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Some Remarks on a Paper of M. S. Robertson

Kilka uwag o pewnej pracy M. S. Robertsona

Несколько заметок об одной работе М. С. Робертсона

1. Introduction. The aim of this paper is to establish two theorems A′ and B′ which are analogues of theorems A and B proved recently by M. S. Robertson, cf. [1]. In the statement of the Theorem B (due to M. S. Robertson) the notion of subordination plays a basic role, whereas in the statement of the Theorem B′ an analogous role plays the inequality between the moduli of two functions regular in the unit disc.

2. Main results.

Theorem A′. Let $w(z, t) = \sum_{n=0}^{\infty} b_n(t) z^n$ be regular in $|z| < 1$ for any $t \in \langle 0, \delta \rangle$ and let $|w(z, t)| < 1$ in $|z| < r$ for any $r \in (0,1)$ and for $t \in \langle 0, \delta(r) \rangle$ whereas $w(z, 0) = 1$. If the limit

$$w(z) = \lim_{t \to 0+} \frac{w(z,t)-1}{t^{\varrho}} \tag{1}$$

exists for a positive ϱ, then

$$Rw(z) \leqslant 0 \tag{1'}$$

in $|z| < 1$.

Proof. The function $u(z, t) = [w(z, t)-1]/[w(z, t)+1]$ is regular in $|z| < 1$ and of negative real part in $|z| < r$. We have

$$R \frac{w(z, t)-1}{t^{\varrho}} \frac{2}{w(z, t)+1} = R \left\{ \frac{2u(z, t)}{t^{\varrho}} \right\} < 0. \tag{2}$$

The condition (1) implies the equality $\lim_{t \to 0+} w(z, t) = w(z, 0)$, therefore in view of (2), we have $Rw(z) \leqslant 0$ in $|z| < 1$. The Theorem A′ is proved.

Theorem B′. Suppose $F(z,t)$ is a function regular in $|z|<1$ for any $t\in\langle 0,\delta\rangle$, vanishing at the origin for any $t\in\langle 0,\delta\rangle$. If $f(z) = F(z,0)$ is univalent in $|z|<1$, if $|F(z,t)|\leqslant|f(z)|$ in $|z|<r$ for any $r\in(0,1)$, for any $t\in\langle 0,\delta(r)\rangle$ and if the limit

$$F(z)=\lim_{t\to 0+}\frac{F(z,t)-F(z,0)}{t^{\varrho}} \tag{3}$$

exists for a real and positive ϱ, then

$$R\left\{\frac{F(z)}{f(z)}\right\}\leqslant 0 \tag{4}$$

in $|z|<1$.

Proof. The inequality $|F(z,t)|\leqslant|f(z)|$ is equivalent to the identity $F(z,t)=f(z)\,w(z,t)$, where $|w(z,t)|<1$ in $|z|<r$. Since $F(z,0)=f(z)$, we have $w(z,0)=1$. Hence

$$\frac{F(z,t)-F(z,0)}{t^{\varrho}}=f(z)\frac{w(z,t)-1}{t^{\varrho}}. \tag{5}$$

The left hand side in (5) has a limit $F(z)$ for $t\to 0+$, therefore the limit $\lim\limits_{t\to 0+}[w(z,t)-1]/t^{\varrho}=w(z)$ exists. Since $f'(0)\neq 0$, we have in view of Theorem A′, $R\{F(z)/f(z)\}\leqslant 0$.

Corollary 1. It is easy to see that, if $w(z)$ (resp. $F(z)$) are regular in $|z|<1$ and $Rw(0)\neq 0$ (resp. $R\{F(0)/f(0)\}\neq 0$) then the sign of equality in (1′) and (4) is imposible (the maximum principle for harmonic functions).

3. Applications

Let S be the class of functions $f(z)=z+a_2z^2+\ldots$ regular and univalent in $|z|<1$ and let $\check{S}$ be the subclass of functions mapping the unit disc on spiral-like domains. It is well known [2] that $f\in\check{S}$ if and only, if the real part of $e^{-i\varphi}zf'(z)/f(z)$ is positive for some real constant φ. For $\varphi=0$ we obtain the subclass S^* of functions mapping the unit disc on domains starshaped w.r.t. origin.
We now prove the following

Theorem C′. If $f\in S$ then $f\in\check{S}$ if and only, if there exists a $\delta(r)>0$ such that for any $t\in\langle 0,\delta(r)\rangle$ the inequality

$$|f[z(1-te^{-i\varphi})]|\leqslant|f(z)| \tag{6}$$

holds in the disc $|z|<r$, $r\in(0,1)$ (for some real constant φ).

Proof. Necessity. Put $F(z,t)=f[z(1-te^{-i\varphi})]$. We have $F(0,t)=0$, $F(z,0)=f(z)$ and

$$F(z)=\lim_{t\to 0+}\frac{f[z(1-te^{-i\varphi})]-f(z)}{t}=\lim_{t\to 0+}\frac{-ze^{-i\varphi}[f(z-zte^{-i\varphi})-f(z)]}{-tze^{-i\varphi}}=$$
$$=-e^{-i\varphi}zf'(z).$$

From the Theorem B′ we have $R\{e^{-i\varphi}zf'(z)/f(z)\}>0$ in $|z|<1$, hence $f\in\hat{S}$.

Sufficiency. Let now $f\in\hat{S}$ and let $F(z,t)=f[z(1-te^{-i\varphi})]$. We have for $|z|<r$ and $t>0$: $\{\partial F(z,t)/\partial t\}/F(z,t)|_{t=0}=-ze^{-i\varphi}f'(z)/f(z)$ thus $R\{F'_t/F\}_{t=0}<0$ in $|z|<r$ because $f\in\check{S}$. The continuity of the function $F'_t/F(z,t)$ with respect to t, $t\in\langle 0,1\rangle$, implies

$$R\{F'_t/F\}<0 \tag{8}$$

for $t\in\langle 0,\delta(r)\rangle$, $|z|<1$, and δ sufficiently small. The condition (8) implies that for every fixed z, $|z|<r$, $|F(z,t)|$ is a decreasing function of t. Since $F(z,0)=\lim_{t\to 0+}F(z,t)=f(z)$, we have $|F(z,t)|\leqslant|f(z)|$, $t\in\langle 0,\delta(r)\rangle$. The theorem C′ is proved.

Corollary 2. If $f(z)\in S$, then $f\in S^*$ if and only, if there exists a $\delta(r)>0$ such that $|f[z(1-t)]|\leqslant|f(z)|$ in $|z|<r$ for any $t\in\langle 0,\delta(r)\rangle$ and any $r\in(0,1)$.

REFERENCES

[1] Robertson, M. S., Applications of the subordination principle to univalent functions, Pacific Journ. of Math. XI, (1961), p. 315-324.
[2] Špaček, L., Přispěvek k teorii funkci prostych, Časopis Pěst. Mat. 62 (1933), p. 12-19.

Streszczenie

W pracy tej dowodzę dwu podstawowych twierdzeń A′ i B′, które pozwalają na charakteryzację pewnych klas funkcji holomorficznych w kole jednostkowym. W twierdzeniach tych główną rolę gra nierówność modułów funkcji holomorficznych. Twierdzenia te są pewnymi analogonami twierdzeń A i B Robertsona [1]. W zastosowaniu daję nieznane, o ile mi się wydaje, warunki konieczne i dostateczne na to, by funkcja $f(z)$ holomorficzna i jednolistna w kole $|z|<1$ była funkcją spiralną.

Резюме

В этой работе я доказываю две основные теоремы A' и B', которые позволяют характеризовать некоторые классы функций голоморфных в единичном круге. В этих теоремах главную роль играет неравенство модулей голоморфных функций. Эти теоремы являются некоторыми аналогиями теореми A и B Робертсона [1]. Как применение я даю новые, как думаю, необходимые и достаточные условия того, чтобы функция $f(z)$ голоморфная и однолистная в круге $|z| < 1$ была спиральной функцией.