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Some Results Concerning Univalent Majorants

Kilka wyników dotyczących majorant jednolistnych

Некоторые розультаты относящиеся к однолистным мажорантам

The papers of M. Biernacki [6], [7], G. M. Golusin [8], [9], [10] and also some papers due to A. Bielecki and the author [1], [2], [3], [4], [5], [12], [13], indicate that there exists a relation between subordination and modular majorization. The aim of this paper is to point out further relations of this kind. The Theorem 1′ stated below is an analogue of a theorem due to M. Biernacki [6], restated in an improved form by Golusin [8] and Shah Tao-Shing [15]. Similarly, the Theorem 2 is an analogue of a well-known result of M. Schiffer [16]. The supposition of subordination in the results referred to is replaced here by the assumption of modular majorization. The remaining results are of similar type.

In what follows S denotes the class of functions $f(z) = z+a_2z^2+\dots$ regular and univalent in the unit disc $|z|<1$ and $S^* \subset S$ denotes the subclass of functions starshaped w.r.t. the origin.

Let $B(a)$, $0\leqslant a<1$, denote the class of functions $\omega(z) = a+\beta_1 z+ +\beta_2 z^2+\dots$ regular and bounded in the unit disc such that $|\omega(z)|\leqslant 1$ for $|z|<1$ and let B be the union $\bigcup_{0\leqslant a<1} B(a)$ of all classes $B(a)$. We now prove

Theorem 1. *Suppose that $F(z)\in S$ and $f(z) = az+\dots$ $(0\leqslant a<1)$ is regular in the unit disc. If the inequality $|f(z)|<|F(z)|$ holds for any z in the unit disc, then $|f'(z)|<|F'(z)|$ holds for any z in the disc $|z|<r(a)$, where $r(a)$ is the least positive root of the equation $ar^3-3ar^2-3r+1=0$. The radius $r(a)$ is best possible.*

Proof. Obviously, $f(z) = F(z)\omega(z)$, with $\omega(z) \in B(a)$. Besides, we have (see [11], [10])

$$\frac{1-|z|}{1+|z|} \leqslant \left|\frac{zF'(z)}{F(z)}\right| \leqslant \frac{1+|z|}{1-|z|}, \tag{1}$$

$$|\omega(z)| \leqslant \frac{|z|+|\omega(0)|}{1+|\omega(0)|\,|z|}, \tag{2}$$

$$|\omega'(z)| \leqslant \frac{1-|\omega(z)|^2}{1-|z|^2}. \tag{3}$$

Suppose that r is a fixed real number which satisfies $0 \leqslant r < r(a)$ and put $W(r, a) = -ar^3+3ar^2+3r-1$. We have obviously $W(r, a) = r[1+ +a+(1+a)r]-(1-r)^2(1+ar) < 0$ for $r \in [0, r(a))$ and this implies

$$\frac{r}{1-r} < \frac{1-r}{1+\dfrac{r+a}{1+ar}}, \qquad 0 \leqslant r < r(a). \tag{4}$$

From (2) and (4) we obtain for $|z| = r$:

$$r\frac{1+r}{1-r} < \frac{1-r^2}{1+|\omega(z)|}. \tag{5}$$

Using next (1), we get

$$\left|\frac{F(z)}{F'(z)}\right| < \frac{(1-r^2)\big(1-|\omega(z)|\big)}{1-|\omega(z)|^2}. \tag{6}$$

Applying finally inequalities (3) and (6) we have

$$|\omega(z)| + \left|\frac{F(z)}{F'(z)}\right| |\omega'(z)| < 1,$$

that is
$|F'(z)\omega(z)|+|F(z)\omega'(z)| < |F'(z)|$. This means that $|f'(z)| < |F'(z)|$ in the disc $|z| < r(a)$. An easy calculation shows that for $F_0(z) = z(1+z)^{-2}$, $\omega_0(z, a) = (z+a)(1+az)^{-1}$, we have $[F_0(z)\omega_0(z, a)]' = F_0'(z)$ at $z = r(a)$. The theorem is proved.

Since the extremal function in Theorem 1 belongs to S^*, the some results hold for starlike majorants.

It is easy to see that $r(a)$ decreases in $[0, 1)$ and $\lim_{a\to 1} r(a) = 2-\sqrt{3}$. This implies immediately

Theorem 1'. *Suppose that* $F(z) \in S$ *and* $f(z) = az+\ldots\,(0 \leqslant a < 1)$ *is regular in the unit disc. If the inequality* $|f(z)| < |F(z)|$ *holds for any* z

in the unit disc, then $|f'(z)| < |F'(z)|$ holds for any z in the disc $|z| < 2-\sqrt{3}$. The number $2-\sqrt{3}$ is best possible.

We now show that if we confine ourselves to starlike f and F, the radius $2-\sqrt{3}$ cannot be increased.

Theorem 1″. *Suppose that $F(z)\in S^*$ and that $f(z)$ with $f(0)=0$, $f'(0)>0$, is univalent and starshaped (w.r.t. the origin) in the unit disc. If the inequality $|f(z)| < |F(z)|$ holds in the unit disc, then $|f'(z)| < |F'(z)|$ holds for any z in the disc $|z| < 2-\sqrt{3}$. The number $2-\sqrt{3}$ is best possible.*

Proof. Suppose that for any $f(z)$ and $F(z)$ satisfying the assumptions of Theorem 1″ we have $|f'(z)| < |F'(z)|$ in the disc $|z| < R$ with $R > 2-\sqrt{3}$. Put $F_\varrho(z) = z(1+\varrho z)^{-2}$, $\omega_\varrho(z) = (\varrho z+a)(1+a\varrho z)^{-1}$, where $0<\varrho<1$, $0\leqslant a<1$, and consider $f_\varrho(z) = F_\varrho(z)\omega_\varrho(z)$. The functions f_ϱ and F_ϱ satisfy the assumptions of Theorem 1″. Besides,

$$H_\varrho(z) = \frac{zf_\varrho'(z)}{f_\varrho(z)} = \frac{1-\varrho z}{1+\varrho z} + \frac{\varrho z(1-a^2)}{(1+a\varrho z)(\varrho z+a)}. \tag{7}$$

The real part of the first term on the right hand side is at least equal to $(1-\varrho)(1+\varrho)^{-1}$, whereas the second term tends uniformly to 0 if $a\to 1$ Hence for any fixed ϱ, $0<\varrho<1$, there exists t_0, $0<t_0<1$, such. that for $a\in[t_0,1)$ we have $\mathrm{re}\{zf_\varrho'(z)/f_\varrho(z)\}>0$ in $|z|<1$. However, $H(0)=1$, and for $a\in[t_0,1)$ the function $f_\varrho(z)$ is univalent and starlike in the unit disc. On the other hand $f_\varrho'(r) = F_\varrho'(r)$ for the least positive root r of the equation

$$a\varrho^3r^3 - 3a\varrho^2r^2 - 3\varrho r + 1 = 0. \tag{8}$$

From (8) we deduce that $r = r(a)\varrho^{-1}$. Now, $r(a)\searrow 2-\sqrt{3}$ for $a\to 1$ and we can choose a and ϱ so that $2-\sqrt{3} < r < R$ which contradicts Theorem 1′.

Theorem 2. *Let $f(z)$ be regular for $|z|<1$ with $f(0)=0$, $f'(0)\geqslant 0$, and let $F(z)\in S$. If $|f(z)|\leqslant|F(z)|$ for $|z|<1$, then*

$$|f'(z)| \leqslant \frac{1+|z|}{(1-|z|)^3}. \tag{9}$$

Proof. We have $f(z) = F(z)\omega(z)$ with $\omega(z)\in B$. The well known estimations: $|F(z)|\leqslant|z|(1-|z|)^{-2}$, $|F'(z)|\leqslant(1+|z|)(1-|z|)^{-3}$, and (2), (3) imply

$$|f'(z)| = |F'(z)\omega(z)+F(z)\omega'(z)|$$
$$\leqslant \frac{1+|z|}{(1-|z|)^3}|\omega| + \frac{|z|}{(1-|z|)^2}\cdot\frac{1-|\omega|^2}{1-|z|^2} = \frac{1+|z|}{(1-|z|)^3}\left[|\omega| + \frac{|z|(1-|\omega|^2)}{(1+|z|)^2}\right].$$

The expression in square brackets attains a maximum for a fixed ω, $|\omega| < 1$, and z ranging over the unit disc if $|z| = 1$, the maximal value being $\frac{1+|z|}{(1-|z|)^3}[|\omega|+\frac{1}{4}(1-|\omega|^2)]$. Now, the values of $x+\frac{1}{4}(1-x^2)$, $x\in[0,1]$, lie within $[\frac{1}{4}, 1]$, and this proves Theorem 2. Another analogue of a well known result for subordinate functions is

Theorem 3. *If*

$$f(z) = \sum_{k=1}^{\infty} a_k z^k,\ F(z) = \sum_{k=1}^{\infty} A_k z^k$$

are regular in $|z| < 1$ *and if* $|f(z)| \leqslant |F(z)|$ *in* $|z| < 1$, *then*

$$\sum_{k=1}^{n} |a_k|^2 \leqslant \sum_{k=1}^{n} |A_k|^2. \tag{10}$$

Proof. Put

$$s_n = \sum_{k=1}^{n} a_k z^k,\ S_n = \sum_{k=1}^{n} A_k z^k,\ R_n = \sum_{k=n+1}^{\infty} A_k z^k.$$

We have $f(z) = F(z)\omega(z)$ with $\omega(z)\in B$. Hence

$$f(z) = S_n(z)\omega(z)+R_n(z)\omega(z) = S_n(z)\omega(z)+\sum_{k=n+1}^{\infty} a_k' z^k.$$

Besides,

$$S_n(z)\omega(z) = s_n(z)+\sum_{k=n+1}^{\infty} a_k'' z^k. \tag{11}$$

The inequality

$$\int_0^{2\pi} |S_n(z)\omega(z)|^2 d\theta \leqslant \int_0^{2\pi} |S_n(z)|^2 d\theta, \quad z = re^{i\theta},$$

and (11) imply

$$\int_0^{2\pi} |s_n(z)+\sum_{k=n+1}^{\infty} a_k'' z^k|^2 d\theta \leqslant \sum_{k=1}^{n} |A_k|^2 r^{2k}$$

that is

$$\sum_{k=1}^{n} |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |a_k''|^2 r^{2k} \leqslant \sum_{k=1}^{n} |A_k|^2 r^{2k},$$

and for $r \to 1$ we obtain (10).

The relation (10) holds also under the assumption of subordination, (cf. [14]). The proofs in both cases are similar.

Theorem 4. *Under the assumptions of Theorem 3 we have*

$$|a_1| \leqslant |A_1|, \ |a_2| \leqslant 1 + \tfrac{1}{4}|A_2|^2/|A_1|.$$

Proof. Let $f(z) = F(z)\omega(z)$, where $\omega(z) = a_0 + a_1 z + \ldots \in B$. We have $a_1 = a_0 A_1$ and $a_2 = a_0 A_2 + a_1 A_1$. In view of $|a_0| \leqslant 1$ we have $|a_1| \leqslant |A_1|$. Besides from (3) it follows that $|\omega'(0)| \leqslant 1 - |\omega(0)|^2$, i.e. $|a_1| \leqslant 1 - |a_0|^2$. Hence $|a_2| \leqslant |a_0||A_2| + |a_1||A_1| \leqslant |a_0||A_2| + (1 - |a_0|^2)|A_1|$. Now, the right hand side attains a maximum $1 + \frac{1}{4}|A_2|^2/|A_1|$ for $|a_0| = \frac{1}{2}|A_2|/|A_1|$, and the bound for $|a_2|$ follows.

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Streszczenie

W pracy tej zajmuję się zagadnieniem wyznaczenia dokładnej wartości r_0 z przedziału (0,1) promienia koła w którym zachodzi nierówność $|f'(z)'| \leqslant$ $\leqslant |F'(z)|$ przy założeniu, że funkcja f jest modułowo podporządkowana funkcji F w kole jednostkowym. O funkcji F zakłada się, że jest jednolistna i gwiaździsta w tym kole. Prócz tego dowodzę dwu innych twierdzeń będących analogami odpowiednich twierdzeń z teorii majoryzacji funkcji z tym, że założenie podporządkowania obszarowego zastępuje nierównością modułów funkcji f i F.

Резюме

В этой работе автор занимается определением радиуса $r_0 \epsilon (0, 1)$ круга в котором исполняется неравенство $|f'(z)| < |F'(z)|$, если в целом одиночном кругу имеем $|f(z)| < |F(z)|$, где F — однолистная или звездообразная функция. Доказанные еще две теоремы аналогичны хорошо известным теоремам из теории подчиненных функций.