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Department of Mathematics, University of Maryland
College Park, Maryland, USA

WILLIAM E. KIRWAN

The Koebe Constant for a Class of Bounded Domains

Stała Koebego dla klasy obszarów ograniczonych

Константа Кебе для ограниченных областей

In this note we determine the Koebe constant for bounded domains with boundary rotation at most $k\pi (2 \leqslant k \leqslant 4)$. This result extends an earlier result of J. Krzyż for bounded convex domains.

1. Introduction

Let F denote a compact family of functions

$$f(z) = z + a_2 z^2 + \ldots \tag{1.1}$$

that are analytic and univalent in the unit disk U. We denote by $k(F)$ the Koebe constant for the family F. That is, $k = k(F)$ is the largest constant such that $f(U) \supset \{|\omega| < k\}$ for every $f \epsilon F$. The constant $k(F)$ is known for many families of functions. In this note we will be concerned with determining the Koebe constant for some classes of bounded univalent functions.

Let $\mathscr{S}(B)$ denote the class of univalent functions with the normalization (1.1) that satisfy $|f(z)| < B$ $(z \epsilon U)$. The Koebe constant for this class of functions is known [5, p. 224] and the extremal function maps U into the disk $|\omega| < B$ minus the segment from $-B$ to $-k[\mathscr{S}(B)]$. The extremal domain is unique up to rotation.

In [2] J. Krzyż determined the Koebe constant for the class $\mathscr{C}(B)$ of functions in $\mathscr{S}(B)$ that map U onto a convex domain. An extremal function in this case maps U onto a domain containing the origin, bounded by an arc of $|\omega| = B$ and a vertical line through $-k[\mathscr{C}(B)]$. Again the extremal domain is unique up to rotation.

We will denote by V_k the class of functions with normalization (1.1) that map U onto a domain with boundary rotation at most $k\pi$ (see [4] for definitions and basic properties of the class V_k), and by $V_k(B)$ the subclass of V_k consisting of functions bounded by B. The class $V_2(B)$ coincides with $C(B)$ and the extremal function for the Koebe constant for $\mathscr{S}(B)$ belongs to $V_4(B)$. In this paper we will characterize the extremal domain for the Koebe constant of the class $V_k(B)$ $(2 \leqslant k \leqslant 4)$.

2. Let $2 < k \leqslant 4$ and let $D_k(a)$ denote the non-convex domain containing the origin, contained in $|\omega| < B$, and bounded by an arc of $|\omega| = B$ and two half lines emanating from $-a$ $(0 < a < B)$ that are symmetric with respect to the real axis and form an angle of $\frac{1}{2}(k-2)\pi$ at $-a$. The boundary rotation of $D_k(a)$ is $k\pi$ and it is clear that there exists a unique value of a, $\frac{1}{4} < a < 1$, such that the conformal mapping radius of $D_k(a)$ at $\omega = 0$ (in the sequel denoted $r(D_k(a))$) is equal to 1; i.e., such that $D_k(a) = f(U)$ for some $f \epsilon V_k(B)$. This value of a we denote a_k and the corresponding domain we denote simply D_k.

Theorem. *The Koebe constant for* $V_k(B)$ *is* a_k.

Note. The function that maps U onto D_k can be computed (see for example [1, p. 230]) and the value of $a_k = k[V_k(B)]$ can be determined implicitly. We will not do so to avoid the reproduction of the lengthy formulas involved.

3. Proof of the theorem.

Let $f(z) \epsilon V_k(B)$ $(2 \leqslant k \leqslant 4)$ and suppose that $f(z)$ is analytic in $\overline{U} = \{|z| \leqslant 1\}$. If we can show that $D_1 = f(U) \supset \{|\omega| < a_k\}$ then by a standard argument the proof will be complete. Let ξ denote a boundary point of D_1 nearest the origin. There exists a wedge with vertex ξ and opening $\frac{1}{2}(k-2)\pi$ lying in the complement of D_1. For if not, the curve $\omega = f(e^{i\theta})$ $(0 \leqslant \theta < 2\pi)$ crosses both sides of such a wedge and the boundary rotation of D_1 would exceed $k\pi$ which is impossible. Thus if D_2 denotes the domain in $|\omega| < B$ containing the origin, bounded by the sides of the wedge and an arc of $|\omega| = B$, then $D_2 \supseteq D_1$. Consequently

$$r(D_2) \geqslant r(D_1) = 1. \tag{3.1}$$

Further, since ξ is a boundary point of D_1 nearest the origin, ξ is the boundary point of D_2 nearest the origin. By rotating D_2 if necessary, we may assume that the "sides" of D_2 intersect $|\omega| = B$ at points $B_1(\operatorname{Im} B_1 \geqslant 0)$ and $\overline{B}_1$ in the left half-plane that are symmetric with respect to the real axis. We may also assume for definiteness that $\operatorname{Im}\xi \leqslant 0$. We now form the domain D contained in $|\omega| = B$, containing the origin, bounded by an arc of $|\omega| = B$ and the line segments from B_1 to $-|\xi|$

and $\bar{B}_1$ to $-|\xi|$. It is clear that the boundary rotation of D is smaller than $k\pi$, but it is not clear that $r(D) \geqslant 1$. Once we have shown that $r(D) \geqslant 1$, the proof is easily completed. Indeed since the boundary rotation of $D \leqslant k\pi$, if we replace D by $D_k(|\xi|)$ then by the monotonicity of the mapping radius, $r\big(D_k(|\xi|)\big) \geqslant 1$ which again implies by the monotonicity of the mapping radius that $a_k < |\xi|$. Thus $f(U) = D_1 \supset \{|\omega| < a_k\}$ which completes the proof.

It remains to show that $r(D) \geqslant 1$. Let D_3 be the domain obtained from D_2 by circular symmetrization with respect to the real axis. By a result due to Pólya and Szegö [6, p. 44],

$$r(D_3) \geqslant r(D_2). \tag{3.2}$$

D_3 is bounded by an arc of $|\omega| = B$ and two curves, symmetric with respect to the real axis, joining B_1 to $-|\xi|$ and $\bar{B}_1$ to $-|\xi|$ respectively. We will show that $D_3 \subseteq D$. By the monotonicity of the mapping radius, (3.2) and (3.1), it will then follow that $r(D) \geqslant 1$.

Let H and H_2, respectively, denote the complement of D and D_2 in $|\omega| < B$. Let $m(s)$ and $m_2(s)$, respectively, denote the linear measure of the intersection of H and H_2 with $|\omega| = s$ $(|\xi| \leqslant s \leqslant B)$. To show that $D_3 \subset D$ it suffices to show that

$$m(s) \leqslant m_2(s) \quad (|\xi| \leqslant s \leqslant B). \tag{3.3}$$

We note that $m(|\xi|) = 0 = m_2(|\xi|)$ and $m(B) = m_2(B)$. It is easy to see that the area of H_2 is greater than the area of H. Thus there exists an s, $|\xi| < s < B$, such that $m(s) < m_2(s)$. Hence, if we can show that for $|\xi| \leqslant s < B$, $m(s) = m_2(s)$ at most once then (3.3) will follow.

This fact can be seen in the following way. We reflect the segment from $\bar{B}_1$ to ξ about the real axis and denote the reflected segment by $\mathscr{L}_1$. We denote by $\mathscr{L}_2$ the segment from B_1 to $-|\xi|$ and by $\mathscr{L}_3$ the segment from B_1 to ξ. We assume that $\arg\bar{\xi} \leqslant \arg B_1$. If $\arg B_1 < \arg\bar{\xi}$, the argument is not essentially changed. Let $\mathscr{L}_4$ denote the segment from 0 to B_1. Let a denote the angle formed by $\mathscr{L}_1$ and $\mathscr{L}_4$, β the angle formed by $\mathscr{L}_2$ and $\mathscr{L}_4$ and γ the angle formed by $\mathscr{L}_2$ and $\mathscr{L}_3$. It is not difficult to show that

$$a < \beta+\gamma, \quad \gamma < \beta+a$$

and

$$a+\beta+\gamma < \pi.$$

Suppose that $m_2(s) = m(s)$, $|\xi| \leqslant s < B$. Let $|\omega| = s$ intersect $\mathscr{L}_1$ at C_1, $\mathscr{L}_2$ at C_2 and $\mathscr{L}_3$ at C_3. Then $|C_1 - C_2| = |C_2 - C_3|$. Denote the angles $\sphericalangle B_1C_1C_2$, $\sphericalangle 0C_1C_2$ and $\sphericalangle B_1C_3C_2$ by u, t and v respectively. By our assumption that $m(s) = m_2(s)$: $\sphericalangle 0C_3C_2 = t$. We have the following relations between these angles.

$$\frac{\sin u}{|B_1 - C_2|} = \frac{\sin(\alpha+\beta)}{|C_1 - C_2|}$$

$$\frac{\sin v}{|B_1 - C_2|} = \frac{\sin\gamma}{|C_2 - C_3|}$$

and therefore

$$\frac{\sin v}{\sin u} = \frac{\sin\gamma}{\sin(\alpha+\beta)} = \mu < 1. \tag{3.4}$$

Similarly,

$$\frac{\sin(u+t)}{B} = \frac{\sin\alpha}{s}$$

$$\frac{\sin(v+t)}{B} = \frac{\sin(\beta+\gamma)}{s}$$

and therefore

$$\frac{\sin(v+t)}{\sin(u+t)} = \frac{\sin(\beta+\gamma)}{\sin\alpha} = \eta > 1. \tag{3.5}$$

Also, it follows that

$$2t = u+v+\alpha+\beta+\gamma$$

or if we set $q = \alpha+\beta+\gamma$,

$$2t = u+v+q.$$

From (3.4) and (3.5) we derive

$$\tan\frac{v-u}{2} = \frac{\mu-1}{\mu+1}\tan(t-\tfrac{1}{2}q)$$

and

$$\tan\frac{v-u}{2} = \frac{\eta-1}{\eta+1}\tan(2t-\tfrac{1}{2}q)$$

respectively.

Hence for any value of s, $|\xi| \leqslant s < B$, for which $m_2(s) = m(s)$

$$\tan(2t-\tfrac{1}{2}q) = \frac{\eta+1}{\eta-1}\cdot\frac{\mu-1}{\mu+1}\tan(t-\tfrac{1}{2}q). \tag{3.6}$$

Since $\dfrac{\eta+1}{\eta-1}\cdot\dfrac{\mu-1}{\mu+1} < 0$, it is not hard to show that (3.6) has at most one solution for t and hence $m_2(s) = m(s)$ has at most one solution for $|\xi| \leqslant s < B$. This completes the proof of the theorem.

Notes.

1. It seems likely that if $F(z) \epsilon V_k(B)$ and $F(U) = D_k$ then
$$F(-|z|) \leqslant |f(z)| \leqslant F(|z|)$$
for all $f(z) \epsilon V_k(B)$. For the case $k = 2$ this result was proved by Krzyż [3].
2. The author wishes to express his appreciation to Dov Aharanov for several helpful conversations during the preparation of this paper.

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РЕЗЮМЕ

Пусть $V_k (k \geqslant 2)$ класс функций $f(z) = z + a_2 z^2 + \ldots$, которые дают отображение единичного круга U на область с ограниченным вращением не превышающим $k\pi$. Пусть $V_k(B)$ — подкласс класса V_k, состоящий из функций, которые в круге U удовлетворяют неравенства $|f(z)| \leqslant B$. Я. Кжиж вычислил константу Кебе для класса $V_2(B)$. Применяя метод круговой симметризации, автор расширил этот результат на классы $V_k(B)$, $k \geqslant 2$.

STRESZCZENIE

Niech V_k $(k \geqslant 2)$ oznacza klasę funkcji $f(z) = z + a_2 z^2 + \ldots$ odwzorowujących konforemnie koło jednostkowe U na obszar o obrocie brzegowym nie przekraczającym $k\pi$. Niech $V_k(B)$ oznacza podklasę klasy V_k złożoną z funkcji organicznych $f(z)$, takich że $|f(z)| \leqslant B$ w kole U. J. Krzyż wyznaczył stałą Koebego dla klasy $V_2(B)$. Stosując metodę symetryzacji kołowej autor rozszerza ten wynik na klasy $V_k(B)$, $k \geqslant 2$.