma z) \in \mathscr{P}_3$ it is clearly enough to show that $H_2 \subset H_1$. If $K_1(\varphi)$ is the supporting function of H_1 then according to Lemma 5 this holds if and only if $K_2(\varphi) \leqslant K_1(\varphi)$ for $0 \leqslant \varphi < 2\pi$. Hence $zp'(\gamma z) \in \mathscr{P}_3$ if for $0 \leqslant \varphi < 2\pi$

$$2\gamma K(\varphi) \leqslant K_1(\varphi) = \tfrac{1}{2}(1 + 81t^2\gamma^4 + 18t\gamma^2\cos 2\varphi)^{1/2}. \tag{3.1}$$

In order to verify (3.1) we have to distinguish many cases.

Case (i). $0 \leqslant t \leqslant 1/5$.

In this case $K(\varphi)$ is given by (2.1) and we have to show that for $0 \leqslant \varphi < 2\pi$

$$\gamma(1 + 9t^2 + 6t\cos 2\varphi)^{1/2} \leqslant \tfrac{1}{2}(1 + 81t^2\gamma^4 + 18t\gamma^2\cos 2\varphi)^{1/2},$$

or

$$6t\gamma^2\cos 2\varphi \leqslant 1 - 4\gamma^2 + 81t^2\gamma^4 - 36t^2\gamma^2.$$

Hence it is enough to prove that

$$\psi(t) = 1 - 4\gamma^2 - 6t\gamma^2 + 81t^2\gamma^4 - 36t^2\gamma^2 \geqslant 0. \tag{3.2}$$

Since $\frac{d}{dt}\psi(t) < 0$ for $t \geqslant 0$ and $\psi(1/5) > 0$, (3.2) holds for $0 \leqslant t \leqslant 1/5$,

Case (ii). $1/5 < t < 1/3$, $\cos 2\varphi \leqslant \dfrac{1-15t^2}{2t}$.

In this case $K(\varphi) \leqslant \frac{1}{2}(1+9t^2+6t\cos 2\varphi)^{1/2}$ and hence $2\gamma K(\varphi) \leqslant K_1(\varphi)$ if:

$$4\gamma^2(1+9t^2+6t\cos 2\varphi) \leqslant 1+81t^2\gamma^4+18t\gamma^2\cos 2\varphi,$$

or

$$6t\gamma^2\cos 2\varphi \leqslant 1+81t^2\gamma^4-4\gamma^2-36t^2\gamma^2.$$

This latter inequality surely holds if

$$6t\gamma^2\frac{1-15t^2}{2t} \leqslant 1+81t^2\gamma^4-4\gamma^2-36t^2\gamma^2$$

or

$$\psi_1(t) = 1+81t^2\gamma^4+9t^2\gamma^2-7\gamma^2 \geqslant 0.$$

This is certainly true since $\psi_1(1/5) > 0$ and $\frac{d}{dt}\psi_1(t) > 0$ for positive t.

Case (iii). $1/5 < t < 1/3$, $\cos 2\varphi > \frac{1-15t^2}{2t}$.

In this case $K(\varphi) \leqslant [2t(1+t^2-2t\cos 2\varphi)^{1/2}+2t\cos 2\varphi-2t^2]^{1/2}$ and hence $2\gamma K(\varphi) \leqslant K_1(\varphi)$ if

$$16\gamma^2[2t(1+t^2-2t\cos 2\varphi)^{1/2}+2t\cos 2\varphi-2t^2] \leqslant 1+81t^2\gamma^4+18t\gamma^2\cos 2\varphi$$

or

$$A(\cos 2\varphi) = 196t^2\gamma^4\cos^2 2\varphi+[2048t^3\gamma^4-28t\gamma^2(1+32t^2\gamma^2+81t^2\gamma^4)]\cos 2\varphi$$
$$+(1+32t^2\gamma^2+81t^2\gamma^4)^2-1024t^2\gamma^4(1+t^2) \geqslant 0.$$

For a given t in the range the minimum of $A(\cos 2\varphi)$ occurs for

$$\cos 2\varphi = -(288t^2\gamma^2-7-567t^2\gamma^4)/(98t\gamma^2)$$

and is 0. Hence $2\gamma K(\varphi) \leqslant K_1(\varphi)$ for $0 \leqslant \varphi < 2\pi$.

Case (iv). $t = 1/3$.

In this case we have to verify that for $0 \leqslant \varphi < 2\pi$

$$8\sqrt{2}\gamma|\cos\varphi| \leqslant 3(1+9\gamma^4+6\gamma^2\cos 2\varphi)^{1/2}$$

or

$$-9+64\gamma^2+10\gamma^2\cos 2\varphi-81\gamma^4 \leqslant 0,$$

which certainly holds if

$$-9+74\gamma^2-81\gamma^4 \leqslant 0.$$

But indeed $-9+74\gamma^2-81\gamma^4 < 0$ and the proof of the theorem is complete.

The proof of Theorem 3 is analogous to that of Theorem 1 and we therefore omit it. The example $p(z) = z+\frac{2\sqrt{2}}{3}z^2+\frac{1}{3}z^3$ shows that the result is best possible as far as polynomials of degree 3 are concerned.

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STRESZCZENIE

W pracy tej autorzy wykazują, że jeśli wielomian $p(z) = z+\ldots+a_n z^n$ jest jednolistny w kole jednostkowym K, to wielomiany $\int_0^z \zeta^{-1} p(\zeta)d\zeta$, $\frac{2}{z}\int_0^z p(\zeta)d\zeta$ są również jednolistne w kole K o ile stopień p nie przekracza 6 (w pierwszym przypadku), względnie 5 (w drugim przypadku).

РЕЗЮМЕ

В работе доказано, что, если многочлен $p(z) = z+\ldots+a_n z^n$ есть однолистным в единичном круге K, то многочлены $\int_0^z \zeta^{-1}p(\zeta)d\zeta$, $\frac{2}{z}\int_0^z p(\zeta)d\zeta$ также однолистны в K, если степень p не больше 6 (в первом случае) или 5 (во втором случае).