ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA
VOL. XVI, 10 SECTIO A 1962

Z Zakładu Statystyki Matematycznej Wydziału Rolnego
Wyższej Szkoły Rolniczej w Lublinie
Kierownik: doc. dr Wiktor Oktaba

DOMINIK SZYNAL

On the Strong Law of Large Numbers for Random Variables Bounded by Sequences of Numbers

O mocnym prawie wielkich liczb dla zmiennych losowych ograniczonych przez ciągi liczbowe

Об усиленном законе больших чисел для случайных величин ограниченных через числовые последовательности

1. Introduction

Let $\{X_k\}$ $(k=1,2,\ldots)$ be a sequence of random variables and let $S_n=\sum_{k=1}^{n}X_k$.

Definition 1. The strong law of large numbers (SLLN) is said to hold for the sequence $\{X_k\}$ if there exists a sequence of constants $\{c_n\}$ such that

$$P\left[\lim_{n\to\infty}\left(\frac{S_n}{n}-c_n\right)=0\right]=1. \tag{1}$$

Definition 2. The sequence of random variables $\{S_n/n\}$ is said to contain the characteristic subsequence $\{S_{n_k}/n_k\}$ $(k=1,2,\ldots)$ if

$$\begin{cases} 1^\circ \ \lim\limits_{k\to\infty}\dfrac{n_{k+1}}{n_k}=1, \quad n_k<n_{k+1}, \\[2ex] 2^\circ \ P\left[\lim\limits_{k\to\infty}\dfrac{S_{n_k}-E(S_{n_k})}{n_k}=0\right]=1, \end{cases} \tag{2}$$

where $E(S_{n_k})$ is the expectation of S_{n_k}.

A well known though unsolved problem in the theory of probability is to find a set of necessary and sufficient conditions for the validity of the strong law of large numbers. The results of heretofore investigations may be found in the papers of Chung [1], of Loève [5], and of Prohorov [6].

In papers devoted to the necessary and sufficient conditions for SLLN some authors give such conditions for certain classes of random variables, expressing them in terms of moments, in terms of probabilities of sums of some segments of the considered sequence [6], or in terms of characteristic subsequences [3], while other authors indicate instead certain classes of random variables for which necessary and sufficient conditions, expressed e. g. in terms of moments, do not exist [2].

For uniformly limited variables E. Franckx [3] shows that the existence of characteristic subsequence is a necessary and sufficient condition for SLLN.

In this paper it is remarked that the existence of characteristic subsequence is the necessary condition for any sequence of random variables, and it is proved that this condition is also sufficient for such a class of random variables for which there exists such a sequence of numbers $\{L_n\}$ that

$$\sum_{n=1}^{\infty} P[|X_n| \geqslant L_n] < \infty. \tag{3}$$

Furthermore it is shown that the existence of characteristic subsequence is not always sufficient for random variables which do not obey (3). It is also shown that the assumption (3) may be replaced by some other assumptions.

2. Theorem.

Let $\{X_k\}$ be a sequence of random variables such that (3) *holds. The existence of characteristic subsequence is the necessary and sufficient condition for* SLLN.

Proof.

A. The condition is necessary.

The proof of E. Franckx for the class of random variables that are uniformly limited is based on the convergence almost surely (a. s.) of $\{S_n/n\}$ and may be transfered to the class of random variables in the above theorem, and even, to any sequence $\{X_k\}$, since a convergence almost surely of a sequence $\{S_n/n\}$ implies a convergence almost surely of subsequence $\{S_{n_k}/n_k\}$ $(k \to \infty)$.

B. The condition is sufficient.

Without loss of generality we may assume that the median $\mu(X_k \mid X_1, X_2, \ldots, X_{k-1}) = 0$, since if a theorem holds for the sequence of random variables centered at the conditional medians, it follows that it holds also for original sequence of random variables [4]. Let

$$U_k = X_k, \quad V_k = 0 \quad \text{for} \quad |X_k| < L_k,$$
$$U_k = 0, \quad V_k = X_k \quad \text{for} \quad |X_k| \geqslant L_k,$$
$$S'_n = \sum_{k=1}^{n} U_k.$$

With the assumption that $S'_{n_k}/n_k \to 0$ a. s. for $k \to \infty$, we have

$$Y_k = \frac{S'_{n_k} - S'_{n_{k-1}}}{n_k} = \frac{S'_{n_k}}{n_k} - \frac{n_{k-1}}{n_k} \cdot \frac{S'_{n_{k-1}}}{n_{k-1}} \to 0 \text{ a. s.}$$

Putting

$$Z_k = \max_{n_{k-1} < n \leqslant n_k} \frac{|S'_n - S'_{n_{k-1}}|}{n_k}$$

and using extended P. Lévy's inequality [4] we obtain, for every $\varepsilon > 0$, $P[Z_k \geqslant \varepsilon] \leqslant 2P[|Y_k| \geqslant \varepsilon]$, and hence $\Sigma P[Z_k \geqslant \varepsilon] \leqslant 2\, \Sigma P[|Y_k| \geqslant \varepsilon] < \infty$ so that $Z_k \to 0$ a. s. Therefore, for $n_{k-1} < n \leqslant n_k$

$$\left|\frac{S_n}{n}\right| = \left|\frac{S'_n - S'_{n_{k-1}}}{n} + \frac{S'_{n_{k-1}}}{n}\right| \leqslant \frac{n_k}{n} Z_k + \frac{n_{k-1}}{n}\left|\frac{S'_{n_{k-1}}}{n_{k-1}}\right| \leqslant$$
$$\leqslant \frac{n_k}{n} Z_k + \left|\frac{S'_{n_{k-1}}}{n_{k-1}}\right| \to 0 \text{ a. s.}$$

This proves SLLN for U_k, since for $\{X_k\}$, satisfying (3),

$$|\mu(S_n | S_1, S_2, \ldots, S_{n-1}) - E(S_n | S_1, S_2, \ldots, S_{n-1})|/n \to 0,$$

for $n \to \infty$ [4]. Now, we have on the basis of (3)

$$\Sigma P[V_k \neq 0] = \Sigma P[|X_k| \geqslant L_k] < \infty$$

so that SLLN holds for $\{X_k\}$.

3. Remarks.

Remark 1. The existence of characteristic subsequence is not always sufficient for SLLN if $\{X_k\}$ does not obey (3).

For example, let $\{X_k\}$ be a sequence of independent random variables such that $P\{X_k = -k^{1/2}\} = P\{X_k = k^{1/2}\} = 1/\log\log n$, and $P\{X_k = 0\} = 1 - 2/\log\log n$. It is easy to verify that $E(X_k) = 0$, $\sigma^2(X_k) = 2k/\log\log k$.

The sequence $\{X_k\}$ obeys weak law of large numbers (WLLN), because

$$\sum_{k=3}^{n} \sigma^2(X_k)/n^2 = \Big(\sum_{k=3}^{n} 2k/\log\log k\Big)/n^2 < 2/\sqrt{n}\log\log n \to 0, \text{ for } n \to \infty.$$

So there exists a subsequence $\{S_{n_k}/n_k\}$ which converges to zero a. s. But SLLN is not satisfied, which will be shown directly. Because $|X_k| = o(k/\log\log k)$ the necessary and sufficient condition for the validity for SLLN is convergence of the series

$$\sum_{r=1}^{\infty} e^{-\varepsilon/H_r}, \quad H_r = \frac{1}{2^{2r}} \sum_{k>2^r}^{2^{r+1}} \sigma^2(X_k),$$

for any $\varepsilon > 0$ [6]. By elementary computation we obtain

$$\sum_{r=1}^{\infty} \exp(-\varepsilon/H_r) > \sum_{r=1}^{\infty} \exp(-\varepsilon \log(r+1)\log 2) = \infty,$$

which shows that SLLN does not hold for $\{X_k\}$ in spite of the existence of characteristic subsequence.

Remark 2. The assumption (3) may be replaced by some other assumption concerning the sequence $\{X_k\}$, for example, by assumption

$$K_i = \frac{X_{n_i+1} + X_{n_i+2} + \ldots + X_{n_{i+1}}}{n_i} \to 0 \text{ a. s.}$$

or, by assumption

$$R_i = \sup_k \left| \frac{S_k}{k} - \frac{S_{n_i}}{n_i} \right| \to 0 \text{ a. s. [5].}$$

REFERENCES

[1] Chung, K.L., *The strong law of large numbers*, Proc. Second Berkeley Symp. on Stat. and Prob., (1951), p. 341-352.

[2] Fisz, M., *On necessary and sufficient conditions for the validity of the strong law of large numbers expressed in terms of moments*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 7 (1959), p. 221-225.

[3] Franckx, E., *La loi forte des grands nombres des variables uniformement bornées*, Trab. Estad. 9 (1958), p. 111-115.

[4] Loève, M., *Probability Theory*, New York, 1955.

[5] Loève, M., *On almost sure convergence*, Proc. Second Berkeley Symp. on Stat. and Prob., (1951), p. 279-303.

[6] Прохоров, Ю. В., *Об усиленном законе больших чисел*, Изв. АН СССР, сер. матем., 14, (1950), p. 523-536.

Streszczenie

W pracy tej podano warunki konieczne i dostateczne dla spełnienia mocnego prawa wielkich liczb dla ciągu zmiennych losowych ograniczonych przez ciągi liczbowe oraz wykazano, że dla ciągów pewnych zmiennych losowych nieograniczonych nie istnieją warunki konieczne i dostateczne wyrażone przez podciągi charakterystyczne.

Резюме

В этой работе, поданы необходимые и достаточные условия приложимости усиленного закона больших чисел (у. з. б. ч.) к последовательности случайных величин ограниченных через числовые последовательности, а также доказано, что к последовательности некоторых неограниченных случайных величин не существуют необходимые и достаточные условия приложимости у. з. б. ч. выраженные через характерные подпоследовательности.