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# Some Remarks Concerning Close-to-Convex Functions

Pewne uwagi o funkcjach prawie wypukłych

Некоторые заметки о почти выпуклых функциях

1. Let $S$ denote the class of functions of the form $f(z) = z + \ldots$ analytic and univalent in the unit disc $\Delta$ and let $S^c$, $S^*$, $L$ be its subclasses consisting of functions mapping $\Delta$ onto domains convex, starshaped w.r.t. the origin and close-to-convex, respectively.

Let $P$ be the family of functions analytic in $\Delta$ and satisfying the conditions

$$p(0) = 1, \quad \mathscr{R}\{p(z)\} > 0.$$

It is well-known that the following statements hold:

(i) $f \in S^c \Leftrightarrow 1 + \dfrac{zf''(z)}{f'(z)} \in P$,

(ii) $f \in S^* \Leftrightarrow \dfrac{zf'(z)}{f(z)} \in P$,

(iii) $f \in L$ iff there exist a function $g \in S^*$ and a real number $a$, $|a| < \pi/2$ such that

$$\mathscr{R}\left\{e^{ia}\frac{zf'(z)}{g(z)}\right\} > 0$$

in the unit disc.

Our aim is an investigation of the class $G \subset L$ defined as follows.

**Def. 1.** A function $f(z) = z + a_2 z^2 + \ldots$ is said to be an element of the class $G$ if there exist an odd starlike function $\varphi$ and a real number $a$, $|a| < \pi/2$ such that

$$\mathscr{R}\left\{e^{ia}\frac{zf'(z)}{\varphi(z)}\right\} > 0 \tag{1}$$

holds in the unit disc.

In this note we determine the region of variability of the expression $\log f'(z)$, the radius of convexity of the class $G$ and we consider some problems of univalence.

It is my pleasure to express my thanks to Professors J. G. Krzyż and Z. Lewandowski for their helpful remarks.

2. Let $M$ be the family of non-decreasing functions $\mu$ on the interval $\langle 0, 2\pi\rangle$ subject to the condition $\int_0^{2\pi} d\mu = 1$.

**Lemma.** *$f \in G$ iff there exist functions $\mu, \gamma \in M$ such that $f$ has the representation*

$$f(z) = \int_0^z \left\{\left(\int_0^{2\pi} \frac{1+e^{i(2a-t)}u}{1+e^{it}u}\, d\mu(t)\right) \exp\left(-\int_0^{2\pi} \log(1-u^2e^{-it})\,d\nu\right)\right\} du \tag{2}$$

*for $z \in \Delta$ and $a \in \langle 0, \pi/2)$.*

A proof of Lemma follows immediately from (1) and the well-known Herglotz formula and hence will be omitted.

Let $z$ be a fixed point of $\Delta$, and let the region of variability of $\log f'(z)$ be the set $D = \{w\colon w = \log f'(z), f \in G\}$.

**Theorem 1.** *The set $D$ is a closed and convex set bounded by the curve*

$$w(t) = \log\big(1+r\alpha(t)\big)\big(1-r\beta(t)\big)^{-1}\big(1-r^2\gamma(t)\big)^{-1}$$

*where*

$$\begin{aligned} \alpha(t) &= \exp\big(t+\arcsin(r\sin t)\big),\\ \beta(t) &= \exp\big(t-\arcsin(r\sin t)\big),\\ \gamma(t) &= \exp\big(t-\arcsin(r^2\sin t)\big) \end{aligned} \tag{3}$$

*and $t \in \langle 0, 2\pi)$, $|z| = r$.*

**Proof.** First we prove the convexity of $D$. If $\varphi$ and $\psi$ are odd starlike functions, then for each $\lambda \in \langle 0, 1\rangle$, the function

$$\omega_\lambda(z) = z\left(\frac{\varphi}{z}\right)^{\lambda}\left(\frac{\psi}{z}\right)^{1-\lambda}$$

is also odd and starlike. If $g$ and $h$ are elements of $G$, chosen so that

$$\mathscr{R}\left(\frac{zg'}{\varphi}\right) > 0 \quad \text{and} \quad \mathscr{R}\left(\frac{zh'(z)}{\psi}\right) > 0$$

and if $f_\lambda(z)$ is defined by $f'_\lambda(z) = [g'(z)]^\lambda[h'(z)]^{1-\lambda}$, $f(0) = 0$, then we have

$$\left|\arg\frac{zf'_\lambda}{\omega_\lambda}\right| \leqslant \lambda\left|\arg\frac{zg'}{\varphi}\right| + (1-\lambda)\left|\arg\frac{zh'(z)}{\psi}\right| < \pi/2$$

Hence $f_\lambda \in G$ for each $\lambda \in \langle 0, 1\rangle$ and $w_\lambda = \lambda\log g'(z) + (1-\lambda)\log h' \in D$ Thus the convexity of $D$ has been proved.

In view of this fact, in order to determine $D$ it is sufficient to solve the following extremal problem: For a given $z \in \Delta$ and $t \in \langle 0, 2\pi)$ find a function $f \in G$ for which

$$F(f) = \mathscr{R}\{e^{-it}\log f'(z)\}$$

is a maximum.

To do this we shall use a variational technique. Since the functions $\mu, \gamma$ in (2) can be varied independently of each other we are in a position to apply Golusin's variational formulas [2] for both integrals in (2). After routine considerations, we conclude that the extremal functions have the form

$$f'(z) = (1-e^{-it}z)(1-e^{-it}z)(1-e^{-it}z^2)^{-1},$$

where $t_k \in \langle 0, 2\pi)$, $k = 1, 2, 3$.

In view of the convexity of $D$, we have the relations

$$\mathscr{R}\left\{e^{-it}\frac{\partial}{\partial t_k}\log\frac{1+re^{-it_1}}{(1-re^{-it_2})(1-r^2e^{-it_3})}\right\} = 0$$

$k = 1, 2, 3.$ for the boundary points of $D$. The proof of the theorem is complete.

Theorem 1 is an analogue of Krzyż's theorem for the class $L$ [3] (cf. also [1]).

**Corollary.** *If $f \in G$ then*

(i) $$|\arg f'(z)| \leqslant 2(\arcsin r + \arcsin r^2),$$

(ii) $$(1+r)^{-2} \leqslant |f'(z)| \leqslant (1-r)^{-2},$$

(iii) $$r(1+r)^{-1} \leqslant |f(z)| \leqslant r(1-r)^{-1}.$$

*These estimates are sharp.*

**Theorem 2.** *Each function $f \in G$ maps the disc $|z| < r_c$, $r_c = \frac{1}{2}\left(1+\sqrt{5}-\sqrt{2(1+\sqrt{5})}\right)$ onto a convex domain. The constant $r_c$ is the best possible.*

**Proof.** In order to prove Theorem 2 we shall estimate the expression

$$\underset{f\in G}{\text{g.l.b.}}\ \mathscr{R}\left\{1+\frac{zf''(z)}{f'(z)}\right\}$$

According to the Lemma, we have

$$1+\frac{zf''(z)}{f'(z)} = \int_0^{2\pi}\frac{1+z^2e^{-it}}{1-z^2e^{-it}}d\mu + \frac{zq'(z)}{q(z)},$$

where $q(z) = \cos a p(z) + i\sin a$, $p(z) \in P$. Hence

$$\mathscr{R}\frac{zq'(z)}{q(z)} \geqslant -\frac{2r}{1-r^2}$$

and

$$1+\mathscr{R}\frac{zf''(z)}{f'(z)} \geqslant \frac{r^4-2r^3-2r^2-2r+1}{1-r^4} \equiv K(f)$$

both hold. Moreover, $K(f) \geqslant 0$ iff $|z| \leqslant r_c$ where $r_c$ is the smallest positive root of the equation $x^4-2x^3-2x^2-2x+1 = 0$. The value $r_c$ holds for the function $f(z) = \log(1+z)(1+z^2)^{-1/2}$

**Theorem 3.** *If* $f(z) = z+a_2z^3+\ldots \in G$, *then*

(5.1) $\frac{1}{2}\big(f(z)-f(-z)\big) \in G$

(5.2) $|a_n| \leqslant 1, \quad n = 2, 3, \ldots$

**Proof.** It follows from (1) that we have

$$\mathscr{R}\left\{e^{i\alpha}\frac{z}{\varphi(z)}f'(z)\right\} > 0 \quad \text{and} \quad \mathscr{R}\left\{e^{i\alpha}\frac{z}{\varphi(z)}f(-z)\right\} > 0$$

Thus

$$\mathscr{R}\left\{e^{i\alpha}\frac{z}{\varphi(z)}\big(f(z)-f(-z)\big)'\right\} > 0$$

and (5.1) has been established.

Let $f(z) = z+a_2z^2+\ldots$, $\varphi(z) = z+\sum_{k=1}^{\infty} b_{2k-1}z^{2k-1} \in S^*$ and $p(z) = 1+ + \sum_{k=1}^{\infty} c_kz^k \in P$. It is well-known that

$$|b_{2k-1}| \leqslant 1, \quad |c_k| \leqslant 2, \quad k = 1, 2, \ldots$$

According to (1) we have now

$$|na_n| \leqslant \sum_{0}^{n-1} |c_k|\,|b_{n-k}|, \; b_{2k} = 0.$$

Thus

$$|a_n| \leqslant 1.$$

It is clear that $G$ contains all odd close-to-convex functions so that (5.2) is a well-known result for these functions [4].

**Theorem 4.** *If* $f \in L$ *and if* $h = \int_0^z u^{-1}f(u)du$ *then*

$$\tfrac{1}{2}\big(h(z)-h(-z)\big) \in G$$

**Proof.** Suppose that $f \in L$ and $\varphi \in S^c$ is such that $\mathscr{R}\left\{e^{i\alpha}\frac{f'}{\varphi'}\right\} > 0$. Then we have

$$z[h(z)-h(-z)]' = f(z)-f(-z).$$

If $w_1, w_2 \in \varphi(\Delta)$ and if set $F(w) = f \circ \varphi^{-1}(w)$, $a = \varphi(w_1)$, $b = \varphi(w_2)$, then we obtain

$$0 < \mathscr{R}\, e^{i\alpha} \frac{F(w_1)-F(w_2)}{w_1-w_2} = \mathscr{R}\, e^{i\alpha} \frac{f(b)-f(a)}{\varphi(b)-\varphi(a)} = \mathscr{R} \int_0^1 e^{i\alpha} F'\big(w_1+t(w_1-w_2)\big)$$

If we let $z = a$, $b = -z$ then we have

$$\mathscr{R}\, e^{i\alpha} \frac{[h(z)-h(-z)]'}{\varphi(z)-\varphi(-z)} = \mathscr{R} \frac{f(z)-f(-z)}{\varphi(z)-\varphi(-z)} e^{i\alpha} > 0$$

Now, it is well-known that if $f \in S^c$ then $\frac{1}{2}\big(f(z)-f(-z)\big) \in S^*$ thus $\frac{1}{2}\big(h(z)-h(-z)\big) \in G$.

## REFERENCES

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## STRESZCZENIE

W pracy tej rozpatruję pewną podklasę funkcji prawie wypukłych. Znaleziono obszar wartości funkcjonału $\log f'(z)$ i dokładną wartość promienia wypukłości.

## РЕЗЮМЕ

В работе изучается некоторый подкласс почти выпуклых функций. Найдена область значений функционала $\log f'(z)$ и точное значение радиуса выпуклости.