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On some Classes of Close-to-convex Functions

O pewnych podklasach funkcji prawie wypukłych

О некоторых подклассах почти выпуклых функций

1. Introduction

Let S be the class of functions regular and univalent in the unit disk K of the form

(1) $$f(z) = z + a_2 z^2 + \dots$$

Let C_k denote the class of functions φ_k convex and k-symmetric in K with the power series expansion

(2) $$\varphi_k(z) = z + a_{k+1} z^{k+1} + a_{2k+1} z^{2k+1} + \dots$$

and let P_m be the class of functions of positive real part in K with the power series expansion

(3) $$p_m(z) = 1 + a_m z^m + a_{2m} z^{2m} + \dots$$

I.E. Bazilevic was concerned [1] with the class B of univalent functions f satisfying some special Löwner-Spencer-Kufarev equation. These functions can be also defined as follows: each $f \epsilon B$ satifies

(4) $$f'(z) = \varphi'(z) p(z)$$

with $\varphi \epsilon C_1$ and $p \epsilon P_1$. Some authors (see eg. [4], [7], [8]) assumed that the class B is identical with the class L of normalized close-to-convex functions introduced independently and in a formally different manner by M. Biernacki [2] and W. Kaplan [3]. This is not, however, true [5]. In [6] some special classes of L were investigated. In this paper we are concerned with some subclasses B_{km} of B which are defined as follows.

We say that $f \in B_{km}$ iff there exist the functions $\varphi_k \in C_k$ and $p_m \in P_m$ such that

(5) $$f'(z) = \varphi_m'(z) p_m(z).$$

In this paper we find the domain of variability of $\log f'(z)$ for f ranging over B_{km} and solve some extremal problems associated with this class.

2. The domain of variability of $\log f'(z)$ within B_{km}

It is easily verified that the domains of variability of $\log f'(z)$ and $\log f'(|z|)$ coincide. Hence we are led to determine the domain of variability $E(r, k, m)$ of $\log f'(r)$ with $0 < r < 1$.

Theorem 1. *The set $E(r, k, m)$ is closed and convex.*

Proof. The class B_{km} is compact and therefore $E(r, k, m)$ is closed. Suppose now that $\varphi, \psi \in C_k$. Then also

(6) $$\varphi(z) = \int_0^z [\varphi'(z)]^\lambda [\psi'(z)]^{1-\lambda} dz$$

belongs to C_k for any real $\lambda \in [0, 1]$.
Moreover, if $p, q \in P_m$, then

(7) $$Q(z) = [p(z)]^\lambda [q(z)]^{1-\lambda}$$

also belongs to P_m.

If $f, g \in B_{km}$ then by our previous remarks also the function

(8) $$h(z) = \int_0^z [f'(z)]^\lambda [g'(z)]^{1-\lambda} dz$$

belongs to B_{km}. From (8) we obtain

(9) $$\log h'(r) = \lambda \log f'(r) + (1-\lambda) \log g'(r).$$

Hence, if $w_1, w_2 \in E(r, k, m)$ and $0 \leqslant \lambda \leqslant 1$, then $w = \lambda w_1 + (1-\lambda) w_2 \in E(r, k, m)$. This proves the convexity of $E(r, k, m)$.

Theorem 2. *The domain $E(r, k, m)$ of variability of $\log f'(r)$ for a fixed $r \in (0, 1)$ and f ranging over B_{km} is a convex, closed domain symmetric with respect to the real axis and the straight line* $\operatorname{re} w = -\frac{1}{k}\log(1-r^{2k})$ *whose boundary consists of an arc Γ determined by the equation*

(10) $$w = \log \frac{1 + r^m e^{i\gamma_m(\beta)}}{(1 - r^m e^{i\gamma_m(\beta)})(1 - r^k e^{i\theta_k(\beta)})}, \quad 0 \leqslant \beta \leqslant \pi,$$

where

(11) $$\theta_k(\beta) = \beta - \arcsin(r^k \sin\beta)$$

(12) $$\gamma_k(\beta) = \pi + \operatorname{arcctg}\left(\frac{1+r^{2m}}{1-r^{2m}}\operatorname{ctg}\beta\right)$$

and its reflection Γ^* in the real axis.

The functions corresponding to the boundary points of $E(r, k, m)$ have the form

(13) $$F(z) = \int_0^z \frac{1+z^m e^{i\gamma_m(\beta)}}{(1-z^m e^{i\gamma_m(\beta)})(1-r^k e^{i\theta_k(\beta)})}\, dz,$$

or

(14) $$G(z) = \overline{F(\bar{z})}$$

where θ_k, γ_k are given by (11), (12), *resp.*

Proof. We first determine the domains of variability $E_1(r, k)$, $E_2(r, m)$ of $\log\varphi'(r)$ and $\log p(r)$ within the classes C_k and P_m, resp.

The domain $E_1(r, k)$ is obtained from $E_1(r^k, 1)$ by a homothety with ratio k^{-1} which is easily verified in a similar way as in [6]. It is convex and has the real axis and the straight line $\operatorname{re} w = -k^{-1}\log(1-r^{2k})$ as symmetry axes. The boundary points correspond to the functions

(15) $$\int_0^z \frac{dz}{(1-z^k e^{i\theta})^{2/k}}.$$

On the other hand, $E_2(r, m)$ is symmetric w.r.t. both coordinate axes and its boundary corresponds to the functions

(16) $$p(z) = \frac{1+z^m e^{i\gamma}}{1-z^m e^{i\gamma}}.$$

Hence $E(r, k, m)$ has the following form:

(17) $$E(r, k, m) = \{w\colon\ w = w_1 + w_2,\ w_1 \in E_1(r, k),\ w_2 \in E_2(r, m)\}.$$

This means that $E(r, k, m)$ has the real axis and the straight line $\operatorname{re} w = -k^{-1}\log(1-r^{2k})$ as symmetry axes and its boundary points correspond to φ such that

(18) $$\varphi'(z) = \frac{1}{(1-z^k e^{i\theta})^{2/k}}\,\frac{1+z^m e^{i\gamma}}{1-z^m e^{i\gamma}}$$

with θ, γ suitably chosen.

The parameters θ, γ can be determined as follows. The domain $E(r, k, m)$ being convex, its supporting line subtending the angle β with the imaginary axis becomes vertical after a rotation by an angle $-\beta$. Hence we have to find θ, γ for a given $\beta \in [0, \pi]$ so that the expression

$$(19)\qquad H(\gamma, \theta) = \mathrm{re}\left\{e^{-i\beta}\left[\log\frac{1+r^m e^{i\gamma}}{1-r^m e^{i\gamma}} - \frac{2}{k}\log(1-r^k e^{i\theta})\right]\right\}$$

has a maximum.

Obviously the maximum of (19) corresponds to the maximum of $I_1(\gamma)$ and the minimum of $I_2(\theta)$, where

$$(20)\qquad I_1(\gamma) = \mathrm{re}\left\{e^{-i\beta}\log\frac{1+r^m e^{i\gamma}}{1-r^m e^{i\gamma}}\right\}$$

$$(21)\qquad I_2(\theta) = \mathrm{re}\{e^{-i\beta}\log(1-r^k e^{i\theta})\}.$$

Differentiating (20) we obtain

$$(22)\qquad I_1'(\gamma) = \mathrm{re}\left\{e^{-i\beta}\frac{2ir^m e^{i\gamma}}{1-r^{2m}e^{2i\gamma}}\right\} = 2r^m\, im\left\{\frac{e^{-i\beta}}{e^{-i\gamma}-r^{2m}e^{i\gamma}}\right\}.$$

Hence the equation $I_1'(\gamma) = 0$ is equivalent to

$$\gamma = \gamma(\beta) = \operatorname{arccot}[(1+r^{2m})(1-r^{2m})^{-1}\operatorname{ctg}\beta] + l\pi,\ l = 0,1,$$

and it is easily verified that $l = 1$ gives a maximum. On the other hand

$$\theta(\beta) = \beta - \arcsin(r^k\sin\beta)$$

corresponds to a minimum of $I_2(\theta)$.

Ultimately

$$(23)\qquad \max H(\gamma, \theta) = \log\frac{1+r^m e^{i\gamma_k(\beta)}}{(1-r^m e^{i\gamma_k(\beta)})(1-r^k e^{i\theta_k(\beta)})^{2/k}}$$

which proves that the upper half of the boundary of $E(r, k, m)$ is determined by (10).

Putting $z = r$ in (13), or (14) we obtain the boundary points of $E(r, k, m)$ since $\log F'(r)$ with $\beta \in [0, \pi]$ yields (10). This proves Theorem 2.

As a corollary of Theorem 2 we obtain the estimates of $|f'|$ and $\arg f'$.

Theorem 3. *If $f \in B_{km}$ and $|z| = r$, then*

$$(24)\qquad \frac{1-r^m}{(1+r^m)(1+r^k)^{2/k}} \leqslant |f'(z)| \leqslant \frac{1+r^m}{(1-r^m)(1-r^k)^{2/k}}$$

$$(25)\qquad |\arg f'(z)| \leqslant 2\operatorname{arctg} r^m + \frac{2}{k}\arcsin r^k.$$

The estimates (24), (25) are obtained as bounds of $\mathrm{re}\, w$ and $\mathrm{im}\, w$ for w $E(r, k, m)$, By convexity and symmetry of $E(r, k, m)$ the maximum of

$\operatorname{im} w$ corresponds to $\beta = \pi$ whereas the maximum of $\operatorname{re} w$ corresponds to $\beta = \pi/2$ which gives (24), (25).

The signs of equality are obtained for $z = r$ for the functions (13), (14) with $\beta = \pi, 0, \pi/2$, resp.

3. Some particular cases

Let B_k be a subclass of B consisting of f with the power series expansion (2). It is easily verified similarly as in [6] that

$$B_k = B_{kk}. \tag{28}$$

Corollary 1. *The domain of variability $E(r, k)$ of $\log f'(z)$ for f ranging over B_k is a closed convex domain with the real axis and $\operatorname{re} w = -k^{-1}\log(1-r^{2k})$ as axes of symmetry. Its boundary consists of the arc Γ with the equation*

$$w = \log \frac{1+r^k e^{i\gamma_k(\beta)}}{(1-r^k e^{i\gamma_k(\beta)})(1-r^k e^{i\theta_k(\beta)})^{2/k}} \tag{29}$$

where $\theta_k(\beta)$, $\gamma_k(\beta)$ are given by (11), (12) *resp., and its reflection Γ^* w.r.t. the real axis.*

The boundary points of $E(r, k)$ correspond to the functions (13), (14) with $k = m$.

Corollary 2. *If $f \in B_k$, then*

$$\frac{1-r^k}{(1+r^k)^{k+2/k}} \leqslant |f'(z)| \leqslant \frac{1+r^k}{(1-r^k)^{k+2/k}}$$

$$|\arg f'(z)| \leqslant 2\operatorname{arctg} r^k + \frac{2}{k}\arcsin r^k$$

where $|z| = r$.

The case $m = 1$ corresponds to the class B_{k1} which is the class of close-to-convex functions associated with the class of k-symmetric convex functions.

Two analogous subclasses B_{k1}, B_{1k} would also be considered. Both classes show, however to be different.

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Streszczenie

W pracy [6] autor rozpatruje podklasy L_{km} funkcji prawie wypukłych takich, że pochodna

$$f'(z) = \varphi'(z)\cdot p(z), \quad f'(0) = 1$$

gdzie $\varphi(z) = a_1 z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \ldots, |a_1| = 1$, odwzorowuje koło K_1 na obszar wypukły, a funkcja $p(z) = a_0 + a_m z^m + a_{2m}z^{2m} + \ldots$ spełnia warunki: $|p(0)| = |a_0| = 1, \operatorname{re} p(z) > 0$ dla $|z| < 1$.

Jeżeli zaostrzyć warunki na φ i p a mianowicie położyć $\varphi'(0) = a_1 = 1$ i $p(0) = a_0 = 1$, to otrzymamy podklasę B_{km}. Klasy L_{km} i B_{km} są różne między sobą. Klasa B_{km} jest istotnie węższa od klasy L_{km}. Klasę $B_{1,1}$ rozważał w swej pracy [1] Bazylewicz. Określił on ją jako rozwiązanie pewnego równania Löwnera-Spencera-Kufariowa.

W pracy tej określono obszar zmienności $\log f'(z)$ dla klasy B_{km} oraz oszacowania na $|f'(z)|$ i $|\arg f'(z)|$. Obszar ten nie pokrywa się z analogicznym obszarem w klasie L_{km} oraz inne jest oszacowanie $|\arg f'(z)|$.

Niech B_k będzie podklasą klasy $B = B_{1,1}$ składającą się z funkcji k-symetrycznych. Okazuje się, że klasa B_k jest identyczna z klasą B_{kk}. Wynikają stąd dwa następujące wnioski.

Obszar zmienności $\log f'(z)$ w klasie B_k jest równy obszarowi zmienności $\log f'(z)$ w klasie B_{kk} i określony we wniosku 1 pracy oraz oszacowanie na $|f'(z)|$ i $|\arg f'(z)|$ są takie jak w klasie B_{kk} (wniosek 2).

Jeżeli przyjąć $k = m = 1$ to otrzymamy wyniki z pracy J. Krzyża [4].

Резюме

В работе [6] автор рассматривает подклассы L_{km} почти выпуклых функций, таких, что производная $f'(z) = \varphi'(z)\cdot p(z), f'(0) = 1$, где $\varphi(z) = a_1 z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \ldots, |a_1| = 1$, отображает круг K_1

на выпуклую область, а функция $p(z) = a_0 + a_m z^m + a_{2m} z^{2m} + \ldots$ выполняет условия:

$$|p(0)| = |a_0| = 1, \operatorname{re} p(z) > 0 \text{ для } |z| < 1.$$

Если усилить условия на φ и p, а именно положить $\varphi'(0) = a_1 = 1$ и $p(0) = a_0 = 1$, то получим подкласс B_{km}. Классы L_{km} и B_{km} разные между собой. Класс B_{km} существенно уже, чем класс L_{km}. Класс $B_{1,1}$ рассматривал в своей работе [1] Базилевич. Он определил его как решение определенного уравнения Левнера-Спенцера-Куфарёва.

В этой работе определяется область изменения $\log f'(z)$ для класса B_{km}, а также оценки на $|f'(z)|$ и $|\arg f'(z)|$. Эта область не совпадает с аналогичной областью в классе L_{km}, а также различна и оценка $|\arg f'(z)|$.

Пусть B_k будет подклассом класса $B = B_{1,1}$, состоящим из k-симметрических функций. Оказывается, что класс B_k тождественен классу B_{kk}. Отсюда вытекают два вывода.

Область изменения $\log f'(z)$ в классе B_k равна области изменения $\log f'(z)$ в классе B_{kk} и он определён в выводе 1 этой работы, причём оценка на $|f'(z)|$ и $|\arg f'(z)|$ та же самая как в классе B_{kk}.

Если примем $k = m = 1$, то получим результаты работы Й. Кжижа [4].