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On the Parametrization of Quasiconformal Mappings in an Annulus

O parametryzacji odwzorowań quasi-konforemnych w pierścieniu

О параметризации квазиконформных отображений в круговом кольце

Introduction

The idea of parametrization of conformal mappings of the unit disc was first realized by Löwner [24]. The corresponding results for conformal mappings of an annulus were obtained by Komatu [16] and Golusin [12]. With the appearance of the theory of quasiconformal mappings which was initiated in 1928-32 by Grötzsch [13], [14], [15] and Lavrentieff [20], and contained, in particular, the theory of conformal mappings, there arose the problem of parametrization for this very general class of mappings. In the case of the unit disc the problem has been solved by Shah Tao-shing [34], while in the case of an annulus by Shah Tao-shing and Fan Le-le [35] and by the author [25], with different assumptions restricting the class considered. The aim of the present paper is to parametrize quasiconformal mappings of an annulus in the general case (for a dense subclass of the class of all quasiconformal mappings of an annulus).

There exist several definitions of quasiconformal mappings and it is necessary to decide which one has to be used. The classical definition due to Grötzsch was completed in 1957 by Bers [8] who extended this notion on a considerably wider class of functions. As proved by Gehring and Lehto [11], the definition of Bers may be considerably simplified with maintenance of the same class of mappings. Apart from the analytic definition there exist some other ones. Ahlfors [2], Pfluger [32] and Mori [28] gave the geometric definition, while Lavrentieff [20], [21], [22], Pesin [31] and Bers [8] presented definitions using mappings of infinitesimal circles onto infinitesimal ellipses, or using solutions of Beltrami differential equation. The problem of equivalence of these definitions was considered among others by Bers [8] and Gehring [10]. In the present paper the following *equivalent* definitions will be used:

Definition 1 (Grötzsch, Bers and Gehring). A mapping $w = f(z)$ of a plane domain D onto Δ is said to be Q-quasiconformal ($1 \leqslant Q < \infty$), if (i) f is a sense-preserving homeomorphism in D, (ii) f is absolutely continuous in D on almost all horizontal and vertical lines, (iii) an essential estimate(¹)

$$|f_{\bar{z}}| \leqslant \{(Q-1)/(Q+1)\}|f_z| \tag{0.1}$$

takes place a.e. in D.

Definition 2 (Lavrentieff and Pesin). A mapping $w = f(z)$ of a plane domain D onto Δ is said be Q-quasiconformal ($1 \leqslant Q < \infty$), if (i) f is a sense-preserving homeomorphism in D, (ii) the expression

$$H(z_0) = \overline{\lim_{h\to 0}}\{\sup_{|z-z_0|=h} |f(z)-f(z_0)|/\inf_{|z-z_0|=h} |f(z)-f(z_0)|\} \tag{0.2}$$

is bounded for all $z_0 \in D$, (iii) the essential l.u.b. of H in D is equal to Q.

A mapping $w = f(z)$ is said to be Q-quasiconformal in a closed domain, if it is Q-quasiconformal in its interior and homeomorphic on its closure. Boundary points of a domain and the convergence to them is understood in the sense of the theory of prime ends due to Carathéodory.

In this place I should like to express my sincere gratitude to Professor P. P. Belinskii from Novosibirsk who gave me many advices and hints, and to Professors Z. Charzyński from Łódź and J. Krzyż from Lublin for their remarks which vastly improved my already accomplished paper. I also owe very much to discussions with Dr Krushkal from Novosibirsk, and I profited particularly from Krushkal's papers [17], [18] and [19]. Moreover, Professor Belinskii and Dr Krushkal kindly made accessible to me their yet unpublished results.

§ 1. Existential problems

Parametrization of quasiconformal mappings requires the usage of the theorem on existence and uniqueness of such mappings with given characteristics. For the case of simply connected domains and continuous

(¹) Here $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$ where $z = x+iy$. A function $w = F(z)$ is said to be essentially estimated a.e. in D by $MG(z)$, where M is a constant, if

$$\operatorname*{ess\,sup}_{z\in D} \{F(z)/G(z)\} = M.$$

The essential supremum is defined as

$$\inf_{E} \sup_{z\in D\setminus E} \{F(z)/G(z)\}$$

where the infimum is taken over all sets E with the plane measure equal to zero.

characteristics it was first formulated and proved by Lavrentieff [21], and then generalized by Morrey [29], Belinskii [5] and Belinskii and Pesin [7]:

Theorem 1 (Lavrentieff, Belinskii and Pesin). *Let D be an arbitrary simply connected closed domain, different from the whole plane, and let be given a.e. in D an arbitrary measurable and bounded pair of characteristics $p = p(z)$, $\theta = \theta(z)$, where the essential l.u.b. of p is equal to $Q(1 \leqslant Q < \infty)$. Then for any simply connected closed domain Δ, different from the whole plane, there exists a Q-quasiconformal mapping of D onto Δ, determined uniquely apart from conformal mappings of Δ onto itself and having the characteristics p, θ a.e. in D.*

An analogous theorem for multiply connected domains is an unpublished result of Belinskii which has been placed here together with the proof with his consent:

Theorem 2 (Belinskii). *Let D be an arbitrary closed domain of connectivity n and let be given a.e. in D an arbitrary measurable and bounded pair of characteristics $p = p(z)$, $\theta = \theta(z)$, where the essential l.u.b. of p is equal to Q $(1 \leqslant Q < \infty)$. Then there exists a circular canonical closed domain Δ of connectivity n and a Q-quasiconformal mapping of D onto Δ, determined uniquely apart from conformal mappings of the interior of Δ onto itself and having the characteristics p, θ a.e. in D.*

Proof. Let D^* be an arbitrary closed simply connected domain which is different from the whole plane and includes D. Let further $p = p^*(z)$, $\theta = \theta^*(z)$, where $p^*(z) \leqslant Q$ $(1 \leqslant Q < \infty)$ for $z \in D^*$, be measurable functions defined a.e. in D^* and such that $p^*(z) = p(z)$, $\theta^*(z) = \theta(z)$ a.e. in D. Let finally Δ^* be an arbitrary simply connected closed domain different from the whole plane.

In view of Theorem 1 there exists a quasiconformal mapping of D^* onto Δ^* with characteristics p^*, θ^* a.e. in D^*. This mapping transforms, in particular, the domain D onto a certain domain $\Delta^{**} \subset \Delta^*$ Q-quasiconformally. Applying now the theorem on mapping of multiply connected domains onto circular canonical domains, we see that Δ^{**} can be homeomorphically mapped onto a circular canonical closed domain Δ of connectivity n, and that this mapping is conformal inside of Δ^{**}. However, it is well-known (see e.g. [8]) that the Q-quasiconformal mapping with characteristics p, θ, internally or externally composed with a conformal mapping, gives a Q-quasiconformal mapping with characteristics p, θ. Thus, there exists a circular canonical closed domain Δ of connectivity n and a homeomorphical mapping of D onto Δ, Q-quasiconformal with characteristics p, θ in D. Consequently, this mapping is Q-quasiconformal in the whole domain D.

We have still to prove that the described mapping of D onto Δ is determined uniquely apart from a conformal mapping of Δ onto itself. Let us suppose that there exist two mappings $w = f(z)$ and $w = g(z)$ which fulfil the conditions of our theorem. Hence the mapping $\omega = F(w) = g(f^{-1}(w))$ transforms Δ onto itself as a sense-preserving homeomorphism in such a way that

$$\overline{\lim_{h\to 0}}\{\sup_{|w-w_0|=h} |F(w)-F(w_0)|/\inf_{|w-w|=h} |F(w)-F(w_0)|\} = 1 \tag{1.1}$$

for almost all $w_0 \in \Delta$.

In fact, in order to show (1.1) we verify that $U(w) = 0$ a.e. in Δ, where $F_{\bar{w}}(w) = U(w)F_w(w)$, and this implies (1.1). Let u and v be defined by the relations $f_{\bar{w}}^{-1}(w) = u(w)f_w^{-1}(w)$ and $g_{\bar{z}}(z) = v(z)g_z(z)$ a.e. in Δ and D, respectively. Hence $v(z) = -u(w)\exp(2i\arg f_{\bar{w}}^{-1}(w))$ a.e. in Δ, where $z = f^{-1}(w)$. Consequently,

$$F_w = g_z f_w^{-1} + g_{\bar{z}}\{\overline{f^{-1}}\}_w = g_z f_w^{-1} + g_{\bar{z}}\overline{f_{\bar{w}}^{-1}},$$

$$F_{\bar{w}} = g_z f_{\bar{w}}^{-1} + g_{\bar{z}}\{\overline{f^{-1}}\}_{\bar{w}} = g_z f_{\bar{w}}^{-1} + g_{\bar{z}}\overline{f_w^{-1}},$$

that is, by $F_{\bar{w}}(w) = U(w)F_w(w)$,

$$g_z\{f_{\bar{w}}^{-1} - \overline{f_{\bar{w}}^{-1}}\exp(2i\arg f_{\bar{w}}^{-1})\}u = g_z\{f_w^{-1} - u\bar{u}\overline{f_w^{-1}}\exp(2i\arg f_w^{-1})\}U,$$

whence $U(w) = 0$ a.e. in Δ.

Notice now (cf. [10], p. 13) that Definition 2 implies $H(z)$ to be less than ∞ everywhere in Δ, except perhaps at points of a set of Σ-finite linear measure. Now, applying Corollary 3 of Gehring's paper [10] (p. 15) which is a generalization of a known Menchoff's theorem [26] we see that the mapping $\omega = F(w)$ is conformal and this complets the proof.

An immediate consequence of Theorem 2 is the following

Corollary 1. Let be given a.e. in a nondegenerate annulus $r < |z| < 1$ an arbitrary measurable and bounded pair of characteristics $p = p(z)$, $\theta = \theta(z)$, where the essential l.u.b. of p is equal to Q $(1 \leqslant Q < \infty)$. Then there exists exactly one number R $(0 < R < 1)$ and a Q-quasiconformal mapping of the annulus $r \leqslant |z| \leqslant 1$ onto $R \leqslant |w| \leqslant 1$, determined uniquely apart from reciprocal and rotations around the origin and having the characteristics p, θ a.e. in the annulus $r < |z| < 1$.

§ 2. Remarks on parametrization in the unit disc

We shall use further theorems on parametrization of quasiconformal mappings of the unit disc onto itself, obtained by Shah Tao-shing [34], mostly in the sense of application of analogous methods. We quote here his results in a more precise form than that given in [34] and the most of them under slightly weakened assumptions.

We prove first

Lemma 1. Let f map quasiconformally a closed domain D, containing a circle $|z| = r$, onto Δ so that $(\partial/\partial\vartheta)|f(re^{i\vartheta})| = 0$ for almost all $\vartheta\,(-\pi < \vartheta \leqslant \pi)$. In order that the condition $(\partial/\partial\varrho)\arg f(\varrho e^{i\vartheta})|_{\varrho=r} = 0$ be fulfilled for almost all ϑ, it is necessary and sufficient that the characteristics p, θ, corresponding to f, be such that for almost all ϑ

$$\text{(2.1)} \qquad \text{either } p(re^{i\vartheta}) = 1, \text{ or } \theta(re^{i\vartheta}) = \vartheta \text{ or } \theta(re^{i\vartheta}) = \vartheta + \tfrac{1}{2}\pi.$$

Proof. It is known (see e.g. [8]) that the function f has generalized partial derivatives of the first order which satisfy a.e. the Beltrami equation (see e.g. [8])

$$\text{(2.2)} \qquad f_{\bar z} = \{(1-p)/(1+p)\}e^{2i\theta}f_z,$$

where (p, θ) denotes the pair of characteristics corresponding to f. Moreover, it is easy to verify that

$$\text{(2.3)} \qquad f_z = \tfrac{1}{2}\{|f|_\varrho + (1/\varrho)|f|(\arg f)_\vartheta + i\,|f|(\arg f)_\varrho - i(1/\varrho)\,|f|_\vartheta\}\times \\ \times \exp i(\arg f - \vartheta),$$

$$\text{(2.4)} \qquad f_{\bar z} = \tfrac{1}{2}\{|f|_\varrho - (1/\varrho)\,|f|(\arg f)_\vartheta + i\,|f|(\arg f)_\varrho + i(1/\varrho)\,|f|_\vartheta\}\times \\ \times \exp i(\arg f + \vartheta).$$

From (2.2), (2.3) and (2.4) the necessity as well as sufficiency of the condition (2.1) follows immediately.

Lemma 1 has an obvious geometrical sense.

Let now $\tilde U_Q$ denote the class of all functions f which map Q-quasiconformally the disc $|z| \leqslant 1$ onto itself with $f(0) = 0$ and $f(1) = 1$. Let further U_* denote the class of all measurable and bounded pairs of characteristics (p, θ) defined a.e. in the disc $|z| < 1$. Let in turn $(S)_*$ denote the subclass of U_* consisting of pairs of functions defined in the disc $|z| \leqslant 1$ and belonging to the class C^1, and by S_* the subclass of $(S)_*$ consisting of pairs of functions which have for $|z| \leqslant 1$ partial derivatives of the first order fulfilling a global Hölder condition with a certain exponent δ $(0 < \delta \leqslant 1)$. Let finally $(S)_Q$ and S_Q denote the subclasses of $U_Q = \bigcup_{Q^* < Q}\tilde U_{Q^*}$ consisting of all functions corresponding (by virtue of Theorem 1) to pairs of charac teristics that belong to the classes $(S)_*$ and S_*, respectively.

As it was remarked in [34] (cf. also Remark in the next paragraph of the presentpaper), there takes place the following

Lemma 2 (Shah Tao-shing). The subclasses S_Q and $(S)_Q$ are dense in the class U_Q.

In what follows we consider functions f and the corresponding pairs of characteristics (p, θ) depending on one real parameter t.

The basic part of Shah Tao-shing's paper [34] is the proof of the following integral lemma, analogous to Golusin's lemma [12], and presented here with weakened assumptions, in a slightly different form:

Lemma 3. If a pair of characteristics $(p, \theta) \epsilon (S)_*$, defined in the disc $|z| \leqslant 1$ and in an interval $0 < t \leqslant T$, fulfils in this disc the conditions

$$(1/t)u(z, t) \rightrightarrows \varphi(z) \quad \text{for} \quad t \to 0+, \tag{2.5}$$

$$(1/t)|u_z(z, t)| \leqslant k(z) \quad \text{for} \quad 0 < t \leqslant T, \tag{2.6}$$

where φ and k are bounded and $u = e^{2i\theta}(1-p)/(1+p)$, then for the function $f \epsilon (S)_Q$ which corresponds (by virtue of Theorem 1) to the pair of characteristics (p, θ), the formula

$$(1/t)[f(z, t)-z] \rightrightarrows (1/\pi)z(1-z) \iint\limits_{|\zeta|\leqslant 1} \{\varphi(\zeta)/\zeta(1-\zeta)(z-\zeta) + \tag{2.7}$$

$$+\overline{\varphi(\zeta)}/\bar{\zeta}(1-\bar{\zeta})(1-z\bar{\zeta})\}\, d\xi\, d\eta \quad \text{for } t \to 0+ \qquad (\zeta = \xi + i\eta)$$

is satisfied in the disc $|z| < 1$. (2)

Proof. This lemma differs from that given in [34] by the missing assumption that the condition (2.1) holds for all ϑ $(-\pi < \vartheta \leqslant \pi)$. Moreover, we prove (2.7) in the sense of the footnote (2) only, but this is not essential for further applications.

If (2.1) is fulfilled, the function θ, continued outside the circle $|z| = 1$ according to the formula $\theta(z, t) = \arg z^2 - \theta(1/\bar{z}, t)$, is of the class C^1 and, consequently, we see in view of Lemma 1 that the function f, continued outside the circle $|z| = 1$ by the formula $f(z, t) = 1/\overline{f(1/\bar{z}, t)}$, is of the class $(S)_Q$.

If (2.1) is not fulfilled, the proof runs quite analogously, but we obtain (2.7) in the sense of the footnote (2) only. (3)

From Lemma 3 the following theorem can be deduced (see Shah Tao-shing [34]):

Theorem 3. *If for a pair of characteristics $(p, \theta) \epsilon S_*$, defined in the disc $|z| \leqslant 1$ and in an interval $0 \leqslant t \leqslant T$, the function $u = e^{2i\theta}(1-p)/(1+p)$ has in the disc $|z| \leqslant 1$ and in the interval $0 \leqslant t \leqslant T$ partial derivatives u_t*

(2) We use the notation $\rightrightarrows$ for an open set D in the sense of the so-called almost uniform convergence in D (i. e. the uniform convergence on compact subsets of D) and the convergence of $\operatorname{Re}\{(1/tz)[f(z, t)-z]\}$ on its closure.

(3) It seems to the author that in (2.7) we have the uniform convergence for $|z| < 1$ also in this case. The problem requires a separate publication. An analogous remark concerns also Lemma 6.

and u_{zt}, then the function $f \in S_Q$, corresponding to the pair of characteristics (p, θ), satisfies in the disc $|z| < 1$ the equation

$$(2.8)\qquad \partial f/\partial t = (1/\pi) f(1-f) \iint_{|\zeta|\leqslant 1} \{\varphi(\zeta, t)/\zeta(1-\zeta)(f-\zeta) + \overline{\varphi(\zeta, t)/\bar\zeta(1-\bar\zeta)(1-f\bar\zeta)}\}\, d\xi\, d\eta \qquad (\zeta = \xi + i\eta),$$

where the function φ is defined by the formula

$$(2.9)\ \varphi(\zeta, t) = \Big\{1/\big(1-|u(f^{-1}(\zeta, t), t)|^2\big)\Big\} u_t\big(f^{-1}(\zeta, t), t\big)\exp\big(-2i\arg f_\zeta^{-1}(\zeta, t)\big).$$

If, in particular, $p(z, t) = [p(z)]^t$ and $\theta(z, t) = \theta(z)$, then

$$(2.10)\qquad \varphi(\zeta, t) = -\tfrac{1}{2}\log p\big(f^{-1}(\zeta, t)\big)\exp\big(2i\theta\big(f^{-1}(\zeta, t)\big) - 2i\arg f_\zeta^{-1}(\zeta, t)\big)$$

and the solution $w = f(z, t)$ of the equation

$$(2.11)\qquad \partial w/\partial t = (1/\pi) w(1-w) \iint_{|\zeta|\leqslant 1} \{\varphi(\zeta, t)/\zeta(1-\zeta)(w-\zeta) + \overline{\varphi(\zeta, t)/\bar\zeta(1-\bar\zeta)(1-w\bar\zeta)}\}\, d\xi\, d\eta \qquad (\zeta = \xi + i\eta)$$

satisfies the initial condition $f(z, 0) = z$.

The proof is the same as in [34], but the lemma applied there must be replaced by Lemma 3 from the present paper.

An application of the above formulated theorem, instead of the corresponding one from [34], permits to avoid the condition (2.1) which has been tacitly assumed by Shah Tao-shing in his Theorems 3 and 5 from the same paper. Thus the proofs of the mentioned theorems may be already taken as complete.

Let us finally mention the results of Krushkal [19]. Basing on Lemma 19 of the paper of Ahlfors and Bers [3] he proved that the assertion of Lemma 3 takes place also under the following assumptions: 1° φ is measurable and bounded in the disc $|z| < 1$, 2° $(1/t)(u - t\varphi)$ is measurable and bounded on compact subsets of the disc $|z| < 1$ by a constant common for all t $(0 < t \leqslant T)$, 3° if $t \to 0+$, then $(1/t)u(z, t) \to \varphi(z)$ for almost all z $(|z| < 1)$. He also showed that under these assumptions not only the uniform convergence in (2.7) takes place, but even there exists such a number $p^* > 2$ that for any p from the interval $1 < p < p^*$ there takes place the convergence with respect to the norm in $B_p(|z| \leqslant 1)$, where $B_p(|z| \leqslant 1)$ denotes the Banach space of functions f which are defined in the disc $|z| \leqslant 1$ with the norm

$$(2.12)\qquad \|f(z)\|_{B_p(|z|\leqslant 1)} = \sup_{|z_1|, |z_2| \leqslant 1} \{|f(z_1) - f(z_2)|/|z_1 - z_2|^{1-2/p}\} + \|f_z(z)\|_{L_p(|z|<1)} + \|f_{\bar z}(z)\|_{L_p(|z|<1)}$$

and fulfil the conditions $f(0) = 0$ and $f(1) = 1$. Using this lemma Krushkal proved further in [19] that the assertion of Theorem 3 takes place also if $u(z, t) = tu(z)$ for $|z| \leqslant 1$ and $0 \leqslant t \leqslant T$, where u is measurable and bounded by a constant $k < 1$. Krushkal did not obtain theorems which correspond to Theorems 3 and 4 of Shah Tao-shing's paper [34].

§ 3. Dense subclasses of quasiconformal mappings in an annulus

Let $\tilde{U}_Q^{r,R}$ denote the class of all functions f which map Q-quasiconformally an annulus $r \leqslant |z| \leqslant 1$ onto $R \leqslant |w| \leqslant 1$ with $f(1) = 1$ and let $U_Q^r = \bigcup_{R,Q^* \leqslant Q} \tilde{U}_{Q^*}^{r,R}$. Let further U_*^r denote the class of all measurable and bounded pairs of characteristics (p, θ) defined a.e. in the annulus $r < |z| < 1$. Let in turn $(S)_*^r$ denote the subclass of U_*^r consisting of pairs of functions defined in the annulus $r \leqslant |z| \leqslant 1$ and belonging to the class C^1, and by S_*^r the subclass of $(S)_*^r$ consisting of pairs of functions which have for $r \leqslant |z| \leqslant 1$ partial derivatives of the first order fulfilling a global Hölder condition with a certain exponent δ $(0 < \delta \leqslant 1)$. Let further $(S)_Q^r$ and S_Q^r denote the subclasses of U_Q^r consisting of all functions corresponding (by virtue of Corollary 1) to pairs of characteristics that belong to the classes $(S)_*^r$ and S_*^r, respectively. Let finally $(S)_Q^{r,R}$ and $S_Q^{r,R}$ denote respectively the subclasses of the classes $(S)_Q^r$ and S_Q^r which consist of all functions mapping the annulus $r \leqslant |z| \leqslant 1$ onto $R \leqslant |w| \leqslant 1$. Obviously, $(S)_Q^r = \bigcup_R (S)_Q^{r,R}$ and $S_Q^r = \bigcup_R S_Q^{r,R}$.

We prove first

Lemma 4. If $f \in U_Q^{r,R}$ then

$$(1/16)^Q |z_1 - z_2|^Q \leqslant |f(z_1) - f(z_2)| \leqslant 16\, |z_1 - z_2|^{1/Q} \tag{3.1}$$

and

$$|f(z) - z| < 18 \log Q \tag{3.2}$$

in the whole annulus $r \leqslant |z| \leqslant 1$.

Proof. Let us continue the function f into the inner disc by the formulae

$$f^*(z) = R^{2\nu} / \overline{f(r^{2\nu}/z)} \quad \text{for} \quad r^{2\nu} \leqslant |z| \leqslant r^{2\nu-1} \; (\nu = 1, 2, \ldots), \tag{3.3}$$

$$f^*(z) = R^{2\nu} f(z/r^{2\nu}) \quad \text{for} \quad r^{2\nu+1} \leqslant |z| \leqslant r^{2\nu} \; (\nu = 1, 2, \ldots). \tag{3.4}$$

Obviously, we admit $f^*(z) = f(z)$ for $r \leqslant |z| \leqslant 1$, and $f^*(0) = 0$. It is easy to see that f^* is Q-quasiconformal in the disc $|z| \leqslant 1$. Hence f^* satisfies here the estimates $(1/16)^Q |z_1 - z_2|^Q \leqslant |f^*(z_1) - f^*(z_2)| \leqslant 16\, |z_1 - z_2|^{1/Q}$ and $|f^*(z) - z| < 18 \log Q$ obtained by Mori [27] and Belinskii [6], respectively, and this implies the assertion of our lemma.

We prove now

Lemma 5. A subclass of the class U_Q^r consisting of all real-analytic functions is dense in U_Q^r; in particular the subclasses S_Q^r and $(S)_Q^r$ are dense in U_Q^r.

Proof. The method of our proof is analogous to that used by Bers in [8].

Let us first note that, in view of Lemma 4, the function f satisfies (3.1) in the whole considered annulus. On the other hand, f satisfies here a.e. the Beltrami differential equation (see e.g. [8])

$$f_{\bar z} = u(z) f_z \tag{3.5}$$

where $|u(z)|$ is sharply estimated by $(Q-1)/(Q+1)$. If we define f^* as in the proof of Lemma 4, then the corresponding function u^* will be determined by the formulae

$$u^*(z) = e^{4i\arg z}\,\overline{u(r^{2\nu}/\bar z)} \quad \text{for} \quad r^{2\nu} \leqslant |z| < r^{2\nu-1} \ (\nu = 1, 2, \ldots), \tag{3.6}$$

$$u^*(z) = u(z/r^{2\nu}) \quad \text{for} \quad r^{2\nu+1} \leqslant |z| < r^{2\nu} \ (\nu = 1, 2, \ldots), \tag{3.7}$$

and, obviously, $u^*(z) = u(z)$ for $r \leqslant |z| \leqslant 1$.

Now, let $u^{(n)}$ be a sequence of complex valued real-analytic functions such that $|u^{(n)}(z)| \leqslant (Q-1)/(Q+1)$ and $u^{(n)} \to u^*$ a.e. in the disc $|z| \leqslant 1$. Let further

$$\sup_{r\leqslant|z|\leqslant 1} |u^{(n)}(z)| = (Q_n - 1)/(Q_n + 1) \quad (n = 1, 2, \ldots).$$

Obviously, $Q_n \to Q$ as $n \to \infty$. According to the theorem on existence and uniqueness of systems of partial differential equations, any equation

$$\omega_{\bar z} = u^{(n)}(z)\,\omega_z \quad (n = 1, 2, \ldots) \tag{3.8}$$

has exactly one solution $\omega = f^{*(n)}(z)$ which is a real-analytic sense-preserving homeomorphism of the disc $|z| \leqslant 1$ onto itself and fulfils the initial conditions $f^{*(n)}(0) = 0$ and $f^{*(n)}(1) = 1$.

Applying now Corollary 1 we see that there exists a uniquely determined sequence of numbers R_n $(0 < R_n < 1)$ and a uniquely determined sequence of Q_n-quasiconformal mappings $w = f^{(n)}(z)$ of the annulus $r \leqslant |z| \leqslant 1$ onto $R_n \leqslant |w| \leqslant 1$ which have in $r \leqslant |z| \leqslant 1$ the characteristics p_n, θ_n, respectively, given by the relations $\exp(2i\theta_n)\cdot(1-p_n)/(1+p_n) = u_n$ (cf. (2.2) and (3.5)) while $f^{(n)}(1) = 1$. Moreover, Corollary 1 implies also the existence of a uniquely determined sequence of conformal mappings $w = g_n(\omega)$ of the domains $f^{*(n)}(\{z\colon\ r \leqslant |z| \leqslant 1\})$ onto $R_n \leqslant |w| \leqslant 1$, respectively, such that for any n the relation $f^{(n)}(z) = g_n\big(f^{*(n)}(z)\big)$ holds for $r \leqslant |z| \leqslant 1$.

Thus we see that the functions $f^{(n)}$ are real-analytic in the annulus $r \leqslant |z| \leqslant 1$. On the other hand, in view of Lemma 4, we have here

$$(3.9) \qquad (1/16)^{Q_n}|z_1 - z_2|^{Q_n} \leqslant |f^{(n)}(z_1) - f^{(n)}(z_2)| \leqslant 16\,|z_1 - z_2|^{1/Q_n}.$$

Hence we may assume, selecting if need be a subsequence, that the sequence of function $f^{(n)}$ converges uniformly to a sense-preserving homoemorphism $f^{(\infty)}$ which maps the annulus $r \leqslant |z| \leqslant 1$ onto $R \leqslant |w| \leqslant 1$ where R is the limit of R_n for $n \to \infty$.

It remains to prove that $f^{(\infty)}$ is identically equal to f in the annulus $r \leqslant |z| \leqslant 1$. To this end we note first that, by virtue of the condition (iii) of quasiconformality in Definition 1, we have the estimate $|f_{\bar{z}}^{(n)}(z)| \leqslant \{(Q_n - 1)/(Q_n + 1)\}\,|f_z^{(n)}(z)|$. Let $z = x + iy$ and $f^{(n)} = \varphi^{(n)} + i\psi^{(n)}$. Then, squaring both sides of the last inequality, we obtain

$$\varphi_x^{(n)2} + \varphi_y^{(n)2} + \psi_x^{(n)2} + \psi_y^{(n)2} - (\varphi_x^{(n)}\psi_y^{(n)} - \varphi_y^{(n)}\psi_x^{(n)})$$
$$\leqslant [(Q_n - 1)/(Q_n + 1)]^2 [\varphi_x^{(n)2} + \varphi_y^{(n)2} + \psi_x^{(n)2} + \psi_y^{(n)2} - (\varphi_x^{(n)}\psi_y^{(n)} - \varphi_y^{(n)}\psi_x^{(n)})],$$

that is $\varphi_x^{(n)2} + \varphi_y^{(n)2} + \psi_x^{(n)2} + \psi_y^{(n)2} \leqslant (Q_n + 1/Q_n)(\varphi_x^{(n)}\psi_y^{(n)} - \varphi_y^{(n)}\psi_x^{(n)})$, and consequently, after integration,

$$(3.10) \qquad \iint\limits_{r \leqslant |z| \leqslant 1} \{|f_x^{(n)}|^2 + |f_y^{(n)}|^2\}\,dx\,dy \leqslant \pi(Q_n + 1/Q_n).$$

From (3.10) we infer that the sequences of functions $f_x^{(n)}$ and $f_y^{(n)}$ are, after selecting if need be subsequences, weakly convergent to certain functions g and h, measurable and locally integrable with the square in the annulus $r \leqslant |z| \leqslant 1$, and at the same time $g = f_x^{(\infty)}$ and $h = f_y^{(\infty)}$ almost everywhere. Hence we infer that the sequence of functions $u^{(n)}f_z^{(n)}$ is weakly convergent in $r \leqslant |z| \leqslant 1$ to $uf_z^{(\infty)}$. Thus $w = f^{(\infty)}(z)$ is one of solutions of the differential equation

$$(3.11) \qquad w_{\bar{z}} = u(z)\,w_z$$

determined for $r \leqslant |z| \leqslant 1$.

But, as proved by Bers [8], if $w = f_1(z)$ and $w = f_2(z)$ are two solutions of (3.11) in the same domain, and f_1 is a homeomorphism, then f_2 is a holomorphic function of f_1, and thus $f^{(\infty)}(z) = G\big(f(z)\big)$ where G is holomorphic in $r < |z| < 1$. Since f as well as $f^{(\infty)}$ map the annulus $r \leqslant |z| \leqslant 1$ onto $R \leqslant |w| \leqslant 1$ and $f(1) = f^{(\infty)}(1) = 1$, then G must be the identity function in the annulus $r < |z| < 1$, and thus also on its closure.

In this way we have proved that a subclass of U_Q^r consisting of all real-analytic functions is dense in U_Q^r. This implies in particular that the subclasses S_Q^r and $(S)_Q^r$ are also dense in U_Q^r.

Remark. Similarly as in the previous proof it is easy to verify that a subclass of the class U_Q consisting of all real-analytic functions is dense in U_Q and this is a generalization of Lemma 2.

In the following paragraphs we shall consider functions $f \in U_Q^{r,R}$ and the corresponding pairs of characteristics (p, θ) as functions of a complex variable z and a real variable t. In these considerations r will be fixed, while R and Q will be functions of the variable t.

§ 4. Integral Lemma for an annulus

The following lemma on integral representation of quasiconformal mappings of the class $(S)_Q^{r,R}$, depending on one real parameter, has a basic importance for the problem of parametrization of quasiconformal mappings in an annulus. It is in fact a considerable generalization of the corresponding results of [35] and [25] because the uniform convergence in $r \leqslant |z| \leqslant 1$ instead of the convergence in the sense of the footnote (²) is not essential for further applications.

Lemma 6. If a pair of characteristics $(p, \theta) \in (S)_*^r$, defined in an annulus $r \leqslant |z| \leqslant 1$ and in an interval $0 < t \leqslant T$, fulfils in this annulus the conditions (2.5) and (2.6) where φ and k are bounded and $u = e^{2i\theta} \times \times (1-p)/(1+p)$, then for the function $f \in (S)_Q^{r,R}$ which corresponds (by virtue of Corollary 1) to the pair of characteristics (p, θ), the formula

$$(4.1)\qquad (1/t)[f(z,t)-z] \rightrightarrows (\tfrac{1}{2}\pi)\iint_{r\leqslant|\zeta|\leqslant 1} z\sum_{\nu=-\infty}^{+\infty}\left\{\frac{\varphi(\zeta)}{\zeta^2}\left(\frac{z+r^{2\nu}\zeta}{z-r^{2\nu}\zeta}-\frac{1+r^{2\nu}\zeta}{1-r^{2\nu}\zeta}\right)-\right.$$

$$\left.-\frac{\overline{\varphi(\zeta)}}{\bar\zeta^2}\left(\frac{1+r^{2\nu}z\bar\zeta}{1-r^{2\nu}z\bar\zeta}-\frac{1+r^{2\nu}\bar\zeta}{1-r^{2\nu}\bar\zeta}\right)\right\}d\xi\, d\eta \qquad \text{for}\quad t\to 0+ \quad (\zeta = \xi + i\eta)$$

is satisfied in the whole annulus $r < |z| < 1$. (²), (⁴) Moreover,

$$(4.2)\qquad (1/t)[R(t)-r] \to (1/2\pi)\iint_{r\leqslant|\zeta|\leqslant 1} r\{\varphi(\zeta)/\zeta^2+\overline{\varphi(\zeta)}/\bar\zeta^2\}\,d\xi\, d\eta \quad \text{for } t\to 0+.$$

Proof. The method of our proof is analogous to that used by Shah Tao-shing in [34] and Shah Tao-shing and Fan Le-le in [35]. For more clearness the proof is divided into several steps.

Step A. *Reduction to a Dirichlet problem.* Let us put

$$f(z,t)-z = \beta(z,t)+J(z,t)+J_0(z,t) \qquad (r\leqslant|z|\leqslant 1,\ 0<t\leqslant T),$$

(⁴) In the sequel we apply for the sake of simplicity the notation $\sum_{\nu=-\infty}^{+\infty} a_\nu$ instead of $a_0+\sum_{\nu=1}^{+\infty}(a_\nu+a_{-\nu})$ provided the last series converges.

where

$$J(z,t) = (1/\pi) \iint_{r<|\zeta|<1} (z-\zeta)^{-1} u(\zeta,t)\,d\xi d\eta,$$

$$J_0(z,t) = (1/\pi) \iint_{r<|\zeta|<1} (z-\zeta)^{-1} u(\zeta,t)(\partial/\partial\zeta)[f(\zeta,t)-\zeta]\,d\xi d\eta.$$

Applying Lemma 4 we prove, analogously as in [34], that $(1/t)J_0(z,t) \rightrightarrows 0$ and $(1/t)[f(z,t)-z] \rightrightarrows \beta(z)+J(z)$ for $t\to 0+$ in the annulus $r<|z|<1$, where β is holomorphic in the above annulus and continuous on its closure, and $J(z) = (1/\pi) \iint_{r<|\zeta|<1} (z-\zeta)^{-1}\varphi(\zeta)\,d\xi d\eta$. Hence, putting(5)

(4.3) $$J^*(z) = (1/\pi) \iint_{r<|\zeta|<1} \left\{ \frac{(1-z)\varphi(\zeta)}{\zeta(1-\zeta)(z-\zeta)} + \frac{(1-z)\overline{\varphi(\zeta)}}{\bar\zeta(1-\bar\zeta)(1-z\bar\zeta)} \right\} d\xi d\eta$$

we obtain

(4.4) $$(1/t)[f(z,t)-z] \rightrightarrows z\beta^*(z)+zJ^*(z) \quad \text{for } t\to 0+ \quad (r<|z|<1),$$

where, as it is easily seen, β^* is holomorphic in the annulus $r<|z|<1$ and continuous on its closure.

Since $|f(z,t)| = 1$ on the circle $|z| = 1$, we have

$$2\,\mathrm{Re}\{(1/tz)[f(z,t)-z]\} = (1/tz)[f(z,t)-z]^2/f(z,t) \quad (|z|=1).$$

Hence, in view of (4.4),

(4.5) $$\mathrm{Re}\{(1/tz)[f(z,t)-z]\} \to 0 \quad \text{for } t\to 0+ \ (|z|=1).$$

Similarly, we have $|f(z,t)| = R(t)$ on the circle $|z| = r$. Hence, in view of the identity

$$\mathrm{Re}\{(1/tz)[f(z,t)-z]\} = [R(t)/r]\,\mathrm{Re}\{(1/tzr^{-1})[f(rzr^{-1},t)/R(t)-zr^{-1}]\} + \\ +(1/tr)[R(t)-r] \quad (|z|=r)$$

and (4.5), we obtain

(4.6) $$\mathrm{Re}\{(1/tz)[f(z,t)-z]\} \to \varrho \quad \text{for } t\to 0+ \ (|z|=r),$$

where

(4.7) $$\varrho = \lim_{t\to 0+}\{(1/tr)[R(t)-r]\}.$$

From (4.4), (4.5) and (4.6) we get

(4.8) $$\begin{aligned} \mathrm{Re}\,\beta^*(z) &= -\mathrm{Re}\,J^*(z) \quad (|z|=1),\\ \mathrm{Re}\,\beta^*(z) &= \varrho - \mathrm{Re}\,J^*(z) \quad (|z|=r). \end{aligned}$$

---

(5) The function J^* is convenient for further calculations because $\mathrm{Re}\,J^*(z) = 0$ for $|z| = 1$.

In consequence, we see, by virtue of a well-known theorem, that β^*, being holomorphic in the annulus $r < |z| < 1$ and continuous on its closure, can be expressed in it by the values of $\operatorname{Re} J^*(z)$ on the circles $|z| = 1$ and $|z| = r$. Thus we have reduced our problem to a certain boundary problem equivalent in fact to the Dirichlet problem for an annulus. Further calculations are analogous to that of the author's paper [25].

Step B. *Solution of the boundary problem and transformations of the integrals.* The solution of the boundary problem formulated above is given by Villat's formula (see e.g. [1], p. 226). This formula will be written in a form more convenient for our purposes and used in [25]. Put

$$c = (1/2\pi i) \int_{|z'|=1} (1/z')\beta^*(z')\,dz', \tag{4.9}$$

$$F_1(z) = (1/2\pi i) \int_{|z'|=1} (z'-z)^{-1} 2\operatorname{Re}\beta^*(z')\,dz', \tag{4.10}$$

$$F_2(z) = (1/2\pi i) \int_{|z'|=r} (z'-z)^{-1} 2\operatorname{Re}\beta^*(z')\,dz'. \tag{4.11}$$

Then, for $r < |z| < 1$,

$$\beta^*(z) = c - 2\operatorname{Re} c + \overline{F_1(1/\bar{z})} + \sum_{\nu=0}^{+\infty} \{F_1(r^{2\nu} z) - \overline{F_1(r^{2\nu}/\bar{z})} - F_2(z/r^{2\nu}) + \overline{F_2(1/r^{2\nu}\bar{z})}\}. \tag{4.12}$$

Note now that (4.3) yields

$$\overline{J^*(1/\bar{z})} = -J^*(z) \quad (z \neq 0) \text{ and } \operatorname{Re} J^*(z) = 0 \quad (|z| = 1). \tag{4.13}$$

From (4.9), (4.10) and (4.13) we obtain

$$\operatorname{Re} c = 0 \quad \text{and} \quad F_1(z) = 0 \quad (|z| = 1). \tag{4.14}$$

We admit for the elegance of further calculations

$$c = c^* + (1/\pi) \iint_{r<|\zeta|<1} \{\varphi(\zeta)/\zeta(1-\zeta) - \overline{\varphi(\zeta)}/\bar{\zeta}(1-\bar{\zeta})\}\,d\xi\,d\eta; \tag{4.15}$$

it is easily seen that the above integral exists.

In turn we transform the integral (4.11). First, in view of (4.8), it is possible to write this integral in the form

$$F_2(z) = (1/2\pi i) \int_{|z'|=r} (z-z')^{-1} [J^*(z') + \overline{J^*(z')} - 2\varrho]\,dz'$$

whence, in view of (4.13), we have

$$F_2(z) = (1/2\pi i) \int_{|z'|=r} (z-z')^{-1} [J^*(z') - J^*(z'/r^2)]\,dz'. \tag{4.16}$$

For the further transformation of (4.16) we consider the difference $\Delta(z,z') = J^*(z') - J^*(z)$. By virtue of (4.3) we have

$$\Delta(z,z') = (1/\pi) \iint\limits_{r\leqslant|\zeta|\leqslant 1} \left\{ \frac{(z-z')\varphi(\zeta)}{\zeta(z-\zeta)(z'-\zeta)} + \frac{(z-z')\overline{\varphi(\zeta)}}{\bar\zeta(1-z\bar\zeta)(1-z'\bar\zeta)} \right\} d\xi d\eta,$$

and thus we may replace (4.16) by an equivalent formula

$$(4.17)\quad F_2(z) = (1/2\pi i) \int\limits_{|z'|=r} \Big\{ (z-z')^{-1}[J^*(z) - J^*(z/r^2)] + \\ +(1/\pi) \iint\limits_{r\leqslant|\zeta|\leqslant 1} \left[\frac{\varphi(\zeta)}{\zeta(z-\zeta)(z'-\zeta)} - \frac{r^2\varphi(\zeta)}{\zeta(z-r^2\zeta)(z'-r^2\zeta)} + \right. \\ \left. + \frac{\overline{\varphi(\zeta)}}{\bar\zeta(1-z\bar\zeta)(1-z'\bar\zeta)} - \frac{r^2\overline{\varphi(\zeta)}}{\bar\zeta(r^2-z\bar\zeta)(r^2-z'\bar\zeta)} \right] d\xi d\eta \Big\} dz' \quad (r<|z|<1).$$

Step C. *Integration under the sign of integration by the method of shifting the contour beyond the singular points.* The present phase of transformation of the integral $F_2(z)$ we begin with an analysis of the different terms in the formula (4.17). Let for this end

$$g_0(z') = (z-z')^{-1}[J^*(z) - J^*(z/r^2)],$$

$$g_1(z') = (1/\pi) \iint\limits_{r\leqslant|\zeta|\leqslant 1} [\varphi(\zeta)/\zeta(z-\zeta)(z'-\zeta) - \overline{\varphi(\zeta)}/\bar\zeta(1-z\bar\zeta)(1-z'\bar\zeta)] d\xi d\eta,$$

$$g_2(z') = (1/\pi) \iint\limits_{r\leqslant|\zeta|\leqslant 1} [r^2\varphi(\zeta)/\zeta(z-r^2\zeta)(z'-r^2\zeta) - \\ - r^2\overline{\varphi(\zeta)}/\bar\zeta(r^2-z\bar\zeta)(r^2-z'\bar\zeta)] d\xi d\eta.$$

It is seen at once that g_0 is holomorphic in the whole disc $|z'|<|z|$, and so in particular for $|z'|\leqslant r$. Similarly (see e.g. [37], p. 45), g_1 is holomorphic in the disc $|z'|<r$ and continuous on its closure. Thus, applying Cauchy's integral theorem, we obtain

$$(4.18)\qquad \int\limits_{|z'|=r} g_0(z')dz' = 0, \qquad \int\limits_{|z'|=r} g_1(z')dz' = 0.$$

A similar consideration is impossible in the case of g_2. One can verify that the integral of this function over the circle $|z'|=r$ cannot be calculated even by application of the theorem on residues. Therefore an idea arises to integrate along the circle $|z'|=r$ under the sign of double integration.

In order to accomplish this idea we note that a known theorem on inversion of a repeated integral (see e.g. [9], vol. II, p. 753) requires assumptions that the integrand is (i) integrable in both variables separately, and (ii) bounded in the Cartesian product of both integration sets. It is evident that in our case (ii) is not satisfied.

The way out of this difficulty can be achieved by a method that may be called *the method of shifting the contour beyond the singular points*. The idea of this method was suggested to the author by a paper of Vekua [36] who applied it on p. 223 in proving a theorem connected with differential equations of the elliptic type. It seems to be very useful in the theory of multiple integrals in general, and in the theory of quasiconformal mappings, connected with double integrals by a well-known integral formula of Nevanlinna [30], in particular.

The point of the idea of this method is that we replace the considered curvilinear integral by a curvilinear integral over another contour in such a way that the Cartesian product which appears in the formulation of the above quoted theorem on inversion should satisfy (ii). The choice of a new curve of integration depends on the particular properties of the considered integral.

We shall prove that in our case we have

$$\int_{|z'|=r} g_2(z')dz' = \int_{|z'|=1} g_2(z')dz'. \tag{4.19}$$

In view of Cauchy's integral theorem it is sufficient for this end to show that g_2 is holomorphic in the annulus $r < |z'| < 1$ and continuous on its closure. The corresponding reasoning runs quite analogously to the case of the function g_1 in a disc.

Note that it is now possible to apply the above quoted theorem on inversion, because for $|z'| = 1$ and $r \leqslant |\zeta| \leqslant 1$ with fixed z $(r < |z| < 1)$ we have

$$|r^2\varphi(\zeta)/\zeta(z-r^2\zeta)(z'-r^2\zeta)| \leqslant K,\ |r^2\overline{\varphi(\zeta)}/\bar{\zeta}(r^2-z\bar{\zeta})(r^2-z'\bar{\zeta})| \leqslant K\ (K < +\infty).$$

Hence

$$(1/2\pi i)\int_{|z'|=r} g_2(z')dz'$$
$$=(1/\pi)\iint_{r\leqslant|\zeta|\leqslant 1}(1/2\pi i)\int_{|z'|=1}\left\{\frac{r^2\varphi(\zeta)}{\zeta(z-r^2\zeta)(z'-r^2\zeta)}+\frac{r^2\overline{\varphi(\zeta)}}{\bar{\zeta}(r^2-z\bar{\zeta})(r^2-z'\bar{\zeta})}\right\}dz'd\xi d\eta,$$

whence, after application of Cauchy's integral formula,

$$(1/2\pi i)\int_{|z'|=r} g_2(z')dz' = (1/\pi)\iint_{r\leqslant|\zeta|\leqslant 1}\left\{\frac{r^2\varphi(\zeta)}{\zeta^2(z-r^2\zeta)}-\frac{r^2\overline{\varphi(\zeta)}}{\bar{\zeta}(r^2-z\bar{\zeta})}\right\}d\xi d\eta. \tag{4.20}$$

The obtained formulae (4.18), (4.19) and (4.20) permit, in view of the definitions of g_0, g_1 and g_2, to write (4.17) in the form

$$F_2(z) = -(1/\pi)\iint_{r\leqslant|\zeta|\leqslant 1}\left\{\frac{r^2\varphi(\zeta)}{\zeta(z-r^2\zeta)}-\frac{r^2\overline{\varphi(\zeta)}}{\bar{\zeta}^2(r^2-z\bar{\zeta})}\right\}d\xi d\eta\quad (r \leqslant |z| \leqslant 1). \tag{4.21}$$

Step D. *Transformations of the series in the formula for β^*.* The considerations of previous steps of the proof enabled us to obtain the formulae (4.14), (4.15) and (4.21), from which it easily follows that the formula (4.20) may be replaced by the formula

$$\beta^*(z) = c^* + (1/\pi) \iint\limits_{r \leqslant |\zeta| \leqslant 1} \left\{ \frac{\varphi(\zeta)}{\zeta(1-\zeta)} - \frac{\overline{\varphi(\zeta)}}{\bar\zeta(1-\bar\zeta)} \right\} d\xi d\eta + \sum_{\nu=0}^{+\infty} (1/\pi) \iint\limits_{r \leqslant |\zeta| \leqslant 1} \times$$

$$\times \left\{ \frac{r^2\varphi(\zeta)}{\zeta(z/r^{2\nu} - r^2\zeta)} - \frac{r^2\overline{\varphi(\zeta)}}{\bar\zeta^2(r^2 - z\bar\zeta/r^{2\nu})} - \frac{r^2\overline{\varphi(\zeta)}}{\bar\zeta(1/r^{2\nu}z - r^2\bar\zeta)} - \frac{r^2\varphi(\zeta)}{\zeta^2(r^2 - \zeta/r^{2\nu}z)} \right\} d\xi d\eta$$

$$(r < |z| < 1)$$

that is

$$\beta^*(z) = c^* + (1/\pi) \iint\limits_{r \leqslant |\zeta| \leqslant 1} \left\{ \frac{\varphi(\zeta)}{\zeta(1-\zeta)} - \frac{\overline{\varphi(\zeta)}}{\bar\zeta(1-\bar\zeta)} \right\} d\xi d\eta + \tag{4.22}$$

$$+ \sum_{\nu=0}^{+\infty} (1/\pi) \iint\limits_{r \leqslant |\zeta| \leqslant 1} \left\{ \frac{\varphi(\zeta)}{\zeta^2} \left(\frac{1}{1-\zeta/r^{2\nu}z} - \frac{1}{1-z/r^{2\nu}\zeta} \right) - \right.$$

$$\left. - \frac{\overline{\varphi(\zeta)}}{\bar\zeta^2} \left(\frac{1}{1-z\bar\zeta/r^{2\nu}} - \frac{1}{1-1/r^{2\nu}z\bar\zeta} \right) \right\} d\xi d\eta \quad (r < |z| < 1).$$

We shall prove now that the function β^* may be, with preserved continuity, defined by (4.22) also on the circles $|z| = 1$ and $|z| = r$, and that in this formula it is possible to change the order of integration and summation for every z such that $r \leqslant |z| \leqslant 1$. Let us note for this purpose that all integrals, appearing in this formula, exist and are continuous with respect to z in the whole annulus $r \leqslant |z| \leqslant 1$ (cf. e.g. [37], p. 45). Next, we prove that the series of the integrands in (4.22) is uniformly convergent in the whole Cartesian product of the annuli $r \leqslant |\zeta| \leqslant 1$ and $r \leqslant |z| \leqslant 1$.

To prove the uniform convergence we apply the well-known Weierstrass' test. Since $r \leqslant |\zeta| \leqslant 1$, $r \leqslant |z| \leqslant 1$ and $0 < r < 1$, we have

$$|1 - \zeta/r^{2\nu}z| \geqslant |\zeta/r^{2\nu}z| - 1 \geqslant r^{1-2\nu} - 1 > r^{1-2\nu}(1-r),$$

$$|1 - z/r^{2\nu}\zeta| \geqslant |z/r^{2\nu}\zeta| - 1 \geqslant r^{1-2\nu} - 1 > r^{1-2\nu}(1-r),$$

$$|1 - z\bar\zeta/r^{2\nu}| \geqslant |z\bar\zeta/r^{2\nu}| - 1 \geqslant r^{2-2\nu} - 1 > r^{2-2\nu}(1-r),$$

$$|1 - 1/r^{2\nu}z\bar\zeta| \geqslant |1/r^{2\nu}z\bar\zeta| - 1 \geqslant r^{-2\nu} - 1 > r^{-2\nu}(1-r),$$

and, consequently,

$$|1/(1-\zeta/r^{2\nu}z) - 1/(1-z/r^{2\nu}\zeta)| \leqslant 1/|1-\zeta/r^{2\nu}z| + 1/|1-z/r^{2\nu}\zeta| < 2r^{2\nu-1}/(1-r),$$

$$|1/(1-z\bar\zeta/r^{2\nu}) - 1/(1-1/r^{2\nu}z\bar\zeta)| \leqslant 1/|1-z\bar\zeta/r^{2\nu}| + 1/|1-1/r^{2\nu}z\bar\zeta| < 2r^{2\nu-2}/(1-r).$$

Hence

$$|(1/\zeta^2)\varphi(\zeta)\{1/(1-\zeta/r^{2\nu}z)-1/(1-z/r^{2\nu}\zeta)\}-(1/\bar{\zeta}^2)\overline{\varphi(\zeta)}\{1/(1-z\bar{\zeta}/r^{2\nu})-$$
$$-1/(1-1/r^{2\nu}z\bar{\zeta})\}| \leqslant |(1/\zeta^2)\varphi(\zeta)|\{|1/(1-\zeta/r^{2\nu}z)-1/(1-z/r^{2\nu}\zeta)|+$$
$$+|1/(1-z\bar{\zeta}/r^{2\nu})-1/(1-1/r^{2\nu}z\bar{\zeta})|\} < 4r^{2\nu-4}\max_{r\leqslant|\zeta|\leqslant 1}|\varphi(\zeta)|$$

which proves the uniform convergence, as desired.

Thus the formula (4.22) holds in the whole annulus $r \leqslant |z| \leqslant 1$ and the order of integration and summation may be changed (see e.g. [9], vol. II, p. 437, 663 and 439). In order to simplify further (4.22), we subtract and add $\frac{1}{2}$ in each of the parentheses of this formula. In this way, after introducing the above described sign of summation from $-\infty$ to $+\infty$ (see footnote (4)), we obtain the formula

$$\beta^*(z) = c^* + (1/\pi)\iint\limits_{r\leqslant|\zeta|\leqslant 1}\left\{\frac{\varphi(\zeta)}{\zeta(1-\zeta)} - \frac{\overline{\varphi(\zeta)}}{\bar{\zeta}(1-\bar{\zeta})}\right\}d\xi d\eta + \tag{4.23}$$
$$+(1/2\pi)\iint\limits_{r\leqslant|\zeta|\leqslant 1}\sum_{\nu=-\infty}^{+\infty}\left\{\frac{\varphi(\zeta)}{\zeta^2}\cdot\frac{z+r^{2\nu}\zeta}{z-r^{2\nu}\zeta} - \frac{\overline{\varphi(\zeta)}}{\bar{\zeta}^2}\cdot\frac{1+r^{2\nu}z\bar{\zeta}}{1-r^{2\nu}z\bar{\zeta}}\right\}d\xi d\eta -$$
$$-(1/2\pi)\iint\limits_{r\leqslant|\zeta|\leqslant 1}\left\{\frac{(z+\zeta)\varphi(\zeta)}{\zeta^2(z-\zeta)} - \frac{(1+z\bar{\zeta})\overline{\varphi(\zeta)}}{\bar{\zeta}^2(1-z\bar{\zeta})}\right\}d\xi d\eta \qquad (r \leqslant |\zeta| \leqslant 1).$$

Step E. *An integral formula for the functions f and R.* The formula (4.23) enables us to get the integral formula mentioned in the statement of our lemma. In fact, from (4.4), (4.23) and (4.3) we obtain

$$(1/zt)[f(z,t)-z] \rightrightarrows c^* - (1/2\pi)\iint\limits_{r\leqslant|\zeta|\leqslant 1}\{\varphi(\zeta)/\zeta^2 - \overline{\varphi(\zeta)}/\bar{\zeta}^2\}d\xi d\eta + \tag{4.24}$$
$$+(1/2\pi)\iint\limits_{r\leqslant|\zeta|\leqslant 1}\sum_{\nu=-\infty}^{+\infty}\left\{(1/\zeta^2)\varphi(\zeta)\frac{z+r^{2\nu}\zeta}{z-r^{2\nu}\zeta} - (1/\bar{\zeta}^2)\overline{\varphi(\zeta)}\frac{1+r^{2\nu}z\bar{\zeta}}{1-r^{2\nu}z\bar{\zeta}}\right\}d\xi d\eta$$
$$\text{for } t\to 0+ \qquad (r<|\zeta|<1).$$

Hence, in view of the initial condition $f(1,t) = 1$, the formula (4.1) follows.

From the formula (4.1), applied for the values z situated on the circle $|z| = r$, we obtain easily (4.2) using (4.6) and (4.7).

§ 5. Basic Theorem on parametrization in an annulus

By means of Lemma 6 presented above we can now prove the following basic theorem which considerably generalizes the corresponding theorems of the papers [35] and [25], and which is the main result of this paper:

Theorem 4. *If for a pair of characteristics $(p, \theta)\in S_*^T$, defined in an annulus $r \leqslant |z| \leqslant 1$ and in an interval $0 \leqslant t \leqslant T$, the function $u = e^{2i\theta} \times \times (1-p)/(1+p)$ has in the annulus $r \leqslant |z| \leqslant 1$ and in the interval $0 \leqslant t \leqslant T$ partial derivatives u_t and u_{zt}, then the function $f\in S_Q^{T,R}$, corresponding to the pair of characteristics (p, θ), satisfies in the annulus $r \leqslant |z| \leqslant 1$ the equation*(4)

$$(5.1)\qquad \partial f/\partial t = (1/2\pi) \iint\limits_{R(t)\leqslant|\zeta|\leqslant 1} f \sum_{\nu=-\infty}^{+\infty} \left\{\frac{\varphi(\zeta, t)}{\zeta^2}\left(\frac{f+R^{2\nu}(t)\zeta}{f-R^{2\nu}(t)\zeta} - \frac{1+R^{2\nu}(t)\zeta}{1-R^{2\nu}(t)\zeta}\right) - \right.$$

$$\left. - \frac{\overline{\varphi(\zeta, t)}}{\bar\zeta^2}\left(\frac{1+R^{2\nu}(t)f\bar\zeta}{1-R^{2\nu}(t)f\bar\zeta} - \frac{1+R^{2\nu}(t)\bar\zeta}{1-R^{2\nu}(t)\bar\zeta}\right)\right\} d\xi d\eta \qquad (\zeta=\xi+i\eta)$$

where $R^{2\nu}(t) = \{R(t)\}^{2\nu}$ and the function φ is defined by the formula (2.9); *moreover, the function R is of the class C^1 in the interval $0 \leqslant t \leqslant T$, and*

$$(5.2)\qquad dR/dt = (1/2\pi) \iint\limits_{R\leqslant|\zeta|\leqslant 1} R\,\{\varphi(\zeta, t)/\zeta^2 + \overline{\varphi(\zeta,t)/\zeta}\}\,d\xi d\eta .$$

If, in particular, $p(z, t) = [p(z)]^t$ and $\theta(z, t) = \theta(z)$, then (2.9) *takes the from* (2.10), *and the solution $w = f(z, t)$ of the equation*

$$(5.3)\qquad \partial w/\partial t = (1/2\pi) \iint\limits_{R(t)\leqslant|\zeta|\leqslant 1} w \sum_{\nu=-\infty}^{+\infty} \left\{\frac{\varphi(\zeta, t)}{\zeta^2}\left(\frac{w+R^{2\nu}(t)\zeta}{w-R^{2\nu}(t)\zeta} - \right.\right.$$

$$\left.\left. - \frac{1+R^{2\nu}(t)\zeta}{1-R^{2\nu}(t)\zeta}\right) - \frac{\overline{\varphi(\zeta, t)}}{\bar\zeta^2}\left(\frac{1+R^{2\nu}(t)w\bar\zeta}{1-R^{2\nu}(t)w\bar\zeta} - \frac{1+R^{2\nu}(t)\bar\zeta}{1-R^{2\nu}(t)\bar\zeta}\right)\right\} d\xi d\eta \qquad (\zeta = \xi+i\eta)$$

satisfies the initial condition $f(z, 0) = z$.

Proof. The proof runs analogously to the case of the corresponding theorems of the papers [34], [35] and [25]. As the proof was only outlined in the first and the second paper, and omitted in the third one, we give it here in a detailed form. For clearness it is divided into several steps.

Step A. *Construction of a suitable function satisfying the assumptions of Lemma 6.* In order to find a differential equation for functions belonging to the class S_Q^T and to apply Lemma 6, we construct a suitable function satisfying the assumptions of this lemma; we denote this function by F. By a *suitable* function we understand any function of the variable w $\big(r^*(t) \leqslant |w| \leqslant 1\big)$ depending on two real parameters t $(0 \leqslant t \leqslant T)$ and $\tau (0 < \tau \leqslant T^*)$, and fulfilling in the annulus $r^*(t) < |w| < 1$ the condition

$$(5.4)\qquad (1/\tau)[F(w, t, \tau)-w] \rightrightarrows \partial f\big(g(w, t), t\big)/\partial t \qquad \text{for} \qquad \tau\to 0+,$$

where f is a function of the class $S_Q^{r,R}$ corresponding to a pair of characteristics $(p,\theta)\in S_*^r$, and g is a certain function chosen in such a way that $F\in(S)_{Q^*}^{r^*}$ $(1\leqslant Q^*<+\infty)$.

The simplest way is to choose the function F so that the expression on the left-hand side of (5.4) be a difference quotient corresponding to the partial derivative $\partial f/\partial t$ i.e. so that

$$F(w,t,\tau)=f\big(g(w,t),t+\tau\big), \tag{5.5}$$

$$w=f\big(g(w,t),t\big). \tag{5.6}$$

Since f is invertible as belonging to $S_Q^{r,R}$, then (5.6) yields

$$f^{-1}(w,t)=g(w,t). \tag{5.7}$$

In view of (5.5) this means that the most convenient it is to admit $r^*(t)=R(t)$ and

$$F(w,t,\tau)=f\big(f^{-1}(w,t),t+\tau\big). \tag{5.8}$$

There remains to verify whether the function F, defined in the annulus $R(t)\leqslant|w|\leqslant 1$ and in the intervals $0\leqslant t\leqslant T$, $0<\tau\leqslant T-t$ by (5.6), satisfies the assumptions of Lemma 6.

Step B. *Evaluation of the functions U and Φ.* From editirial reasons let us begin from expressing the functions U and Φ, defined by

$$U(w,t,\tau)=F_{\bar w}(w,t,\tau)/F_w(w,t,\tau), \tag{5.9}$$

$$\Phi(w,t)=\lim_{\tau\to 0+}[(1/\tau)U(w,t,\tau)] \tag{5.10}$$

in the annulus $R(t)\leqslant|w|\leqslant 1$, in dependence on the functions f, p and θ. The derivatives in (5.9) exist, because, in view of Theorem 7.3 of [38], the assumption $(p,\theta)\in S_*^r$ implies that f belongs to C^2, and thus, in view of (5.8), the function F must also be of the class C^2.

Note first the identities

$$\overline{w}_z=\overline{w_{\bar z}},\quad \overline{w}_{\bar z}=\overline{w_z} \tag{5.11}$$

which can be easily verified. Since in our case the functions $w-f$ and $\overline{w}-\bar f$, considered as functions of the variables z, $\bar z$, w, $\overline{w}$, satisfy the assumptions of a well-known theorem on implicit functions (see e.g. [9], vol. I, p. 454), then we have

$$f_z z_w+f_{\bar z}\bar z_w=1, \tag{5.12}$$

$$\bar f_z z_w+\bar f_{\bar z}\bar z_w=0, \tag{5.13}$$

where (5.13) may be, in view of (5.11) replaced by

$$\bar{f}_{\bar{z}} z_w + \bar{f}_z \bar{z}_w = 0. \tag{5.14}$$

The equations (5.12) and (5.14) yield

$$z_w = \bar{f}_z/(|^2 - |f_{\bar{z}}|^2), \tag{5.15}$$

$$\bar{z}_w = -\bar{f}_{\bar{z}}/(|f_z|^2 - |f_{\bar{z}}|^2), \tag{5.16}$$

whence in view of (5.11) we get

$$\bar{z}_{\bar{w}} = f_z/(|f_z|^2 - |f_{\bar{z}}|^2), \tag{5.17}$$

$$z_{\bar{w}} = -f_{\bar{z}}/(|f_z|^2 - |f_{\bar{z}}|^2). \tag{5.18}$$

The relations above obtained permit to accomplish differentiation in the formula (5.9). In view of (5.8) we have

$$F_w(w, t, \tau) = f_z(z, t+\tau) z_w + f_{\bar{z}}(z, t+\tau) \bar{z}_w, \tag{5.19}$$

$$F_{\bar{w}}(w, t, \tau) = f_z(z, t+\tau) z_{\bar{w}} + f_{\bar{z}}(z, t+\tau) \bar{z}_{\bar{w}}. \tag{5.20}$$

Now, applying the relations (5.15) and (5.16) to the formula (5.19), and the relations (5.17) and (5.18) to the formula (5.20), we obtain

$$(|f_z(z, t)|^2 - |f_{\bar{z}}(z, t)|^2) F_w(w, t, \tau) = f_z(z, t+\tau)\overline{f_z(z, t)} - f_{\bar{z}}(z, t+\tau)\overline{f_{\bar{z}}(z, t)},$$

$$(|f_z(z, t)|^2 - |f_{\bar{z}}(z, t)|^2) F_{\bar{w}}(w, t, \tau) = -f_z(z, t+\tau) f_{\bar{z}}(z, t) + f_{\bar{z}}(z, t+\tau) f_z(z, t).$$

Hence, after putting to (5.9), we get

$$U(w, t, \tau) = \frac{-f_z(z, t+\tau) f_{\bar{z}}(z, t) + f_{\bar{z}}(z, t+\tau) f_z(z, t)}{f_z(z, t+\tau)\overline{f_z(z, t)} - f_{\bar{z}}(z, t+\tau)\overline{f_{\bar{z}}(z, t)}}.$$

Note finally that in accordance with the notations of our theorem, by virtue of (2.2), we have

$$f_{\bar{z}}(z, t)/f_z(z, t) = u(z, t) \tag{5.21}$$

whence

$$U(w, t, \tau) = \frac{u(z, t+\tau) - u(z, t)}{1 - u(z, t+\tau)\overline{u(z, t)}} \exp\big(2i \arg f_z(z, t)\big). \tag{5.22}$$

Dividing both sides of (5.22) by τ and letting $\tau \to 0+$, in view of (5.10) and the assumed existence of the derivative u_t, we obtain immediately

$$\Phi(w, t) = \{1/(1 - |u(z, t)|^2)\} u_t(z, t) \exp\big(2i \arg f_z(z, t)\big). \tag{5.23}$$

The formulae (5.22) and (5.23) where $z = f^{-1}(w, t)$ give the expressions for U and Φ, as desired.

Step C. *Verification of some properties of the function F.* Note first that $f\in(S)_Q^{r,R}$; this follows immediately from the obvious relation $S_Q^{r,R}\in(S)_Q^{r,R}$. Thus, by virtue of (5.8), the function F belongs to C^2 and fulfils the conditions (i) and (ii) given in Definition 1 of quasiconformality, and maps the annulus $R(t)\leqslant|w|\leqslant 1$ onto itself so that $F(1,t,\tau)=1$. Using next the formulae (0.1) and (5.21), in view of the above proved relation $f\in(S)_Q^{r,R}$, we have

$$(5.24)\qquad 0\leqslant|u(z,t)|\leqslant(Q-1)/(Q+1)<1\quad\big(Q=Q(t)\big).$$

Hence, in view of (5.9) and (5.22), we infer that F fulfils also the condition (iii) of quasiconformality; so this function represents a Q^*-quasiconformal mapping where, as it can be easily verified,

$$Q^*=\sup_{r\leqslant|z|\leqslant 1}\frac{|1-u(z,t+\tau)\overline{u(z,t)}|+|u(z,t+\tau)-u(z,t)|}{|1-u(z,t+\tau)\overline{u(z,t)}|-|u(z,t+\tau)-u(z,t)|},$$

and $z=f^{-1}(w,t)$.

From the above we obtain immediately that $F\in(S)_{Q^*}^{R}$ and that we may associate to this function a uniquely determined pair of characteristics $P=P(w,t,\tau)$ and $\Theta=\Theta(w,t,\tau)$. Simultaneously from the formula (5.9) and from the formula (5.21) applied to the function F we obtain the relation $U=e^{2i\Theta}(1-P)/(1+P)$. In view of the conditions (2.5) and (2.6) of Lemma 6 it means that there only remains to verify the existence of the bounded functions φ and k which fulfil the conditions

$$(5.25)\qquad (1/\tau)\,U(w,t,\tau)\rightrightarrows\varphi(w,t)\quad\text{for}\quad t\to 0+,$$

$$(5.26)\qquad (1/\tau)\,|U_w(w,t,\tau)|\leqslant k(w,t)\quad\text{for}\quad 0<\tau\leqslant T-t$$

in the annulus $R(t)\leqslant|w|\leqslant 1$ and the interval $0\leqslant t\leqslant T$.

To this end note that (5.10) implies

$$(5.27)\qquad (1/\tau)\,U(w,t,\tau)\to\Phi(w,t)\quad\text{for}\quad t\to 0+,$$

and the limit function is given by (5.23) where $z=f^{-1}(w,t)$. Thus, it is necessary to prove that in (5.27) there takes place the uniform convergence. So, let ε, w, t be arbitrary numbers fulfilling the conditions $\varepsilon>0$, $R(t)\leqslant|w|\leqslant 1$, $0\leqslant t\leqslant T$, respectively, and let $z=f^{-1}(w,t)$. By (5.24) we have

$$(5.28)\qquad 1/\big(1-|u(z,t)|^2\big)\leqslant(Q+1)^2/4Q.$$

Moreover, from the assumption on existence of the derivative u_t we easily infer that for a certain η there is

$$(5.29)\qquad \big|1/\big(1-u(z,t+\tau)\overline{u(z,t)}\big)-1/\big(1-|u(z,t)|^2\big)\big|<\varepsilon/2M\quad(0<\tau<\eta).$$

On the other hand, it can be easily seen that $|u_t(z, t)| \leqslant M$, where $M < +\infty$, whence, in view of a well-known theorem,

$$|u(z, t+\tau) - u(z, t)| \leqslant M\tau \quad (0 < \tau < \eta); \tag{5.30}$$

moreover, we have for a certain η^*

$$|(1/\tau)\{u(z, t+\tau) - u(z, t)\} - u_t(z, t)| < 2Q\varepsilon/(Q+1)^2 \quad (0 < \tau < \eta^*). \tag{5.31}$$

From the inequalities (5.28), (5.29), (5.30), (5.31) we obtain immediately that in the interval $0 < \tau < \min(\eta, \eta^*)$ there is

$$\begin{aligned} &\left|(1/\tau)\{u(z, t+\tau) - u(z, t)\}/\{1 - u(z, t+\tau)\overline{u(z, t)}\} - u_t(z, t)/\big(1 - |u(z, t)|^2\big)\right| \\ &\quad \leqslant |(1/\tau)\{u(z, t+\tau) - u(z, t)\}|\left|1/\big(1 - u(z, t+\tau)\overline{u(z, t)}\big) - 1/(1 - |u(z, t)|^2)\right| \\ &\qquad + |(1/\tau)\{u(z, t+\tau) - u(z, t)\} - u_t(z, t)|/\big(1 - |u(z, t)|^2\big) < \varepsilon. \end{aligned}$$

Hence, by (5.22) and (5.23), in the same interval there is

$$|(1/\tau)\, U(w, t, \tau) - \Phi(w, t)| < \varepsilon,$$

that is the uniform convergence takes place in (5.27). Thus, there exists a function φ which fulfils (5.25); it is uniquely determined and is expressed by the formula

$$\varphi(w, t) = \Phi(w, t). \tag{5.32}$$

In concern with the question of existence of the function k which fulfils (5.26) let us notice first that the left-hand side of this inequality exists in view of (5.22) and of the previously shown appertenance of the function f to C^2. There is also an opportunity to notice that only in this place the above property is used in full, and that with application of the present method of proving it is not possible to weaken the assumptions on regularity of characteristics (p, θ). The existence of the function k fulfilling (5.26) follows from the appertenance of the function f to C^2 and from the assumption that the derivative u_{zt} exists. In fact, if for $r \leqslant |z| \leqslant 1$ and $0 \leqslant t \leqslant T$ the derivative u_{zt} exists, then, in view of (5.22) and (5.24), there exists also the derivative U_{wt}, and we have $|U_{wt}(w, t, \tau)| \leqslant M^*$ for $0 < \tau \leqslant T-t$, where $M^* < +\infty$. Hence we infer, that for $R(t) \leqslant |w| \leqslant 1$, $0 \leqslant t \leqslant T$, $0 < \tau \leqslant T-t$ there takes place an estimate $|U_w(w, t, \tau) - U_w(w, t, 0)| \leqslant M^*\tau$, where the existence of $U_w(w, t, \tau)$ for $\tau = 0$ follows immediately from (5.22) and, as it is easily seen, we have $U_w(w, t, 0) = 0$. Thus we may write the last inequality in the form $|(1/\tau)\, U_w(w, t, \tau)| \leqslant M^*$. This means that the estimate (5.26) holds, and that we may put $k(w, t) = M^*$ identically.

In this way we have proved that the function F, constructed in Step A of our proof, satisfies all assumptions of Lemma 6.

Step D. *The differential equation for the class S_Q^r.* In the previous parts of proof we have constructed the function F determined by the formula (5.8) and fulfilling the condition (5.4), and we have verified that this function satisfies the assumptions of Lemma 6. Therefore, applying this lemma for the function F we obtain, in view of (5.4) and (5.8), the differential equation (5.1), as desired.

According to Step C of our proof the function φ, that appears in the obtained equation (5.1), is determined by (5.32) and (5.23), where $z = f^{-1}(w, t)$. For finishing the proof of our theorem there remains to reduce the formulae obtained for the function φ to the form (2.9), and to derive (5.2). For the first question it is sufficient to verify that if $z = f^{-1}(w, t)$, then

$$\arg f_z(z, t) = -\arg f_w^{-1}(w, t)$$

that is

$$\arg w_z = -\arg f_w^{-1}(w, t). \tag{5.33}$$

Applying then a known theorem on implicit functions to the functions $z - f^{-1}$ and $\bar{z} - \overline{f^{-1}}$, considered as functions of the variables w, $\bar{w}$, z, $\bar{z}$, we obtain, similarly as in Step B of our proof, the formula

$$w_z = \overline{f_{\bar{w}}^{-1}(w, t)}/\{|f_w^{-1}(w, t)|^2 - |f_{\bar{w}}^{-1}(w, t)|^2\} \tag{5.34}$$

which is analogous to (5.15). Since, as it was stated in Step C of our proof. f belongs to $(S)_Q^{r,R}$, then f^{-1} belongs to $(S)_Q^{R,r}$, and consequently, by (0.1), we have

$$|f_w^{-1}(w, t)|^2 - |f_{\bar{w}}^{-1}(w, t)|^2 \geqslant \left\{\left(\frac{Q+1}{Q-1}\right)^2 - 1\right\} |f_{\bar{w}}^{-1}(w, t)| = 4Q(Q-1)^{-2} \times \\ \times |f_{\bar{w}}^{-1}(w, t)|^2 \geqslant 0 .$$

Thus, by (5.34), we obtain the formula

$$\arg w_z = \arg \overline{f_{\bar{w}}^{-1}(w, t)}$$

equivalent to (5.33).

The formula (5.2) can be easily obtained in a way analogus to that applied in the proof of Lemma 6 for obtaining (4.2) from (4.1). Obviously, R belongs to C^1. Similarly, it is easily verified that if in particular $p(z, t) = [p(z)]^t$ anf $\theta(z, t) = \theta(z)$, then (2.9) takes the from (2.10) and the solution $w = f(z, t)$ of the equation (5.3) corresponds to the initial condition $f(z, 0) = z$. In this way the proof of Theorem 4 is completed.

§ 6. Further theorems on parametrization

Now we obtain two further theorems on parametrization; they correspond to the theorem on parametrization of conformal mappings and to its converse, respectively, as obtained by Löwner [24], Komatu [16] and Golusin [12]. They also correspond to Theorems 3 and 4 obtained by Shah Tao-shing [34]. One of the announced theorems was in a particular case formulated in [35], but this formulation requires substantial supplements.

Theorem 5. *Let* $w = f(z)$ *belong to* $S_Q^{r,R}$. *Then there exists a function* $\omega = \varphi(w, t)$, *defined for* $|w| \leqslant 1$ *and* $0 \leqslant t \leqslant T = \log Q$, *bounded by* $\frac{1}{2}$, *and such that*: (i) *the solution* $\varrho = R(t)$ *of*

$$\varrho' = (1/2\pi) \iint_{\varrho \leqslant |\zeta| \leqslant 1} \varrho \{\varphi(\zeta, t)/\zeta^2 + \overline{\varphi(\zeta, t)}/\bar{\zeta}^2\} d\xi d\eta \tag{6.1}$$

with the initial condition $R(0) = r$ *satisfies* $R(T) = R$, (ii) *the derivatives* φ_w *and* $\varphi_{\bar{w}}$ *are continuous for* $R(t) \leqslant |w| \leqslant 1$ $(0 \leqslant t \leqslant T)$, (iii) *the solution* $w = f(z, t)$ *of* (5.3) *with the initial condition* $f(z, 0) = z$ *is identically equal to* f *for* $t = T$.

The proof is omitted as very easy and analogous to that of Theorem 3 in [34].

Theorem 6. *Let* $\omega = \varphi(w, t)$ *be a function defined for* $|w| \leqslant 1$ *and* $0 \leqslant t \leqslant T$, *and bounded by* $\frac{1}{2}$. *Then there exists a unique solution* $\varrho = R(t)$ *of* (6.1) *with the initial condition* $R(0) = r$. *Moreover, there exists a unique solution* $w = f(z, t)$ *of* (5.3) *with the initial condition* $f(z, 0) = z$ *which represents a mapping belonging to* $U_{Q(t)}^{r, R(t)}$ *where* $Q(t) \leqslant \exp t$.

Proof. For more clearness the proof is divided into three steps.

Step A. *Existence of the unique solution of* (6.1). Let $H(\varrho, t)$ denote the right-hand side of the equation (6.1). It is easily seen (cf. [37], p. 44) that for every t $(0 \leqslant t \leqslant T)$ the function H is continuous with respect to ϱ in the interval $0 < \varrho < 1$. In consequence the assumptions of a known theorem of Peano are fulfilled, and thus there exists at least one solution $\varrho = R(t)$ of (6.1) that fulfils the initial condition $R(0) = r$.

In order to prove the uniqueness of (6.1) we verify that the assumptions of a known theorem of Osgood are fulfilled. In fact, let $0 < \varrho_1 < \varrho_2 < 1$. Then

$$|H(\varrho_1, t) - H(\varrho_2, t)| \leqslant (1/2\pi) \iint_{\varrho_2 \leqslant |\zeta| \leqslant 1} (\varrho_2 - \varrho_1) |\varphi(\zeta, t)/(\zeta^2 + \overline{\varphi(\zeta, t)}/\bar{\zeta}^2| d\xi d\eta +$$
$$+ (1/2\pi) \iint_{\varrho_2 \leqslant |\zeta| \leqslant \varrho_1} \varrho_1 |\varphi(\zeta, t)/\zeta^2 + \overline{\varphi(\zeta, t)}/\bar{\zeta}^2| d\xi d\eta \leqslant M_0 |\varrho_1 - \varrho_2|$$

and, consequently, there exists at most one solution $\varrho = R(t)$ of (6.1) that fulfils the initial condition $R(0) = r$.

Summing up, there exists exactly one solution $\varrho = R(t)$ of (6.1) that fulfils the initial condition $R(0) = r$.

Step B. *Existence of the unique solution of* (5.3). Let $F(w, t)$ denote the right-hand side of the equation (5.3). It is easily seen (cf. [37], p. 44) that for every t $(0 \leqslant t \leqslant T)$ the function F is continuous with respect to w in the annulus $R(t) \leqslant |w| \leqslant 1$. In consequence the assumptions of a known theorem of Peano are fulfilled and thus there exists at least one solution $w = f(z, t)$ of (5.3) that fulfils the initial condition $f(z, 0) = z$.

In order to prove the uniqueness of the solution $w = f(z, t)$ of (5.3) we verify that the assumptions of a known theorem of Osgood are fulfilled. In fact, it can be easily verified, that if $w = f(z, t)$ is a solution of (5.3) in the annulus $r \leqslant |z| \leqslant 1$ that fulfils the initial condition $f(z, 0) = z$, then (2.11) has the solution $w = f^*(z, t)$, determined in the disc $|z| \leqslant 1$ and corresponding to the same initial condition; this solution is defined by the formulae

$$(6.2)\qquad f^*(z, t) = R^{2\nu}(t)/\overline{f(r^{2\nu}/\bar{z}, t)} \quad \text{for} \quad r^{2\nu} \leqslant |z| \leqslant r^{2\nu-1} \quad (\nu = 1, 2, \ldots),$$

$$(6.3)\qquad f^*(z, t) = R^{2\nu}(t) f(z/r^{2\nu}, t) \quad \text{for} \quad r^{2\nu+1} \leqslant |z| \leqslant r^{2\nu} \quad (\nu = 1, 2, \ldots),$$

where $R^{2\nu}(t) = \{R(t)\}^{2\nu}$. Obviously, we admit $f^*(z, t) = f(z, t)$ for $r \leqslant |z| \leqslant 1$, and $f^*(0, t) = 0$. The corresponding function $u^* = f^*_{\bar{z}}/f^*_z$ is determined by the formulae

$$(6.4)\qquad u^*(z, t) = e^{4i\arg z}\overline{u(r^{2\nu}/\bar{z}, t)} \quad \text{for} \quad r^{2\nu} \leqslant |z| < r^{2\nu-1} \quad (\nu = 1, 2, \ldots),$$

$$(6.5)\qquad u^*(z, t) = u(z/r^{2\nu}, t) \quad \text{for} \quad r^{2\nu+1} \leqslant |z| < r^{2\nu} \quad (\nu = 1, 2, \ldots)$$

and, obviously, $u^*(z, t) = u(z, t)$ for $r \leqslant |z| \leqslant 1$. Similarly, if $w = f^*(z, t)$ is a solution of (2.11) in the disc $|z| \leqslant 1$ that can be expressed in the form (6.2), (6.3), and that fulfils the initial condition $f^*(z, 0) = z$, then $w = f(z, t)$ is a solution of (5.3), determined in the annulus $r \leqslant |z| \leqslant 1$ and corresponding to the same initial condition. Consequently, if $F^*(w, t)$ denotes the right-hand side of the equation (2.11), we can replace our consideration of the expression $|F(w_1, t) - F(w_2, t)|$ that appears in the theorem of Osgood, by the consideration of $|F^*(w_1, t) - F^*(w_2, t)|$. Here we define φ^* in the same way as φ, replacing u by u^*.

An estimate of $|F^*(w_1, t) - F^*(w_2, t)|$ can be obtained as in the paper [19] of Krushkal. We have

$$w(1-w)/\zeta(1-\zeta)(w-\zeta) = 1/(w-\zeta) + (w-1)/\zeta + w/(\zeta-1),$$
$$w(1-w)/\bar{\zeta}(1-\bar{\zeta})(1-w\bar{\zeta}) = -w^3/(1-w\bar{\zeta}) + w(1-w)/\bar{\zeta} + w/(1-\bar{\zeta}).$$

Hence, for any w_1 and w_2 taken from the unit disc, we obtain

$$(6.6)\qquad |F^*(w_1, t) - F^*(w_2, t)| \leqslant \frac{\frac{1}{2}}{\pi(1-\frac{1}{4})}|w_1 - w_2| \left\{ \iint\limits_{|\zeta|\leqslant 1} \frac{d\xi d\eta}{|\zeta - w_1||\zeta - w_2|} + \right.$$
$$\left. + 4 \iint\limits_{|\zeta|\leqslant 1} \frac{d\xi d\eta}{|\zeta|} + 2 \iint\limits_{|\zeta|\leqslant 1} \frac{d\xi d\eta}{|\zeta - 1|} + 6 \iint\limits_{|\zeta|\leqslant 1} \frac{d\xi d\eta}{|1 - w_1\zeta||1 - w_2\zeta|} \right\}.$$

The first integral at the right-hand side is estimated by $M_1|\log|w_1-w_2||$. This is a consequence of a result presented in [37] (Chapter I, § 6), according to which the same integral taken over an arbitrary domain D is estimated by $M_1(D)|\log|w_1-w_2||$ where $M_1(D)$ depends only on D. Applying now Lemma 1 presented in Chapter I, § 5 of [37] we state that the second and the third integrals in (6.6) are bounded; let M_2 and M_3 denote these bounds, respectively. In order to estimate the fourth integral we distinguish two cases. If $|w_1|$ or $|w_2| \leqslant \frac{1}{2}$ and both are $\leqslant 1$, then this integral is bounded by a constant M_4 as a function continuous in w_1 and w_2. If $|w_1|$ and $|w_2|$ are both $\geqslant \frac{1}{2}$ and $\leqslant 1$, then we get an estimate $M_5|\log|w_1-w_2||$, analogous to that obtained in the case of the first integral. Hence we have finally

$$(6.7) \qquad |F^*(w_1,t)-F^*(w_2,t)| \leqslant M_6|w_1-w_2|\{1+M_7|\log|w_1-w_2||\}.$$

Let $G(|w_1-w_2|)$ denote the right-hand side of (6.7). In order to apply the theorem of Osgood we verify easily that $G(\eta) > 0$ for $\eta > 0$, and that

$$\int_c^d \{1/G(\eta)\}\,d\eta \to \infty \text{ as } d \to \infty \quad (c > 0).$$

Consequently, there exists at most one solution $w = f(z, t)$ of (5.3) that fulfils the initial condition $f(z, 0) = z$.

Summing up, there exists exactly one solution $w = f(z,t)$ of (5.3) that fulfils the initial condition $f(z, 0) = z$. Moreover, f is continuous with respect to z, t being fixed (cf. [37], p. 44-45).

Step C. *Properties of the found unique continuous solution of* (5.3). From the uniqueness and continuity of the solution $w = f(z,t)$ of (5.3) we infer that f must be univalent in the whole annulus $r \leqslant |z| \leqslant 1$. In fact, the number of solutions of the equation $f(z,t) = w_0(t)$, where w_0 is continuous in t, $R(t) \leqslant |w_0(t)| \leqslant 1$, $0 \leqslant t \leqslant T$, is equal to the index of the point $w_0(t)$ with respect to the cycle formed by the boundary curves of the domain considered, i.e. $(1/2\pi i)\int_{C_1(t)}\{w-w_0(t)\}^{-1}dw-(1/2\pi i)\int_{C_2(t)}\{w-w_0(t)\}^{-1}dw$, where $C_1(t)$: $w = f(e^{i\theta}, t)$, $0 \leqslant \theta \leqslant 2\pi$, $C_2(t)$: $w = f(re^{i\theta}, t)$, $0 \leqslant \theta \leqslant 2\pi$. Let $n(t, w_0(t))$ denote this index. The function n is continuous in $w_0(t)$, t being fixed, and $f(1, t) = 1$ for $0 \leqslant t \leqslant T$, so $n(t, w_0(t)) = n(t, 1)$ for any $w_0(t)$ taken from the annulus $R(t) \leqslant |w| \leqslant 1$, where $0 \leqslant t \leqslant T$. But n is also continuous in t, so $n(t, 1) = n(0, 1) = 1$, and consequently, $n(t, w_0(t)) = 1$. Summing up, f must be univalent in the whole annulus $r \leqslant |z| \leqslant 1$.

Now we show that f transforms the annulus $r \leqslant |z| \leqslant 1$ onto $R(t) \leqslant |w| \leqslant 1$. In order to do this, in view of the continuity and univalence of f, it is sufficient to verify that $|f(z,t)| = 1$ for $|z| = 1$and $|f(z,t)| = R(t)$ for $|z| = r$ $(0 \leqslant t \leqslant T)$.

To prove the relation $|f(z,t)| = 1$ on the circle $|z| = 1$ let us note that on the same circle we have $\operatorname{Re}\{(1/w)F(w,t)\} = 0$. Then, introducing for $r \leqslant |z| \leqslant 1$ and for $0 \leqslant t \leqslant T$ the notation

$$\varepsilon(z,t) = \overline{f(z,t)} - 1/f(z,t),$$

and applying an easily verified identity

$$2\operatorname{Re}\{[1/f(z,t)](\partial/\partial t)f(z,t)\} = (\partial/\partial t)\{f(z,t)\varepsilon(z,t)\}/[1+f(z,t)\varepsilon(z,t)],$$

we obtain by (5.1), after letting $|z| \to 1-$,

$$(\partial/\partial t)\{f(z,t)\varepsilon(z,t)\} = 0 \quad (|z| = 1,\ 0 \leqslant t \leqslant T).$$

From the above it follows that for any t $(0 \leqslant t \leqslant T)$ we have $\varepsilon(z,t) = c(z)/f(z,t)$ on the circle $|z| = 1$, where c does not depend on t. Hence, in view of the definition of ε, we obtain

$$|f(z,t)|^2 = 1 + c(z) \quad (|z| = 1,\ 0 \leqslant t \leqslant T).$$

Now, taking into account the initial condition $f(z,0) = z$, we see that $c(z) = 0$ identically, and thus $|f(z,t)| = 1$ for $|z| = 1$.

Similarly, by virtue of the relation $\operatorname{Re}\{(1/w)F(w,t)\} = R'(t)/R(t)$ on the circle $|z| = r$ (cf. (4.6) and (4.7)), we prove that on the same circle we have $|f(z,t)| = R(t)$. Noticing finally that the mapping $w = f(z,t)$ is sense-preserving for every t $(0 \leqslant t \leqslant T)$, we see that it fulfils the condition (i) in Definition 1 of quasiconformality.

Next, similarly as in an analogous proof of the paper [34], we verify that there are fulfilled the remaining conditions which warrant quasiconformality, that $f(1,t) = 1$ $(0 \leqslant t \leqslant T)$, as remarked before, and that $Q(t) \leqslant \exp t$ in the whole interval $0 \leqslant t \leqslant T$. In this way the proof of Theorem 6 is completed.

Added in proof. During preparation of this paper for print there has appeared a monograph on quasiconformal mappings by Lehto and Virtanen, and also some results on the parametric method and its applications due to Gehring, Reich and others. As some terms and notations become commonly used, it is worth to present them here to compare with those used by the author who was following mostly Shah Tao-shing's terminology. The author hopes to adopt terms and notations presented below in subsequent papers.

(i) It is convenient to speak about the *complex dilatation* μ (or $\varkappa$) of a quasiconformal mapping $w = f(z)$ instead of the *complex characteristics* p and θ. The complex dilatation means the same as the function u in our paper.

(ii) The notation S_Q (or S_K) become commonly used instead of U_Q used in our paper. S_Q seems to be more convenient in extremal problems than S_K, because K can be misunderstood with complete elliptic integrals. Consequently, the author will adopt the notations S_Q, S_*, $(S)_Q$, $(S)_*$, $\hat{S}_Q$, $\hat{S}_*$, $S_Q^{r,R}$, S_Q^r, S_*^r, $(S)_Q^{r,R}$, $(S)_Q^r$, $(S)_*^r$, $\hat{S}_Q^{r,R}$, $\hat{S}_Q^r$ and $\hat{S}_*^r$ instead of U_Q, U_*, $(S)_Q$, $(S)_*$, S_Q, S_*, $U_Q^{r,R}$, U_Q^r, U_*^r, $(S)_Q^{r,R}$, $(S)_Q^r$, $(S)_*^r$, $S_Q^{r,R}$, S_Q^r and S_*^r, respectively.

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Streszczenie

W pracy niniejszej przedstawiam metodę parametryczną dla odwzorowań quasi-konforemnych w pierścieniu w przypadku ogólnym (dla podklasy gęstej klasy wszystkich odwzorowań quasi-konforemnych w pierścieniu). Uzyskana metoda stanowi uogólnienie wcześniejszych wyników Shah Tao-shinga, Fan Le-le i moich. Metoda parametryczna jest podstawowym narzędziem badań w wielu zagadnieniach ekstremalnych. Ponadto praca zawiera twierdzenie nieopublikowane dotychczas, a podane przez Bielińskiego (twierdzenie 1), które zamieszczam w tej pracy za zgodą autora.

Резюме

В работе представлен параметрический метод для квазиконформных отображений в круговом кольце в общем случае (для плотного подкласса класса всех квазиконформных отображений в круговом кольце). Этот метод является обобщением результатов предыдущих исследований, проведенных Ся До-шином, Фан Ле-лем и автором. Параметрический метод является основным орудием для исследований многих экстремальных задач. Кроме того, в работе содержится нигде до сих пор не опубликованное доказательство одной теоремы Белинского (теорема 1), приведенное с согласия автора.