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On a Class of Hyperbolic Quasiconformal Mappings

O pewnej klasie odwzorowań quasi-konforemnych typu hiperbolicznego

О некотором классе квазиконформных отображений гиперболического типа

Introduction

In his recent paper J. Ławrynowicz [4] discusses in detail the class E_Q of Q-quasiconformal mappings f of the unit disc onto itself of the form $f(z) = e^{i\arg z} f(|z|)$, satisfying conditions $f(0) = 0, f(1) = 1$. He also discusses an analogous class E_Q^* of Q-quasiconformal mappings of the complex plane onto itself. At the end of this paper some suggestions concerning further research in this direction are given. E.g. one may consider various classes of quasiconformal mappings which are solutions of Beltrami differential equations with separated variables. In connection with this Ławrynowicz proposes to consider the class H_Q of mappings f of the upper half-plane which are locally Q-quasiconformal and have the form $f(z) = |z| f(e^{i\arg z})$, as well as the corresponding class H_Q^* of mappings locally Q-quasiconformal of the complex plane slit along the positive real axis.

The present paper deals with the classes mentioned above. The definitions, as well as some theorems and proofs, are similar to those in [4], and therefore we omit some of them.

At the beginning we give the definition of the class H_Q and five other equivalent conditions for f to be of the class H_Q and then we formulate two theorems on estimates of $|f(z)|/|z|$ and $\arg f(z)$ for $f \in H_Q$. In Section 3 two theorems on parametric representation are proved. In the next section a general extremal problem in H_Q is considered. Here two theorems are given on sufficient conditions for existence of an extremum of a real functional, as well as the analogous conditions for existence of an extremum of the real part of a complex-valued functional with the fixed imaginary part. Further, we present two theorems which are applications of previous results. In the last section the class H_Q^* is discussed.

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1. The class H_Q

Let $\mathscr{E}$ denote the complex plane, $\mathscr{E}^+$ — the closed upper half-plane, and Δ — the closed unit disc. Throughout the paper $\mathrm{cl}E$ denotes the closure of E in $\mathscr{E}$, E being a set.

Definition 1. A function f is said to be of the class H_Q if it is defined on $\mathscr{E}^+$, and

(i) maps $\mathscr{E}^+$ onto the Riemann surface of $s = w^{t'/\pi}, w \in \mathscr{E}^+$ (t' being positive, and $s = f(-1)$ corresponding to some $w = -|w|$),

(ii) $s = f(0)$, $s = f(1)$, and $s = f(\infty)$ correspond to $w = 0, w = 1$, and $w = \infty$, respectively.

(iii) $f(z) = |z|f(e^{i\arg z})$ for $z \in \mathscr{E}^+ \setminus \{0, \infty\}$, where if $f(e^{i\arg z})$ is a point corresponding to some $w_0 \in \mathscr{E}^+$, then $|z|f(e^{i\arg z})$ denotes the point corresponding to $|z|w_0$.

Since the correspondence between the Riemann surface of $s = w^{t'/\pi}$, $w \in \mathscr{E}^+$ and the half-plane $\mathscr{E}^+$ is one-one, we shall write

$$f(z) = |w|^{t'/\pi}\exp\left(i\frac{t'}{\pi}\arg w\right)$$

instead of

$$s = f(z) \text{ corresponds to } w.$$

In particular, we shall write

$$f(z) = |w|^{t'/\pi} \text{ for } w \in \mathrm{cl}\{w: \arg w = 0\}.$$

Of course we may confine ourselves to the case where the Riemann surface of $s = w^{t'/\pi}$, $w \in \mathscr{E}^+$, reduces to $\mathscr{E}^+$, but this corresponds to the case where $\arg f(-1) = \pi$ only. Obviously this condition restricts the class considered, and it is inconvenient for estimations.

Definition 1 implies

$$\left|\frac{f(z)}{z}\right| = |f(e^{i\arg z})| \equiv R(\arg z), \tag{1}$$

$$\arg f(z) = \arg f(e^{i\arg z}) \equiv \theta(\arg z)\ (z \neq 0, \infty).$$

Now we give five other equivalent conditions for f to be of the class H_Q. The proofs of equivalence, except for the proof of Theorem 1, are analogous to the corresponding proofs in [4], p. 311-315, so we shall give the above mentioned proof only.

Theorem 1. *$f \in H_Q$ if it is defined on $\mathcal{E}^+$, and $f(z) = f^*(z)$ identically for $z \in \mathcal{E}^+ \setminus \mathrm{cl}\{z: \arg z = \pi\}$, where f^* satisfies conditions:*

(i′) *it is defined on $\mathcal{E} \setminus \mathrm{cl}\{z: \arg z = \pi\}$, maps it into the Riemann surface of $s = w^{t'/\pi}$, $w \in \mathcal{E}$ ($t' > 0$ and $f^*(e^{it_n}) = |w_n| e^{it'_n}$ for some $t_n \to \pi-$ and $t'_n \to \pi-$ as $n \to +\infty$),*

(ii′) *$f^*(z_n) \to 0$, $f^*(1) = 1$, $f^*(\tilde{z}_n) \to \infty$ for some $z_n \to 0$ and $\tilde{z}_n \to \infty$ as $n \to +\infty$,*

(iii′) *$a_m \overline{f^*(z)} = f^*(a_m \bar{z})$, where $\{a_m\}$ is a sequence of real numbers such that $a_m \to 1$ as $m \to +\infty$.*

Theorem 2. *$f \in H_Q$ if it is defined on $\mathcal{E}^+$, and $f(z) = f^*(z)$ identically for $z \in \mathcal{E}^+ \setminus \mathrm{cl}\{z: \arg z = \pi\}$ where f^* satisfies conditions (i′) and (ii′) in Theorem 1, as well as $\mu^*(z) = \overline{\mu^*(a_m \bar{z})}$ a.e. in $\mathcal{E}$, where μ^* is the complex dilatation of f^*, and a_m ($m = 1, 2, \ldots$) are the same as in Theorem 1.*

Theorem 3. *$f \in H_Q$ if it is defined on $\mathcal{E}^+$, and satisfies conditions (i) and (ii) in Definition 1, as well as $\mu^*(z) = \mu^*(e^{i \arg z})$ a.e. in $\mathcal{E}^+$.*

Theorem 4. *$f \in H_Q$ if it is defined on $\mathcal{E}^+$, and satisfies conditions (i) and (ii) in Definition 1, as well as $z f_z(z) + \bar{z} f_{\bar{z}}(z) = f(z)$ a.e. in $\mathcal{E}^+$.*

Theorem 5. *$f \in H_Q$ if it is given by the formulae*

$$f(z) = |z| \exp \int_{e^{i \arg z}}^{1} \frac{\mu(\varepsilon) - \varepsilon^2}{\mu(\varepsilon) + \varepsilon^2} \cdot \frac{d\varepsilon}{\varepsilon} \quad \text{for } z \in \mathcal{E}^+, z \neq 0, \infty,$$

$$f(z) = z \quad \text{for } z = 0, \infty,$$

f being continuous and having its values on the Riemann surface of $s = w^{t'/\pi}$, $w \in \mathcal{E}^+$, where

$$t' = \pi \int_{-1}^{+1} \frac{\mu(\varepsilon) - \varepsilon^2}{\mu(\varepsilon) + \varepsilon^2} \frac{d\varepsilon}{\varepsilon},$$

where μ is measurable with $\sup_{0 \leqslant \varphi \leqslant \pi} |\mu(e^{i\varphi})| < 1$ and $\operatorname{ess\,sup}_{0 \leqslant \varphi \leqslant \pi} |\mu(e^{i\varphi})| \leqslant \frac{Q-1}{Q+1}$, ε ranging over the unit circle from $e^{i \arg z}$ to 1.

Remark 1. The correctness of the definition of f as a function having its values on a Riemann surface follows from the well known theorem on existence and uniqueness (see e.g. [3], p. 204).

Remark 2. In general μ is assumed to be complex-valued.

Proof of Theorem 1. It follows from Definition 1 that f^* defined by $f^*(z) = f(z)$ for $z \in \mathscr{E}^+ \setminus \mathrm{cl}\{z\colon \arg z = \pi\}$ and by $f^*(z) = \overline{f(\bar z)}$ for $z \notin \mathscr{E}^+$, $z \neq \infty$, maps $\mathscr{E} \setminus \mathrm{cl}\{z\colon \arg z = \pi\}$ into the Riemann surface of $s = w^{t'/\pi}$, $w \in \mathscr{E}$, $t' > 0$, Q-quasiconformally with $f^*(0) = 0, f^*(1) = 1, f^*(\infty) = \infty$. Moreover f^* satisfies

$$f^*(a\bar z) = \overline{f^*(az)} = a|z|\overline{f^*(e^{i\arg z})} = a\overline{f^*(z)}$$

for $\mathrm{im} z \geqslant 0, z \neq 0, \infty$, and any positive a. Besides,

$$f^*(a\bar z) = a|z| f^*(e^{-i\arg z}) = af^*(\bar z) = a\overline{f^*(z)}$$

for $\mathrm{im} z < 0, z \neq 0, \infty$, and any positive a. Hence $f \in H_Q$ according to Theorem 1.

Conversely, suppose that $f \in H_Q$ according to Theorem 1. We see that putting $\zeta = a_m \bar z$ in the equation $a_m \overline{f^*(\zeta)} = f^*(a_m \bar\zeta)$ we get $a_m^2 f^*(z) = f^*(a_m^2 z)$. It is easy to verify that $a_m^{2n} f^*(z) = f^*(a_m^{2n} z), (n = 2, 3, \ldots)$. There exist subsequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that $1/a_{m_k}^{2n_k} \to \delta$ as $k \to +\infty$, where δ can be an arbitrary positive number. In fact, we can choose as $\{a_{m_k}\}$ an arbitrary non-decreasing subsequence of $\{a_m\}$, and n_k $(k = 1, 2, \ldots)$ is chosen so that $(1/a_k)^{n_k} \leqslant \delta \leqslant (1/a_k)^{n_k+1}$. Therefore we can assume that $1/a_{m_k}^{n_k} \to |z|$ as $k \to +\infty$. Consequently, we have $f^*(z) = |z| f^*(e^{i\arg z})$. Hence, by our hypotheses, it follows that $f \in H_Q$ according to Definition 1.

2. Bounds for R and θ

By the use of Theorem 5 it is easy to find sharp estimates of R and θ when f ranges over H_Q. We first state

Lemma 1. *With the notation of formula* (1) *we have*

$$\arg f^{-1}(w) = \theta^{-1}(\arg w)$$

(2)

$$\left|\frac{f^{-1}(w)}{w}\right| = \frac{1}{R\big(\theta^{-1}(\arg w)\big)}.$$

This is an immediate consequence of Definition 1.

Theorem 6. *For any $f \in H_Q$ and any $z \in \mathscr{E}^+$, $z \neq 0, \infty$, we have*

$$(1/Q)\arg z \leqslant \arg f(z) \leqslant Q \arg z,$$

where $\arg z, \arg f(z)$ *change in a continuous manner from the initial value* $\arg f(1) = 0 = \arg 1$. *Both estimates are sharp for any $z \in \mathscr{E}^+$, $z \neq 0, \infty$, and $Q \in \langle 1, +\infty)$. The only extremal functions are $f(s) = |s| e^{iQ\arg s}$ and $f(s) = |s| e^{i(1/Q)\arg s}$ for the upper and lower bound, respectively.*

Theorem 7. *For any $f \in H_Q$ and $z \in \mathscr{E}^+$, $z \neq 0, \infty$, we have*

$$\exp\left[-\frac{1}{2}\left(Q-\frac{1}{Q}\right)\arg z\right] \leqslant \left|\frac{f(z)}{z}\right| \leqslant \exp\frac{1}{2}\left[\left(Q-\frac{1}{Q}\right)\arg z\right].$$

both estimates are sharp for any $z \in \mathscr{E}^+$, $z \neq 0, \infty$, and $Q \in \langle 1, +\infty)$. The only extremal functions are $f(s) = |s| e^{i\beta \arg s}$ and $f(s) = |s| e^{i\bar{\beta} \arg s}$ for the upper and the lower bound, respectively. Here $\beta = \frac{1}{2}(1-i)Q + \frac{1}{2}(1+i)(1/Q)$.

The proofs of Theorems 6 and 7 are analogous to the proofs of Theorems 1 and 2 in [4], p. 315–316, respectively.

3. Parametric representation

We now give two theorems on parametric representation for $f \in H_Q$. We introduce one-parameter family of functions $g(z, t) \in H_Q$, $0 \leqslant t \leqslant 1$, with the same t', cf. e.g. Definition 1, such that $g(z, 0) = z^{t'/\pi}$, $g(z, 1) = f(z)$, and find a relation between $\partial g / \partial t$ and the complex dilatation ν^* of the inverse mapping g^{-1} (Theorem 8). An analogous theorem involving the complex dilatation of f can be also proved (Theorem 9).

Now define two classes of mappings, cf. [5] p. 150. The class R_Q consists of all functions f which map Δ Q-quasiconformally onto itself with $f(1) = 1, f(i) = i, f(-1) = -1$. T_Q is the subclass of R_Q containing all functions f with continuous partial derivatives of the second order, and such that the partial derivatives of the first order of $(1+|\mu|)/(1-|\mu|)$, $\frac{1}{2}\arg\mu$, satisfy Hölder conditions, μ denoting the complex dilatation of f.

Theorem 8. *Suppose that $w = f(z)$ belongs to H_Q and has $u = \mu(z)$ as its complex dilatation. Moreover, suppose that the functions $w = g(z, t)$, $0 \leqslant t \leqslant 1$, with the corresponding t' being fixed* (*cf. e.g. Definition* 1), *belong to H_Q and have complex dilatations*

$$\nu(z, t) = t\mu(z). \tag{3}$$

Then $w = g(z, t)$, considered as a function of z and t, satisfies on $\{\mathscr{E}^+ \setminus \{\infty\}\} \times \{t\colon 0 \leqslant t \leqslant 1\}$ the equation

$$\frac{\partial w}{\partial t} = \frac{2iw}{t} \int_0^{\arg w} \frac{\nu^*(e^{i\vartheta}, t)}{e^{2i\vartheta}(1-|\nu^*(e^{i\vartheta}, t)|^2)}\, d\vartheta \tag{4}$$

subject to the initial condition $g(z, 0) = z^{t'/\pi}$, where ν^* is the complex dilatation of g^{-1}.

Proof. Put

$$\begin{aligned} f &= h^{t'/\pi} \circ \tilde{f} \circ h^{-1}, \\ g &= h^{t'/\pi} \circ \tilde{g} \circ h^{-1}, \end{aligned} \tag{5}$$

where h denotes the homography $h(s) = \frac{1}{i}\,\frac{s-1}{s+1}$, and $t'/\pi = \lim_{t\to\pi-} \arg f(e^{it})$. Clearly, the functions $\omega = \tilde{f}(\zeta)$ and $\omega = \tilde{g}(\zeta, t)$, $0 \leqslant t \leqslant 1$ belong to the class R_Q, and $\tilde{g}(\zeta, 0) = \tilde{f}(\zeta)$, $|\zeta| \leqslant 1$. Suppose that these functions belong to T_Q, and denote by $\tilde{u} = \tilde{\mu}(\zeta)$ and $\tilde{v} = \tilde{\nu}(\zeta, t)$ their complex dilatations, respectively. It is easily obtained from (5) and (3) that $\tilde{\nu}(\zeta, t) = t\tilde{\mu}(\zeta)$. According to Theorem 1 in [5], p. 150, we have

$$\frac{\partial\omega}{\partial t} = -\frac{1}{\pi}\iint\limits_{|\zeta|\leqslant 1}\left\{\frac{(\omega-i)(\omega^2-1)}{(\omega-\zeta)(1-\zeta^2)(\zeta-i)}\psi(\zeta, t) + \right. \tag{6}$$

$$\left. + \frac{(1+i\omega)(\omega^2-1)}{(1-\bar{\zeta}\omega)(1-\bar{\zeta}^2)(\bar{\zeta}+i)}\overline{\psi(\zeta, t)}\right\}d\xi\, d\eta \qquad (\zeta = \xi + i\eta),$$

where

$$\psi(\omega, t) = \frac{\tilde{\mu}_t(\tilde{g}^{-1}(\omega, t), t)}{1-|\tilde{\mu}(\tilde{g}^{-1}(\omega, t), t)|^2}\exp\left(-2i\arg \tilde{g}_\omega^{-1}(\omega, t)\right) \tag{7}$$

$$= \frac{(1/t)\tilde{\nu}(\tilde{g}^{-1}(\omega, t), t)}{1-|\tilde{\nu}(\tilde{g}^{-1}(\omega, t), t)|^2}\exp\left(-2i\arg \tilde{g}_\omega^{-1}(\omega, t)\right)$$

$$= \frac{(1/t)\tilde{\nu}^*(\omega, t)}{1-|\nu^*(\omega, t)|^2},$$

and $\tilde{\nu}^*$ denotes the complex dilatation of $\tilde{g}^{-1}$. Hence formula (6) takes the form

$$\frac{\partial\omega}{\partial t} = \frac{\omega^2-1}{\pi t}\iint\limits_{|\zeta|\leqslant 1}\frac{1}{1-|\tilde{\nu}^*(\zeta, t)|^2}\times$$

$$\times\left\{\frac{(\omega-i)\tilde{\nu}^*(\zeta, t)}{(\zeta-i)(1-\zeta^2)(\omega-\zeta)} + \frac{(1+i\omega)\overline{\tilde{\nu}^*(\zeta, t)}}{(\bar{\zeta}+i)(1-\bar{\zeta}^2)(1-\bar{\zeta}\omega)}\right\}d\xi\, d\eta$$

$$= \frac{(\omega-i)(\omega^2-1)}{\pi t}\iint\limits_{|\zeta|\leqslant 1}\frac{1}{1-|\tilde{\nu}^*(\zeta, t)|^2}\times$$

$$\times\left\{\frac{\tilde{\nu}^*(\zeta, t)}{(\zeta-i)(1-\zeta^2)(\omega-\zeta)} + i\,\frac{\overline{\tilde{\nu}^*(\zeta, t)}}{(\bar{\zeta}+i)(1-\bar{\zeta}^2)(1-\bar{\zeta}\omega)}\right\}d\xi\, d\eta$$

$$= \frac{(\omega-i)(\omega^2-1)}{\pi t}\iint\limits_{|\zeta|<+\infty}\frac{\hat{\nu}(\zeta, t)}{1-|\hat{\nu}(\zeta, t)|^2}\,\frac{d\xi\, d\eta}{(\zeta-i)(\zeta^2-1)(\zeta-\omega)},$$

where $\hat{\nu}(\omega, t) = \tilde{\nu}^*(\omega, t)$ for $\omega \in \Delta$ and $\hat{\nu}(\omega, t) = e^{4i\arg\omega}\tilde{\nu}^*(1/\overline{\omega}, t)$ for $\omega \notin \Delta$.

Next, we notice that, by (5), we have

$$\nu^*(z, t) = \exp\left(-2i\arg(h^{-1})'\circ(z)\right)\tilde{\nu}^*\left(h^{-1}(z), t\right). \tag{8}$$

From Theorem 3 we see that

$$(9)\qquad \hat{\hat{\nu}}(\zeta, t) = \hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)\exp 2i\arg\left(\frac{1-i\exp i\vartheta}{1-ir\exp i\vartheta}\right)^2,$$

where $\dfrac{1}{i}\,\dfrac{\zeta-1}{\zeta+1} = re^{i\vartheta}$.

Hence, by (6), we obtain

$$\frac{\partial\omega}{\partial t} = \frac{(\omega-i)(\omega^2-1)}{\pi t}\times$$

$$\times\int_0^{\pi}\int_{-\infty}^{+\infty}\frac{\hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)\exp 2i\arg\left(\frac{1-i\exp i\vartheta}{1-ir\exp i\vartheta}\right)^2}{\left(1-\left|\hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)\right|^2\right)\left(1-i+(1+i)(ir\exp i\vartheta\right)}\times$$

$$\times\frac{(1-ir\exp i\vartheta)^4\,dr\,d\vartheta}{i\left(1-\omega+ir(1+\omega)\exp i\vartheta\right)|1-ir\exp i\vartheta|^4\exp i\vartheta}$$

$$= \frac{(\omega-i)(\omega^2-1)}{\pi t}\int_0^{\pi}\left[\frac{\hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)\exp i\arg(1-i\exp i\vartheta)^4}{\left(1-\left|\hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)\right|^2\right)\exp i\vartheta}\times\right.$$

$$\left.\times\int_{-\infty}^{+\infty}\frac{dr}{(1-i+(1+i)ir\exp i\vartheta)(1-\omega+(1+\omega)ir\exp i\vartheta)}\right]d\vartheta$$

$$= -\frac{(\omega-i)(\omega-1)}{\pi t(1+i)}\int_0^{\pi}\left[\frac{\hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)\exp i\arg(1-\exp i\vartheta)^4}{\left(1-\left|\hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)\right|^2\right)\exp 2i\vartheta}\times\right.$$

$$\left.\times\int_{-\infty\exp i\vartheta}^{+\infty\exp i\vartheta}\frac{d\zeta}{(\zeta-1)\left(\zeta-\frac{1}{i}\,\frac{\omega-1}{\omega+1}\right)}\right]d\vartheta.$$

By the theorem of residues we have

$$\int_{-\infty\exp i\vartheta}^{+\infty\exp i\vartheta}\frac{d\zeta}{(\zeta-1)\left(\zeta-\frac{1}{i}\,\frac{\omega-1}{\omega+1}\right)} = \begin{cases}\dfrac{-2\pi(\omega+1)}{\omega-1-i(\omega+1)} & \text{for } 0<\vartheta<\arg\dfrac{1}{i}\,\dfrac{\omega-1}{\omega+1}\\[2ex] 0 \quad \text{for} & \arg\dfrac{1}{i}\,\dfrac{\omega-1}{\omega+1}<\vartheta<\pi\end{cases}$$

Consequently,

$$\frac{\partial\omega}{\partial t} = -\frac{2i(\omega-i)(\omega^2-1)}{t(1+i)\big(\omega-1-i(\omega+1)\big)}\times$$

$$\times\int\limits_0^{\arg\frac{1}{i}\frac{\omega-1}{\omega+1}} \frac{\hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)\exp i\arg(1-i\exp i\vartheta)^4}{\left(1-\left|\hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)\right|^2\right)\exp 2i\vartheta}\,d\vartheta,$$

whence, by (5), we have

$$\frac{\partial w}{\partial t} = -\frac{2iw}{t}\int\limits_0^{\arg w} \frac{\hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)\exp i\arg(1-i\exp i\vartheta)^4}{\left(1-\left|\hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)\right|^2\right)\exp 2i\vartheta}\,d\vartheta. \tag{10}$$

Next, using (8) it is easy to show, that

$$\nu^*(\exp i\vartheta, t) = -\exp 2i\arg(1-i\exp i\vartheta)^2\tilde{\nu}^*\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right).$$

Hence, since $\hat{\nu}\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right) = \tilde{\nu}^*\left(\frac{1+i\exp i\vartheta}{1-i\exp i\vartheta}, t\right)$ for $0\leqslant\vartheta\leqslant\pi$ and (10) holds, the assertion of Theorem 8 follows.

Theorem 9. *Under the hypotheses of Theorem* 8 *the function* $w = g(z, t)$, *considered as a function of* z *and* t, *satisfies on* $\{\mathcal{E}^+\setminus\{\infty\}\}\times\{t\colon 0\leqslant t\leqslant 1\}$ *the equation*

$$\frac{\partial w}{\partial t} = 2|z|\exp\int\limits_{e^{i\arg z}}^{1}\frac{t\mu(\varepsilon)-\varepsilon^2}{t\mu(\varepsilon)+\varepsilon^2}\frac{d\varepsilon}{\varepsilon}\int\limits_{e^{i\arg z}}^{1}\frac{\mu(\varepsilon)}{\big(t\mu(\varepsilon)-\varepsilon^2\big)^2}\frac{d\varepsilon}{\varepsilon} \tag{11}$$

subject to the initial condition $g(z, 0) = z^{\Gamma/\pi}$.

The proof follows immediately from Theorem 5.

4. A general extremal problem in H_Q

In Theorems 6 and 7 we have given sharp estimates for $|f(z)/z|$ and $\arg f(z)$ when f ranges over H_Q. Now we proceed to more general extremal problems. First we determine the extremal functions for any sufficiently regular real-valued functional $U = F(z_1, \ldots, z_n; w_1, \ldots, w_n)$ with fixed $z_1, \ldots, z_n \in \mathcal{E}^+$, $w_1 = f(z_1), \ldots, w_n = f(z_n)$, and f ranging over H_Q. Next we determine the extremal functions in an analogous problem with the additional condition that another real-valued functional G, satisfying the same regularity conditions, admits a given fixed value. This enables us to find in several cases the region of variability of the complex-valued functional $F+iG$.

Theorem 10. *Let* $U = F(\zeta_1, \ldots, \zeta_n; \omega_1, \ldots, \omega_n)$ *be a real-valued function defined for* $\zeta_k \in D_k$, $\omega_k \in D_{Q,k}$, *where* $D_k \subset \mathcal{E}^+$, $D_{Q,k} \supset \bigcup_{g \in H_Q} g(D_k)$ $(k = 1, \ldots, n)$. *Suppose that* F *has continuous partial derivatives with respect to* $\omega_1, \ldots, \omega_n$. *Then there exists a function* $f \in H_Q$ *for which the functional* $U = F(z_1, \ldots, z_n; g(z_1), \ldots, g(z_n))$ *attains its maximum when* g *ranges over* H_Q; z_k *being fixed points of* D_k *such that* $\arg z_k \geqslant \arg z_{k-1}$, $(k = 1, \ldots, n)$, $z_0 = 1$. *The maximum is also attained for any function* f_1 *defined by* $f_1(s) = f(s)$ *if* $\arg z_n > \arg s \geqslant 0$ *and by* $f_1(s) = f(z_n) f^*(s/z_n)$ *if* $\arg z_n \leqslant \arg s \leqslant \pi$, *where* $f^* \in H_Q$. *Moreover, if* f *is not the identity function and if*

$$(12) \qquad \sum_{k=m+1}^{n} f(z_k) F_{\omega_k}(z_1, \ldots, z_n; f(z_1), \ldots, f(z_n)) \neq 0 \qquad (m = 0, \ldots, n-1),$$

then we have

$$(13) \qquad f(s) = w_m |s/z_m| e^{i\beta_m(z_1, \ldots, z_n; \varepsilon_m) \arg(s/z_m)}$$

for $\arg z_m \leqslant \arg s \leqslant \arg z_{m+1}$, $(m = 0, 1, \ldots, n-1)$, *where*

$$(14) \qquad \beta_m(z_1, \ldots, z_n; \varepsilon_m) = \frac{1}{2}\left(Q + \frac{1}{Q}\right) - \frac{1}{2} i \varepsilon_m \left(Q - \frac{1}{Q}\right) \times$$

$$\times \exp\left(-i \arg \sum_{k=m+1}^{n} w_k F_{\omega_k}(z_1, \ldots, z_n; w_1, \ldots, w_n)\right),$$

$$\varepsilon_m = 1 \text{ or } -1, w_0 = 1, w_1 = f(z_1), \ldots, w_n = f(z_n),$$

and $\arg s$, $\arg f(s)$ *change in a continuous manner for* $\arg z_m \leqslant \arg s \leqslant \arg z_{m+1}$ *so that* $f(s) \to w_m$ *as* $s \to z_m$. *The theorem remains valid if "maximum" is replaced by "minimum".*

The proof is analogous to the proof of Theorem 5 in [4], p. 319–321.

Theorem 11. (i) *Let* $w = F(\zeta_1, \ldots, \zeta_n; \omega_1, \ldots, \omega_n)$ *be a complex-valued function defined for* $\zeta_k \in D_k$, $\omega_k \in D_{Q,k}$, *where* $D_k \subset \mathcal{E}^+$, $D_{Q,k} \supset \bigcup_{g \in H_Q} g(D_k)$, $(k = 1, \ldots, n)$. *Suppose that* F *has continuous partial derivatives with respect to* $\omega_1, \ldots, \omega_n$. *Then there exists a function* $f \in H_Q$ *for which the functional* $U = \operatorname{re} F(z_1, \ldots, z_n; g(z_1), \ldots, g(z_n))$ *attains its maximum when* $\operatorname{im} F(z_1, \ldots, z_n; g(z_1), \ldots, g(z_n)) = \tau$, *and* g *ranges over* H_Q; z_k *being fixed points of* D_k *such that* $\arg z_k \geqslant \arg z_{k-1}$, $(k = 1, \ldots, n)$, $z_0 = 1$, *and* τ *being a real number such that*

$$\min_{g \in H_Q} \operatorname{im} F(z_1, \ldots, z_n; g(z_1), \ldots, g(z_n)) < \tau < \max_{g \in H_Q} \operatorname{im} F(z_1, \ldots, z_n; g(z_1), \ldots, g(z_n)).$$

The maximum is also attained for any function f_1 *defined by* $f_1(s) = f(s)$ *if* $\arg z_n > \arg s \geqslant 0$, *and by* $f_1(s) = f(z_n) f^*(s/z_n)$ *if* $\arg z \leqslant \arg s \leqslant \pi$, *where* $f^* \in H_Q$.

(ii) *Suppose that all assumptions given in* (i) *are fulfilled. Let* ε *denote a sequence* $\{\varepsilon_0, \ldots, \varepsilon_{n-1}\}$, *where* $\varepsilon_m = 1$ *or* -1 $(m = 0, \ldots, n-1)$, *and let* Λ_ε *denote the set of real numbers* λ *satisfying*

$$\sum_{k=m+1}^{n} f^{(\lambda,\varepsilon)}(z_k) F_{\omega_k}^{(\lambda)}\big(z_1, \ldots, z_n; f^{(\lambda,\varepsilon)}(z_1), \ldots, f^{(\lambda,\varepsilon)}(z_n)\big) \neq 0 \quad (m = 0, \ldots, n-1).$$

Here $F^{(\lambda)} = \operatorname{re} F + \lambda \operatorname{im} F$, *the functions* $f^{(\lambda,\varepsilon)}$ *are defined by the formulae* $f^{(\lambda,\varepsilon)}(s) = s$ *for* $\arg s = 0$, *and by*

$$f^{(\lambda,\varepsilon)}(s) = w_m^{(\lambda,\varepsilon)} |s/z_m| e_m^{i\beta_m(z_1, \ldots, z_n; \lambda, \varepsilon) \arg(s/z_m)}$$

for $\arg z_m < \arg s \leqslant \arg z_{m+1}$ $(m = 0, 1, \ldots, n-1)$, *where*

$$\beta_m(z_1, \ldots, z_n; \lambda, \varepsilon) = \frac{1}{2}\left(Q + \frac{1}{Q}\right) - \frac{1}{2} i \varepsilon_m \left(Q - \frac{1}{Q}\right) \times$$

$$\times \exp\Big(-i \arg \sum_{k=m+1}^{n} w_k^{(\lambda,\varepsilon)} F_{\omega_k}^{(\lambda)}\big(z_1, \ldots, z_n; w_1^{(\lambda,\varepsilon)}, \ldots, w_n^{(\lambda,\varepsilon)}\big)\Big),$$

$$w_0^{(\lambda,\varepsilon)} = 1,\ w_1^{(\lambda,\varepsilon)} = f^{(\lambda,\varepsilon)}(z_1), \ldots, w_n^{(\lambda,\varepsilon)} = f^{(\lambda,\varepsilon)}(z_n),$$

and $\arg s, \arg f^{(\lambda,\varepsilon)}(s)$ *change in a continuous manner for* $\arg z_m < \arg s \leqslant \arg z_{m+1}$, *so that* $f^{(\lambda,\varepsilon)}(s) \to w_m^{(\lambda,\varepsilon)}$ *as* $s \to z_m$ *in the case of every* $\lambda \in \Lambda_\varepsilon$ *and each* ε. *Next let* λ_ε^τ, $\lambda_\varepsilon^\tau \subset \Lambda_\varepsilon$, *denote the set of numbers* $\lambda(\tau, \varepsilon)$ *such that*

$$\operatorname{im} F\big(z_1, \ldots, z_n; f^{(\lambda(\tau,\varepsilon),\varepsilon)}(z_1), \ldots, f^{(\lambda(\tau,\varepsilon),\varepsilon)}(z_n)\big) = \tau.$$

Finally let Λ_*^τ, $\Lambda_*^\tau \subset \bigcup_\varepsilon \Lambda_\varepsilon^\tau$, *denote the set of numbers* $\lambda_*(\tau)$ *for which* $\operatorname{re} F\big(z_1, \ldots, z_n; f^{(\lambda(\tau,\varepsilon),\varepsilon)}(z_1), \ldots, f^{(\lambda(\tau,\varepsilon),\varepsilon)}(z_n)\big)$ *attains its maximum when* $\lambda(\tau, \varepsilon)$ *ranges over* $\bigcup_\varepsilon \Lambda_\varepsilon^\tau$. *Suppose additionally that the extremal function is not the identity function and that*

$$\sum_{k=m+1}^{n} f(z_k) F_{\omega_k}^{(\lambda_*(\tau))}\big(z_1, \ldots, z_n; f(z_1), \ldots, f(z_n)\big) \neq 0 \quad (m = 0, \ldots, n-1)$$

for some $\lambda_*(\tau) \in \Lambda_*^\tau$. *Then there are: a sequence* $\varepsilon^0 = \{\varepsilon_0^0, \ldots, \varepsilon_{n-1}^0\}$, *where* $\varepsilon_m^0 = 1$ *or* -1 $(m = 0, \ldots, n-1)$, *and a number* $\lambda_0(\tau) = \lambda(\tau, \varepsilon^0)$, $\lambda_0(\tau) \in \Lambda_*^\tau$, *such that*

$$f(s) = f^{(\lambda_0(\tau),\varepsilon^0)}(s) \text{ for } \arg z_n \geqslant \arg s \geqslant 0.$$

Moreover, each other function $f^{(\lambda_*(\tau)\varepsilon^*)}$, *where* $\lambda_*(\tau) \in \Lambda_*^\tau$, $\lambda_*(\tau) = \lambda(\tau, \varepsilon^*)$, $\varepsilon^* = \{\varepsilon_0^*, \ldots, \varepsilon_{n-1}^*)\}$, $\varepsilon_m^* = 1$ *or* -1 $(m = 0, \ldots, n-1)$, *is also an extremal* *function for the problem under consideration in* H_Q *provided it is continued into the sector* $\{s\colon \arg z_n \leqslant \arg s \leqslant \pi\}$ *by any way described in* (i).

(iii) *The theorem remains valid if "maximum" is replaced by "minimum".*

Proof. We apply the well known method of multipliers of Lagrange together with Theorem 10. Hence Theorem 11 follows immediately.

5. Some applications

We apply now Theorems 10 and 11 in order to find

(a) the region of variability of the functional $F(w) = \log w, \log 1 = 0$, $w = f(z)$, z being fixed, and f ranging over H_Q (Theorem 12 below),

(b) the sharp estimate of $|\tilde{f}(0)|$, where $\tilde{f}$ is defined by $\tilde{f} = h^{-1}\circ f\circ h$, f ranging over H_Q and h denoting the homography $h(s) = \dfrac{1}{i}\dfrac{s-1}{s+1}$ (Theorem 13 below).

Theorem 13 is an analogue of a result of O. Teichmüller [6], which has been generalized by J. Krzyż [2].

Theorem 12. (i) *For any* $f \in H_Q$ *and* $z \in \mathscr{E}^+, z \neq 0, \infty$, *we have*

$$(1/Q)\arg z \leqslant \arg f(z) \leqslant Q\arg z,$$

where $\arg z, \arg f(z)$ *change in a continuous manner from the initial value* $\arg f(1) = 0 = \arg 1$.

(ii) *Moreover, the condition*

$$\arg f(z) = \left[\frac{1}{2}\left(Q+\frac{1}{Q}\right)+\frac{1}{2}\left(Q-\frac{1}{Q}\right)\sin\varphi\right]\arg z \quad \left(-\frac{1}{2}\pi \leqslant \varphi \leqslant \frac{1}{2}\pi\right) \tag{15}$$

implies

$$-\frac{1}{2}\left(Q-\frac{1}{Q}\right)(\arg z)\cos\varphi \leqslant \log|f(z)| - \log|z| \leqslant \frac{1}{2}\left(Q-\frac{1}{Q}\right)(\arg z)\cos\varphi. \tag{16}$$

All the given estimates are sharp for any $z \in \mathscr{E}^+, z \neq 0, \infty$, *and* $Q \in \langle 1, +\infty)$. *Given* φ, $-\frac{1}{2}\pi \leqslant \varphi \leqslant \frac{1}{2}\pi$, *the only extremal functions for every* z *in* (16) *are* $f(s) = |s|e^{i\beta_1 \arg s}$ *and* $f(s) = |s|e^{i\beta_2 \arg s}$ *for the upper and lower bound, respectively, where* $\beta_1 = \dfrac{1}{2}\left(Q+\dfrac{1}{Q}\right) - \dfrac{i}{2}\left(Q-\dfrac{1}{Q}\right)e^{i\varphi}$, $\beta_2 = \dfrac{1}{2}\left(Q+\dfrac{1}{Q}\right)+\dfrac{i}{2}\left(Q-\dfrac{1}{Q}\right)e^{-i\varphi}$, *and in each case* $\arg z, \arg f(z)$ *change in a continuous manner from the initial value* $\arg f(1) = 0 = \arg 1$.

(iii) *Furthermore,* (15) *and* (16) *give all points of the variability region of the functional* $F(w) = \log w$, *where* $\log 1 = 0$, $w = f(z)$, f *ranges over* H_Q, *and* $z\,(z \in \mathscr{E}^+, z \neq 0, \infty)$ *is fixed.*

The proof is analogous to the corresponding proof in [4], p. 326.

Theorem 13. *If* $\tilde{f} = h^{-1}\circ f\circ h, f\in H_Q$, *and* h *denotes the homography* $h(s) = \frac{1}{i}\frac{s-1}{s+1}$, *then*

$$|\tilde{f}(0)| < |\hat{w}|,$$

where $\hat{w}$ *denotes any solution of the equation*

$$i\tilde{w} = \operatorname{tg}\frac{1}{4}\pi\left(1-\frac{1}{2}\left(Q+\frac{1}{Q}\right)+\frac{1}{2}i\varepsilon\left(Q-\frac{1}{Q}\right)\exp i\arg\frac{2\tilde{w}}{\tilde{w}-1}\right),\ \varepsilon = 1,\ or\ -1,$$

with the greatest modulus. The estimate is sharp for any $Q\in\langle 1, +\infty)$. *The only extremal functions are of the form* $\tilde{f} = h^{-1}\circ f\circ h$, *where* $f(s) = |s|\exp\left(\left(\frac{1}{2}\left(Q+\frac{1}{Q}\right)-\frac{1}{2}i\varepsilon\left(Q-\frac{1}{Q}\right)\exp i\arg\frac{2\hat{w}}{\hat{w}-1}\right)i\arg s\right)$ *for* $0\leqslant \arg s \leqslant \frac{1}{2}\pi$, $f(s) = \frac{1}{i}\frac{\hat{w}-1}{\hat{w}+1}f^*\left(\frac{1}{i}s\right)$ *for* $\frac{1}{2}\pi < \arg s\leqslant\pi$, *where* $\arg s, \arg f(s)$ *change in a continuous manner from the initial value* $\arg f(1) = 0 = \arg 1$, *and* f^* *is an arbitrary function of the class* H_Q.

Proof. We use the notation of Theorem 10. Applying this theorem to $F(\zeta, \omega) = |1+i\omega|/|1-i\omega|$ and $z_1 = i$ we obtain that the only extremal functions are of the form

$$f(s) = |s|\exp\left(\left(\frac{1}{2}\left(Q+\frac{1}{Q}\right)-\frac{1}{2}i\varepsilon_0\left(Q-\frac{1}{Q}\right)\left(-\frac{w_1+1/w_1}{\overline{w}_1+1/\overline{w}_1}\right)^{\frac{1}{2}}\right)i\arg s\right)$$

$$\text{for } 0\leqslant\arg s\leqslant\frac{1}{2}\pi, f(s) = w_1 f^*\left(\frac{1}{i}s\right) \text{ for } \frac{1}{2}\pi<\arg s\leqslant\pi,$$

where $\arg s, \arg f(s)$ change in a continuous manner from the initial value $\arg f(1) = 0 = \arg 1$, f^* is an arbitrary function of the class H_Q, and w_1 is any solution of the equation

$$w = \exp\frac{1}{2}\pi i\left(\frac{1}{2}\left(Q+\frac{1}{Q}\right)-\frac{1}{2}i\varepsilon_0\left(Q-\frac{1}{Q}\right)\left(-\frac{w+1/w}{\overline{w}+1/\overline{w}}\right)^{\frac{1}{2}}\right),\ \varepsilon_0 = 1 \text{ or } -1,$$

such that $|F(i, w)|$ attains its maximal value. Hence, putting $\tilde{w} = (1+iw_1)/(1-iw_1)$, we obtain the assertion of our theorem.

6. The class H_Q^*

The class H_Q^* is an analogue of H_Q for functions defined on $\mathscr{E}\setminus\mathrm{cl}\{z: \arg z = 0\}$. We give here six equivalent conditions for f to be of the class H_Q^*. The proofs of equivalence are omitted since they are analogous to that given in [4], pp. 311–315.

Definition 1*. A function f is said to be of the class H_Q^* if it is defined on $\mathscr{E}\setminus \mathrm{cl}\{z: \arg z = 0\}$, and

(i) maps $\mathscr{E}\setminus \mathrm{cl}\{z: \arg z = 0\}$ into the Riemann surface of $s = w^{t'/2\pi}$, $w \in \mathscr{E}$, ($t' > 0$ and $f(e^{it_n}) = |w_n| e^{it'_n}$ for some $t_n \to 2\pi -$ and $t'_n \to 2\pi -$ as $n \to +\infty$),

(ii) $f(z_n) \to 0$, $f(e^{i\tilde{t}_n}) = |\tilde{w}_n| e^{i\tilde{t}'_n}$, $f(\tilde{z}_n) \to \infty$ for some $z_n \to 0$, $\tilde{t}_n \to 0+$, $\tilde{t}'_n \to 0+$ and $\tilde{z}_n \to \infty$ as $n \to +\infty$,

(iii) $f(z) = |z| f(e^{i\arg z})$ for $z \in \mathscr{E}\setminus \mathrm{cl}\{z: \arg z = 0\}$.

Theorem 1*. *$f \in H_Q^*$ if it is defined on $\mathscr{E}\setminus \mathrm{cl}\{z: \arg z = 0\}$, satisfies conditions* (i) *and* (ii) *in Definition* 1* *and if $a_m f(z) = f(a_m z)$, where $\{a_m\}$ is a sequence of real numbers such that $a_m \to 1$ as $m \to +\infty$.*

Theorem 2*. *$f \in H_Q^*$ if it is defined on $\mathscr{E}\setminus \mathrm{cl}\{z: \arg z = 0\}$, satisfies conditions* (i) *and* (ii) *in Definition* 1* *and if $\mu(z) = \mu(a_m z)$ a.e. in $\mathscr{E}$, where μ is the complex dilatation of f, and a_m $(m = 1, 2, \ldots)$ are the same as in Theorem* 1*.

Theorem 3*. *$f \in H_Q^*$ if it is defined on $\mathscr{E}\setminus \mathrm{cl}\{z: \arg z = 0\}$, satisfies conditions* (i) *and* (ii) *in Definition* 1*, *and if $\mu(z) = \mu(e^{i\arg z})$ a.e. in $\mathscr{E}$.*

Theorem 4*. *$f \in H_Q^*$ if it is defined on $\mathscr{E}\setminus \mathrm{cl}\{z: \arg z = 0\}$, satisfies conditions* (i) *and* (ii) *in Definition* 1* *as well as $z f_z(z) + \bar{z} f_{\bar{z}}(z) = f(z)$ a.e. in $\mathscr{E}$.*

Theorem 5*. *$f \in H_Q^*$ if it is given by the formulae*

$$f(z) = |z| \exp \int_{e^{i\arg z}}^{1} \frac{\mu(\varepsilon) - \varepsilon^2}{\mu(\varepsilon) + \varepsilon^2} \frac{d\varepsilon}{\varepsilon} \quad \textit{for } z \in \mathscr{E}\setminus \mathrm{cl}\{z: \arg z = 0\},$$

where μ is measurable with $\sup_{0 \leqslant \varphi < 2\pi} |\mu(e^{i\theta})| < 1$ *and* $\operatorname{ess\,sup}_{0 \leqslant \varphi < 2\pi} |\mu(e^{i\theta})| \leqslant \dfrac{Q-1}{Q+1}$, *$\varepsilon$ ranging over the unit circle from $e^{i\arg z}$ to* 1.

Definition 1* immediately implies the relations

$$\text{(17)} \qquad \left|\frac{f(z)}{z}\right| = |f(e^{i\arg z})| \equiv R(\arg z),$$

$$\arg f(z) = \arg f(e^{i\arg z}) \equiv \theta(\arg z)$$

and the following

Lemma 1*. *With the notation of formula* (17) *we have*

$$\arg f^{-1}(w) = \theta^{-1}(\arg w),$$

$$\left|\frac{f^{-1}(w)}{w}\right| = \frac{1}{R(\theta^{-1}(\arg w))}.$$

We also notice the following trivial result which gives the correspondence between H_Q and H_Q^*.

Lemma 2*. *If a function f belongs to H_Q^*, then the functions f_1 and f_2 defined by $f_1(z) = f(z), f_2(z) = \overline{f(\bar{z})}$ for $z \in \mathscr{E}^+ \setminus \mathrm{cl}\{z: \arg z = 0\}$ and $f_1(z) = f_2(z) = z$ for $z \in \mathrm{cl}\{z: \arg z = 0\}$ both belong to H_Q.*

Now we give an analogue of Theorems 8 and 9 for the class H_Q^*. Proofs are omitted since they can be performed in the same way as the corresponding proofs in [4], p. 337–338.

Theorem 6*. *Suppose that $w = f(z)$ belongs to H_Q^* and has $u = \mu(z)$ as its complex dilatation. Moreover, suppose that the functions $w = g(z, t)$, $0 \leqslant t \leqslant 1$, with the corresponding t' being fixed (cf. e.g. Definition 1*), belong to H_Q^* and have complex dilatations (3). Then $w = g(z, t)$, considered as a function of z and t, satisfies on $\{\mathscr{E} \setminus \mathrm{cl}\{z: \arg z = 0\} \times \{t: 0 \leqslant t \leqslant 1\}$ the equation (4) subject to the initial condition $g(z, 0) = z^{t'/2\pi}$, where ν^* is the complex dilatation of g^{-1}.*

Theorem 7*. *Under the hypotheses of Theorem 6* the function $w = g(z, t)$, considered as a function of z and t, satisfies on $\{\mathscr{E} \setminus \mathrm{cl}\{z: \arg z = 0\}\} \times \{t: 0 \leqslant t \leqslant 1\}$ the equation (11) subject to the initial condition $g(z, 0) = z^{t'/2\pi}$.*

We see that in the same way as in [4], p. 338–340, we can formulate some analogues of Theorems 10 and 11 for the class H_Q^*.

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STRESZCZENIE

W pracy niniejszej omówiono klasę odwzorowań lokalnie quasi-konforemnych górnej półpłaszczyzny spełniających równanie $f(z) = |z| f(e^{i \arg z})$ oraz analogiczną klasę odwzorowań quasi-konforemnych

płaszczyzny. Klasy te zostały wprowadzone przez J. Ławrynowicza jako odpowiedniki badanych przez niego klas odwzorowań quasi-konforemnych koła jednostkowego na siebie oraz płaszczyzny na siebie, spełniających równanie $f(z) = e^{i \arg z} f(|z|)$.

РЕЗЮМЕ

В работе исследован класс локально квазиконформных отображений верхней полуплоскости, удовлетворяющих уравнению $f(z) = |z| f(e^{i \arg z})$, и аналогичный класс квазиконформных отображений плоскости. Эти классы были введены Ю. Лаврыновичем как аналогоны исследованных им классов квазиконформных отображений единичного круга на себя и плоскости на себя, удовлетворяющих уравнению $f(z) = e^{i \arg z} f(|z|)$.